Introduction to Cluster Algebra

Homework 2 (Due 2017/06/12)

(Q1) Consider the sequence $y_0, y_1, \ldots$ defined by the recurrence

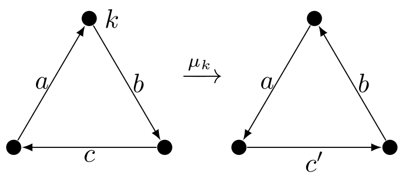
$$y_k = \frac{y_{k-1}^2 + y_{k-1} + 1}{y_{k-2}}$$

Show that every term of this sequence is a Laurent polynomial in y_0 and y_1 with coefficients in $\mathbb{Z}$. In particular, if $y_0 = y_1 = 1$, this is an integer sequence!

Hint: Apply the Caterpillar Lemma with a suitable exchange pattern of rank 2 and exchange polynomial $P \in \mathbb{Z}[x_1, x_2]$ by verifying the conditions of the lemma.

(Q2) Compare the denominators of the cluster variables of the Conway-Coxeter frieze pattern of height $n = 3$ and the root system of type A_3.

(Q3) For a skew-symmetrizable matrix B with diagram $\Gamma(B)$ whose edges have weights $|b_{ij}b_{ji}|$, we have the diagram mutation rule:



where

$$\sqrt{c} + \sqrt{c'} = \sqrt{ab}$$

Show that c' must be an integer. (Hint: There exists integer diagonal matrix D such that DB is skew symmetric.)

(Q4) Show that

(a) Show that $D_4^{(1)}$ is 2-infinite.

(b) Show that $F_4^{(1)}$ is 2-infinite.

(c) Show that $G_2^{(1)}$ is 2-infinite.

(d) Oriented triangle with edge weights $\{2, 2, 1\} \sim B_3$.

(e) Oriented 4-cycle with edge weights $\{2, 1, 2, 1\} \sim F_4$.

(Q5) In type C_3, let $I_+ = \{1, 3\}, I_- = \{2\}$ and the long root be α_3. Calculate the $\langle \tau_+, \tau_- \rangle$-orbits of the almost positive roots of type C_3.

(The positive roots can be found on the C_3-cyclohedron in the lecture notes.)