Positive Representations of Split Real Quantum Groups

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This talk is based on the following papers:


I. Ip, *Positive representations of split real quantum groups of type $B_n$, $C_n$, $F_4$ and $G_2$*, arXiv:1205.2940
Let \( \mathfrak{g} \) be a simple Lie algebra, and \( \mathfrak{g}_c \) its compact form. (e.g. for type \( A_n \), \( \mathfrak{g} = SL(n + 1, \mathbb{C}) \) and \( \mathfrak{g}_c = SU(n + 1) \).)
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Motivation

Let $\mathfrak{g}$ be a simple Lie algebra, and $\mathfrak{g}_c$ its compact form. (e.g. for type $A_n$, $\mathfrak{g} = SL(n+1, \mathbb{C})$ and $\mathfrak{g}_c = SU(n+1)$.)

Representation theory has nice properties:

- Finite dimensional representations $V_\lambda$ parametrized by $\lambda \in P^+ \subset \mathfrak{h}_\mathbb{R}^*$ (positive weights),
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Existence of universal $R$ matrix $\Rightarrow$ Braided Tensor Category.
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Considering the split real form $\mathfrak{g}_R$
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Example:

$(SL(2, \mathbb{R}))$: $\mathcal{P}_\lambda \otimes \mathcal{P}_\mu \simeq \bigoplus P_\nu \bigoplus$ (discrete series)...

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Parametrization: we can restrict to principal series associated to Borel subalgebra $\mathfrak{b}_\mathbb{R}$, parametrized by $\mathbb{R}_+$-span of $P^+$.

(Minimal principal series)
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Let \( q = e^{\pi ib^2}, |q| = 1, b^2 \in (0, 1) \setminus \mathbb{Q} \).

\( \mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R})) \) is the Hopf-* algebra generated by \( E, F, K \) such that

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KE = q^2 KE, \quad KF = q^{-2} FK, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}
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K^* = K, \quad E^* = E, \quad F^* = F
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For higher rank, also $E_iF_j = q^{a_{ij}}F_jE_i$, Serre relations etc.
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unbounded operators on $L^2(\mathbb{R})$. ($p = \frac{1}{2\pi i} \frac{d}{dx}$)

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Together they form the Modular Double.
Ponsot-Teschner’s representation

Special class of representation $\mathcal{P}_\lambda$ for $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ ($\lambda \in \mathbb{R}_+$):

$$E = \left( i q - q - 1 \right) \left( e^{\pi b (x - \lambda - 2p)} + e^{\pi b (-x + \lambda - 2p)} \right)$$
$$F = \left( i q - q - 1 \right) \left( e^{\pi b (x + \lambda + 2p)} + e^{\pi b (-x - \lambda + 2p)} \right)$$
$$K = e^{-2\pi bx}$$

(Note: $i q - q - 1 = (2 \sin \pi b/2)^2 - 1 > 0$.)
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**Theorem**

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$\mathcal{P}_\lambda$ has NO classical limit as $b \longrightarrow 0!$
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1. $\mathcal{P}_\lambda$ is parametrized by $\lambda \in \mathbb{R}_+$.  
2. Represented by positive (essentially) self-adjoint operators on $L^2(\mathbb{R})$.  

Replacing $b$ by $b - 1$, gives $\tilde{E}, \tilde{F}, \tilde{K}$ commuting with $E, F, K$, also a representation of $U_{\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$.

Define $e = (i\tilde{q}^{-1} - q^{-1}) - 1 E, f = (i\tilde{q}^{-1} - q^{-1}) - 1 F$, we have

$e_1 b_2 = \tilde{e}, f_1 b_2 = \tilde{f}, K_1 b_2 = \tilde{K},$

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- Define

$$e = \left( \frac{i}{q - q^{-1}} \right)^{-1} E, \quad f = \left( \frac{i}{q - q^{-1}} \right)^{-1} F,$$

we have

$$e^\frac{b^2}{2} = \tilde{e}, \quad f^\frac{b^2}{2} = \tilde{f}, \quad K^\frac{b^2}{2} = \tilde{K},$$

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- Here $g_b(x)$ is called quantum dilogarithm.
- $g_b(x)$ is the non-compact analogue of the $q$-exponential function
- $|g_b(x)| = 1$ when $x \in \mathbb{R}_{>0}$. 

