Introduction to Cluster Algebra

Homework 1 (Due 2017/05/08)

(Q1) Compute the Conway-Coxeter frieze pattern $(x_1, x_2, x_3, \ldots)$ for $(d_1, d_2) = (1, 3)$, i.e.

$$x_{k+1}x_{k-1} = \begin{cases} x_k + 1 & k \text{ is even} \\ x_k^3 + 1 & k \text{ is odd} \end{cases}$$

or equivalently, the cluster variables of the cluster algebra associated to the initial seed (x_1, x_2) with exchange matrix B:

$$B = \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}.$$

What is the period of this pattern?

(Q2) Proof that there are $\begin{pmatrix} 2n \\ n \end{pmatrix} - 1$ minors $\Delta_{I,J}$ for an $n \times n$ matrix. Recall that a minor $\Delta_{I,J}$ is the determinant of the submatrix restricted to the row index I and column index J with $1 \le |I| = |J| \le n$.

(Q3) Show that the matrices

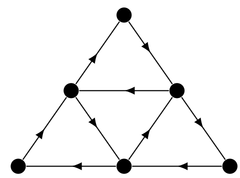
$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

and

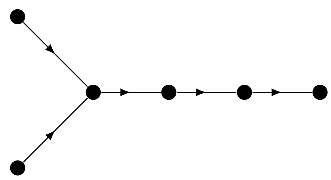
$$B = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 1 & 2 & \cdots & n-1 \\ 0 & 0 & 1 & \cdots & n-2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

are totally positive, by **finding the planar networks** with weight matrix A and B respectively. (Hint: $B = A^2$)

(Q4) Show that the quiver associated to $\mathbb{C}[SL_5/N]$:



is mutation equivalent to some orientation of the $\mathbf{D_6}$ **Dynkin quiver**:



Note: For a tree diagram, any orientation of arrows are mutation equivalent.

(Q5) Let B be skew-symmetric. Recall that the *matrix mutation in direction* k, denoted by μ_k, is defined by

$$B' := \mu_k(B),$$

where

$$b'_{ij} = \begin{cases} -b_{ij} & k \in \{i, j\} \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+ & \text{otherwise} \end{cases} .$$

Show that

(1) B' is again skew-symmetric,

(2) μ_k is an involution, i.e. $\mu_k(B') = B$.