Ivan Ip (Kavli IPMU)
Properties

Closure under tensor product (in the continuous sense):

**Theorem (Ponsot-Teschner (2000))**

We have

\[ \mathcal{P}_\alpha \otimes \mathcal{P}_\beta \simeq \int_{\mathbb{R}^+} \oplus P_\gamma d\mu(\gamma) \]

where \( d\mu(\gamma) \) is expressed in terms of (a variant of) quantum dilogarithm.
Properties

Peter-Weyl type Theorem (proposed by Ponsot-Teschner):

Theorem (Ip (2011))
We have

\[ L^2(SL_q^+(2, \mathbb{R})) \cong \int_{\mathbb{R}_+} \bigoplus P_\alpha \otimes P_\alpha d\mu(\alpha) \]

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Here \( L^2(SL^+_q(2, \mathbb{R})) \) is a Hilbert space constructed from the GNS representation of a \( C^* \)-algebraic version of \( SL^+_q(2, \mathbb{R}) \).
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The action of \( U_q(\mathfrak{sl}(2, \mathbb{R})) \) is obtained by dualizing the regular corepresentation of \( SL_q^+(2, \mathbb{R}) \).
Definition

Can this class of representations be extended to arbitrary type $\mathfrak{g}_R$?
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1. The class of representation is parametrized by $\mathbb{R}^+$-span of $P^+$, or equivalently, $(\mathbb{R}^+)^\text{rank}(\mathfrak{g})$.
2. The action of the generators $E_i$, $F_i$, $K_i$ are represented by positive (essentially) self adjoint operators.
3. Transcendental relations $\tilde{X} = X_1 b_2$ exist, relating $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ to $\mathcal{U}_{\tilde{q}}(\mathfrak{g}_\mathbb{R})$ (Modular Double).

We call these "positive principal series representations" or "positive representations" for short. The answer is YES, and have been constructed for all types of $\mathfrak{g}_\mathbb{R}$.
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Fix a longest element $w_0 = s_{i_1}...s_{i_m} \in W$ in the Weyl group with reduced expression.
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*Fix a longest element $w_0 = s_{i_1}...s_{i_m} \in W$ in the Weyl group with reduced expression. Then the totally positive upper unipotent subgroup $U^+_>0$ is parametrized by*

$$\mathbb{R}^m_{>0} \rightarrow U^+_>0$$

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where $(x_i, \chi_i, y_i)$ is the root subgroup for each root $i$. 
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I will use $\mathcal{U}_q(\mathfrak{sl}(3, \mathbb{R}))$ as a toy model. The general idea is similar.

Theorem (Lusztig Data for total positivity)

Fix a longest element $w_0 = s_{i_1} \cdots s_{i_m} \in W$ in the Weyl group with reduced expression. Then the totally positive upper unipotent subgroup $U^+_{>0}$ is parametrized by

$$\mathbb{R}^m_{>0} \rightarrow U^+_{>0}
\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \mapsto x_{i_1}(a_1) \cdots x_{i_m}(a_m)$$

where $(x_i, \chi_i, y_i)$ is the root subgroup for each root $i$.

Example: choosing $w_0 = s_2 s_1 s_2$ we have

$$U^+_{>0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b & bc \\ 0 & 1 & a + c \\ 0 & 0 & 1 \end{pmatrix}$$
Construction

From this we can apply the regular representation

\[ g \cdot f(g_+) = [f(g+g)] + \chi_\lambda(g+g) \]

to get the action of \( E, F, H \)
Construction

From this we can apply the regular representation

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Example: \( e^{tE_2} \):

\[
\begin{pmatrix}
1 & b & bc \\
0 & 1 & a + c \\
0 & 0 & 1
\end{pmatrix}
\cdot
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & t \\
0 & 0 & 1
\end{pmatrix}
\]
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0 & 0 & 1
\end{pmatrix}
\]

i.e. action on \( \mathbb{C}[U^+_0] : f(a, b, c) \mapsto f(a, b, c + t) \)
**Construction**

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**Example:** \( e^{tE_2} : \begin{pmatrix} 1 & b & bc \\ 0 & 1 & a + c \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \ t \\ 0 & 0 & 1 \end{pmatrix} \)

i.e. action on \( \mathbb{C}[U^+_{>0}] : f(a, b, c) \mapsto f(a, b, c + t) \)

**Action of** \( E_2 : \frac{\partial}{\partial c} : f \mapsto f_c \)
We have

\[
E_1 : f \mapsto \frac{c}{b} f_a + f_b - \frac{c}{b} f_c \\
E_2 : f \mapsto f_c \\
F_1 : f \mapsto -b^2 f_b + baf_a + 2\lambda_1 b \\
F_2 : f \mapsto -a^2 f_a - 2caf_a + bcf_b - c^2 f_c + 2\lambda_2 (a + c) \\
H_1 : f \mapsto af_a - 2b f_b + cf_c + 2\lambda_1 \\
H_2 : f \mapsto -2af_a + b_b - 2c_c + 2\lambda_2
\]

Acting on \( \mathbb{C}[U^+_0] \).
Construction

We have

\[ E_1 : f \mapsto \frac{c}{b} f_a + f_b - \frac{c}{b} f_c \]
\[ E_2 : f \mapsto f_c \]
\[ F_1 : f \mapsto -b^2 f_b + baf_a + 2\lambda_1 b \]
\[ F_2 : f \mapsto -a^2 f_a - 2caf_a + bcf_b - c^2 f_c + 2\lambda_2 (a + c) \]
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\[ H_2 : f \mapsto -2af_a + b_b - 2c_c + 2\lambda_2 \]

Acting on \( \mathbb{C}[U^+] \).

Different from usual regular representation acting on \( \mathbb{C}[U^+] \).
Crucial Step: Mellin Transform

Formally:

\[ f(u) \mapsto F(x) := \int f(u) x^u \, du \]

So that:

\[ x \cdot f(u) \mapsto f(u-1) \]

\[ \frac{\partial}{\partial x} : f(u) \mapsto (u+1) f(u+1) \]

\[ x \frac{\partial}{\partial x} : f(u) \mapsto uf(u) \]

Differential operators become finite difference operators!
Crucial Step: Mellin Transform

Mellin Transform: unitary map $L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}^+)$:

$$f(u) \mapsto F(x) := \frac{1}{2\pi} \int_{\mathbb{R}} f(u)x^{-\frac{1}{2} + iu} du$$
Constructions | Mellin Transform

Crucial Step: Mellin Transform

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Differential operators becomes finite difference operators!
Crucial Step: Mellin Transform

We have \([(a,b,c)\mapsto (u,v,w)\text{ and simplifying notations...}]\)

\[
E_1 : f \mapsto (u + 1)f(u + 1, v + 1, w - 1) + (1 + v - w)f(v + 1)
\]

\[
E_2 : f \mapsto (w + 1)f(w + 1)
\]

\[
F_1 : f \mapsto (2\lambda_1 + u - v + 1)f(v - 1)
\]

\[
F_2 : f \mapsto (2\lambda_2 - u + 1)f(u - 1) + (2\lambda_2 - 2u + v - w + 1)f(w - 1)
\]

\[
H_1 : f \mapsto (u - 2v + w + 2\lambda_1)f
\]

\[
H_2 : f \mapsto (-2u + v - 2w + 2\lambda_2)f
\]

Acting on functions with \(\text{dim}(U^+)\text{ variables.}\)
Crucial Step 2: Quantization

\[ E_1: \quad u + 1 \rightarrow q(f(u + 1, v + 1, w - 1) + 1 + v - w)q(f(v + 1)) \]

\[ E_2: \quad w + 1 \rightarrow q(f(w + 1)) \]

\[ F_1: \quad 2\lambda_1 + u - v + 1 \rightarrow q(f(v - 1)) \]

\[ F_2: \quad 2\lambda_2 - u + 1 \rightarrow q(f(u - 1)) + 2\lambda_2 - 2u + v - w + 1 \rightarrow q(f(w - 1)) \]

\[ K_1 = qH_1 : qu - 2v + w + 2\lambda_1 \]

\[ K_2 = qH_2 : q - 2u + v - 2w + 2\lambda_2 \]

One can check this is a representation for \( U_q(x) \) without the real structure yet.
Crucial Step 2: Quantization

Simply quantizing the weights!!
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\[ E_1 : [u + 1]qf(u + 1, v + 1, w - 1) + [1 + v - w]qf(v + 1) \]
\[ E_2 : [w + 1]qf(w + 1) \]
\[ F_1 : [2\lambda_1 + u - v + 1]qf(v - 1) \]
\[ F_2 : [2\lambda_2 - u + 1]qf(u - 1) + [2\lambda_2 - 2u + v - w + 1]qf(w - 1) \]

\[ K_1 = q^{H_1} : q^{u-2v+w+2\lambda_1} \]
\[ K_2 = q^{H_2} : q^{-2u+v-2w+2\lambda_2} \]
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Simply quantizing the weights!!

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E_1 : [u + 1]qf(u + 1, v + 1, w - 1) + [1 + v - w]qf(v + 1)
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E_2 : [w + 1]qf(w + 1)
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F_1 : [2\lambda_1 + u - v + 1]qf(v - 1)
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\]

One can check this is a representation for \(\mathcal{U}_q(\mathfrak{g})\), without the real structure yet.
Crucial Step 3: Positivity twist

To obtain positive representations, the trick is to induce a "twist" in the quantum weight. 

\[
\begin{align*}
  u + 1 &\quad q \mapsto \left[ Q^2 b - i u b \right] q, \\
  Q &= b + 1, \\
  n &= q^n - q^{n-1}
\end{align*}
\]

Recall that the original variables belong to \( \mathbb{R}^>0 \). Now we use the correct Mellin transform, where the variable includes a complex part.

However, no more classical limit as \( b \to 0 \).

\[ E_2 : \left[ w + 1 \right] q f(w + 1) \to \left[ Q^2 b - i w b \right] q e^{-2\pi bp w} \]

Recall that \( q = e^{\pi ib/2} \), this can be rewritten as:

\[
(i q - q^{-1}) \left( e^{\pi b (w - 2p w)} + e^{-\pi b (w - 2p w)} \right)
\]

which is positive, and can also be shown to be (essentially) self adjoint as required!
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To obtain positive representations, the trick is to induce a ”twist” in the quantum weight.

\[ [u + 1]_q \mapsto \left[ \frac{Q}{2b} - i \frac{u}{b} \right]_q, \quad Q = b + \frac{1}{b}, \quad [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \]

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which is positive, and can also be shown to be (essentially) self adjoint as required!
Construction

Let us denote simply

\[ [u]e(-p) := \left[ \frac{Q}{2b} - i\frac{u}{b} \right]_q e^{-2\pi bp} \]
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It is understood that this is positive as long as \([p, u] = \frac{1}{2\pi i} \).

**Final result:**

\[
\begin{align*}
E_1 & : [u] e(-pu - pv + pw) + [v - w] e(-pv) \\
E_2 & : [w] e(-pw) \\
F_1 & : [2\lambda_1 + u - v] e(pv) \\
F_2 & : [2\lambda_2 - u] e(pu) + [2\lambda_2 - 2u + v - w] e(pw) \\
K_1 & : e^{\pi b (u - 2v + w - 2\lambda_1)} \\
K_2 & : e^{\pi b (-2u + v - 2w - 2\lambda_2)}
\end{align*}
\]

Acting on \(L^2(\mathbb{R}^{\dim U^+})\).
Final Step

So far we have constructed the representation for a particular choice of longest element $w_0$. 
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**Theorem**

The transformation of the operators of $\mathcal{U}_q(\mathfrak{g}_R)$ corresponding to the change of words $\ldots s_is_js_i\ldots = \ldots s_js_is_j\ldots$

$$x_i(u)x_j(v)x_i(w) \leftrightarrow x_j(u')x_i(v')x_j(w')$$
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So far we have constructed the representation for a particular choice of longest element $w_0$. However the representation is indeed canonical:

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is given by

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$$X \mapsto \Phi X \Phi^{-1},$$

where

$$\Phi = T \circ g_b(e^{\pi b(2p_w-2p_u+u-v+w)})g^*_b(e^{\pi b(2p_w-2p_u-u+v-w)}),$$

is a unitary transform. Here $T$ is a linear transformation of $\det=1$. 
Construction

The action of $F_i$ is essentially the Feigin map, and can be obtained directly from any reduced expression of $w_0$. 
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$$F_i = \sum_{k=1}^{n} \left[ \sum_{l=k}^{n} \left( \sum_{j}^{m} u_j^{l,m} - 2u_i^l \right) + u_i^k + 2\lambda_i \right] e(p_i^k)$$

To find action of $E_i$ in general:

Fix any reduced expression of $w_0$. Find the change of words so that the target index $i$ is to the rightmost. The action is $u_i e(-p_i u_i)$ Carry out the (actually very easy) unitary transformation to obtain the desired expression. Positive representations of $U_{q}(g_{R})$ of all simply-laced type can be computed this way.
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The action of $F_i$ is essentially the Feigin map, and can be obtained directly from any reduced expression of $w_0$.

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The action of $F_i$ is essentially the Feigin map, and can be obtained directly from any reduced expression of $w_0$.

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Positive representations of $U_q(g_\mathbb{R})$ of all simply-laced type can be computed this way.
Results

Type $A_n$: (for the best choice of $w_0$)

**Theorem**

The action of $E_i, F_i, K_i$ is given by

\[
E_i = \sum_{k=1}^{n-i+1} [u_{i+k-1}^k - u_{i+k}^k] e \left( \sum_{l=1}^{k} (p_{i+l-1}^{l-1} - p_{i+l}^{l}) \right),
\]

\[
F_i = \sum_{k=1}^{i} \left[ u_{i}^k - \sum_{l=k}^{i} (2u_{i}^l - u_{i-1}^l - u_{i+1}^l) - 2\lambda_i \right] e(p_i^k),
\]

\[
K_i = e^{\pi b (\sum_{k=1}^{i} (u_{i-1}^k + u_{i+1}^k - 2u_i^k) + 2\lambda_i)},
\]
Results

Type $D_n$: (for the best choice of $w_0$)

Theorem

For $i = 0$ or $1$:

$$E_i = \sum_{k=1}^{n-1} \left[ u_{k+i-1}^k - u_{2k-1}^2 \right] e \left( \sum_{l_0=1}^{s_1(k)} (-1)^{l_0} p_{i}^{l_0} - \sum_{l_1=1}^{s_2(k)} (-1)^{l_1} p_{1-i}^{l_1} - \sum_{l_2=1}^{2k-2} (-1)^{l_2} p_{2}^{l_2} \right)$$

$$+ \sum_{k=1}^{n-2} \left[ u_{2k}^2 - u_{k+i}^k \right] e \left( \sum_{l_0=1}^{s_1(k)} (-1)^{l_0} p_{i}^{l_0} - \sum_{l_1=1}^{s_2(k)} (-1)^{l_1} p_{1-i}^{l_1} - \sum_{l_2=1}^{2k} (-1)^{l_2} p_{2}^{l_2} \right)$$

and for $i \geq 2$,

$$E_i = \sum_{k=1}^{2n-2i-1} [(-1)^k (u_{k+i+1}^k - u_i^k)] e \left( \sum_{l_0=1}^{s_1(k)} (-1)^{l_0} p_{i}^{l_0} - \sum_{l_1=1}^{s_2(k)} (-1)^{l_1} p_{i+1}^{l_1} \right),$$

where $\overline{k} := k \pmod{2} \in \{0, 1\}$, and $s_1(k) := 2 \left\lfloor \frac{k}{2} \right\rfloor - 1$, $s_2(k) := 2 \left\lceil \frac{k}{2} \right\rceil$. 
### Number of terms for the action of $E_i$

<table>
<thead>
<tr>
<th></th>
<th>$A_n$</th>
<th>$D_n$</th>
<th>$E_6$</th>
<th>$E_7$</th>
<th>$E_8$</th>
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<tr>
<td>$E_0$</td>
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<td>23</td>
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<tr>
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<td></td>
<td></td>
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<td><strong>Total</strong>:</td>
<td>$\frac{n(n+1)}{2}$</td>
<td>$n^2 - 2$</td>
<td>43</td>
<td>80</td>
<td>175</td>
</tr>
</tbody>
</table>
Fun(?) Facts

Choice of reduced expression for $w_0$ is important.
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Example: For $E_8$, the best choice

$$w_0 = 4 \ 34 \ 034 \ 230432 \ 12340321 \ 5432103243054321 \ 654320345612345034230123456$$

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But from the previous remarks, they are unitary equivalent representations!
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Extending the Feigin map:
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We also have the existence of universal $R$-matrix, essentially replacing $\exp_q$ by $g_b$ in the Reshetikhin model, and showing certain positivity properties. (In preparation)
Properties

Transcendental Relations:

Define $e_i = (i q - q - 1)^{-1}$, $f_i = (i q - q - 1)^{-1}$, we have $e_{1b}^2 = \tilde{e}_i$, $f_{1b}^2 = \tilde{f}_i$, $K_{1b}^2 = \tilde{K}_i$, where $\tilde{E}_i$, $\tilde{F}_i$, $\tilde{K}_i$ generates $U_{\tilde{q}}(g_R)$. (replacing $b \leftarrow b - 1$)

Follows from the "magic Lemma" by Yu. Volkov: For $u, v > 0$: $uv = q^2 vu = \Rightarrow (u + v)^{1b} = u^{1b} + v^{1b}$

However, $(\tilde{E}_i, \tilde{F}_i, \tilde{K}_i)$ commute with $(E_i, F_i, K_i)$ only up to a sign.

$= \Rightarrow$ Need a slight modification in order to define the modular double.
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After modifying the quantum group by some scaling of $K_i$’s, we then have the following result for $U_q(g_{\mathbb{R}})$:
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**Theorem**

The commutant of $U_q(\mathfrak{g}_\mathbb{R})$ is the Langlands dual of the modular double counterpart,

$$(U_q(\mathfrak{g}_\mathbb{R}))' = U_{\tilde{q}}(L\mathfrak{g}_\mathbb{R})$$
Non-simply-laced case

Positive representations constructed using similar techniques. (Different reduced expressions of $w_0 \rightarrow$ transformations are more complicated)

Surprising discovery using transcendental relations. Transcendental relations no longer exchange $b \leftrightarrow b^{-1}$. Instead, it exchanges short roots and long roots!!
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Theorem

Let $q_i = q_i^{\frac{1}{2}}(\alpha_i,\alpha_i) = e^{\pi ib_i^2}$. Define

$$e_i = \left( \frac{i}{q_i - q_i^{-1}} \right)^{-1} E_i,$$

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and define $\tilde{e}_i$, $\tilde{f}_i$, $\tilde{K}_i$ then

$E_i$, $F_i$ generates $U_{\tilde{q}}(LgR)$. If $E_i$ is the short root in $U_q(gR)$, then $\tilde{E}_i$ is the long root in $U_{\tilde{q}}(LgR)$ and vice versa.
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Theorem

Let \( q_i = q^{1/2}(\alpha_i, \alpha_i) = e^{\pi i b_i^2} \). Define

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Note that under this framework, there is still no classical limit.
Future Perspectives

Conjecture

*The class of positive representations is closed under tensor product (in the continuous sense), hence form a (certain kind of) Braided Tensor Category.*
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Indeed using the *magic Lemma*, we have

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Thank you!