Lecture Notes
Introduction to Cluster Algebra

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§1 Examples

1.1 Conway-Coxeter frieze pattern

Frieze is the wide central section part of an entablature, often seen in Greek temples, usually with a horizontal repeating pattern.

Example 1.1 (Frieze of height $n = 3$)**.** *Fill in the blanks with numbers, such that whenever we have 4 numbers arranged as* $\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$*, we have* $ad - bc = 1$*.*

$$\begin{array}{cccccccccccc}
 & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1} \\
\boxed{1} & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & \\
 & \boxed{1} & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot \\
 & & \boxed{1} & & \cdot & & \cdot & & \cdot & & \cdot & \\
 & & & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1}
\end{array}$$

It is easy to solve by putting $d = \frac{bc+1}{a}$*, and fill in the blanks accordingly, we end up with:*

$$\begin{array}{cccccccccccc}
 & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1} \\
\boxed{1} & & 2 & & 2 & & 2 & & 1 & & 4 & \\
 & \boxed{1} & & 3 & & 3 & & 1 & & 3 & & \cdots \\
 & & \boxed{1} & & 4 & & 1 & & 2 & & \cdots & \\
 & & & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1}
\end{array}$$

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We note that the pattern starts repeating itself (with the fundamental domain highlighted in red)

Let's look at a more complicated example by changing the shape of the left boundary

Example 1.2 (Frieze of height $n = 5$)**.** .

$$
\begin{array}{cccccccccccccc}
 & & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1} & \\
 & \boxed{1} & & 3 & & 2 & & 2 & & 2 & & 1 & & \cdots \\
\boxed{1} & & 2 & & 5 & & 3 & & 3 & & 1 & & \cdots & \\
 & \boxed{1} & & 3 & & 7 & & 4 & & 1 & & \cdots & & \\
 & & \boxed{1} & & 4 & & 9 & & 1 & & \cdots & & & \\
 & & & \boxed{1} & & 5 & & 2 & & 1 & & \cdots & & \\
 & & & & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1} & & &
\end{array}
$$

Again we return to a row of 1's! Furthermore, all the entries are positive integers.

Now let us look at a more general pattern in the case of height $n = 2$:

$$
\begin{array}{ccccccccc}
 & \boxed{1} & & \boxed{1} & & \boxed{1} & & \boxed{1} & \\
\boxed{x_1} & & x_3 & & x_5 & & x_7 & & \cdots \\
 & \boxed{x_2} & & x_4 & & x_6 & & \cdots & \\
 & & \boxed{1} & & \boxed{1} & & \boxed{1} & &
\end{array}
$$

Example 1.3. *We can solve for the variables and obtain*

$$
\begin{aligned}
x_3 &= \frac{x_2 + 1}{x_1} \\
x_4 &= \frac{x_3 + 1}{x_2} = \frac{x_1 + x_2 + 1}{x_1 x_2} \\
x_5 &= \frac{x_4 + 1}{x_3} = \frac{x_1 + 1}{x_2} \\
x_6 &= \frac{x_5 + 1}{x_4} = x_1 \\
x_7 &= \frac{x_6 + 1}{x_5} = x_2
\end{aligned}
$$

We see that we return to the initial variables, and the pattern repeat itself. In general, for arbitrary height n,

(1) For $k > n$, the variables x_k can be expressed as Laurent polynomial of the variables $x_1, ..., x_n$

(2) *The variables* $x_1, ..., x_n$ *align themselves again, and the pattern repeat*

(3) *The denominators of the Laurent polynomials are all different*

Example 1.4. *Let us consider again frieze of height* $n = 2$*, but change the rules of the game to*

$$x_{k+1}x_{k-1} = \begin{cases} x_k^{d_1} + 1 & k \text{ is even} \\ x_k^{d_2} + 1 & k \text{ is odd} \end{cases}$$

Previously we have studied the case $(d_1, d_2) = (1, 1)$*. In* $(d_1, d_2) = (1, 2)$*, we obtain*

$$\begin{aligned} x_3 &= \frac{x_2 + 1}{x_1} \\ x_4 &= \frac{x_3^2 + 1}{x_2} = \frac{(x_2 + 1)^2 + x_1^2}{x_1 x_2} \\ x_5 &= \frac{x_4 + 1}{x_3} = \frac{x_1^2 + x_2 + 1}{x_1 x_2} \\ x_6 &= \frac{x_1^2 + 1}{x_2} \\ x_7 &= x_1 \\ x_8 &= x_2 \end{aligned}$$

again we return to the original variables. Also the denominators of the Laurent polynomials are all distinct.

If we take $(d_1, d_2) = (1, 3)$*, and for simplicity we let* $x_1 = x_2 = 1$*, then we obtain the sequence*

$$(x_n) = 1, 1, 2, 9, 5, 14, 3, 2, 1, 1, ...$$

again return to the initial numbers.

However, this phenomenon does not always hold. Take for example $(d_1, d_2) = (1, 4)$*, we obtain*

$$(x_n) = 1, 1, 2, 17, 9, 386, 43, 8857, 206, ...$$

and for $(d_1, d_2) = (2, 2)$

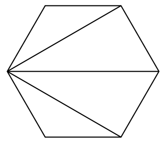
$$(x_n) = 1, 1, 2, 5, 13, 34, 89, 223, 610, 1597, ...$$

In fact, it is known that all x_k *can be expressed as Laurent polynomials in* x_1, x_2*, but we have the periodicity property only in the case when* $(d_1, d_2) = (1, 1), (1, 2)$ *and* $(1, 3)$*.*

In fact the frieze pattern is closely related to triangulations of polygons. More precisely,

Theorem 1.5. *A frieze pattern of height n corresponds to a triangulation of $n+3$-gon, such that the top row represents the number of triangles incident to the vertex of the polygons in the clockwise direction.*

$$(1, 2, 2, 2, 1, 4, ...) \longrightarrow$$

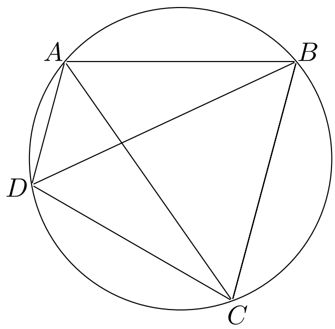


1.2 Triangulations

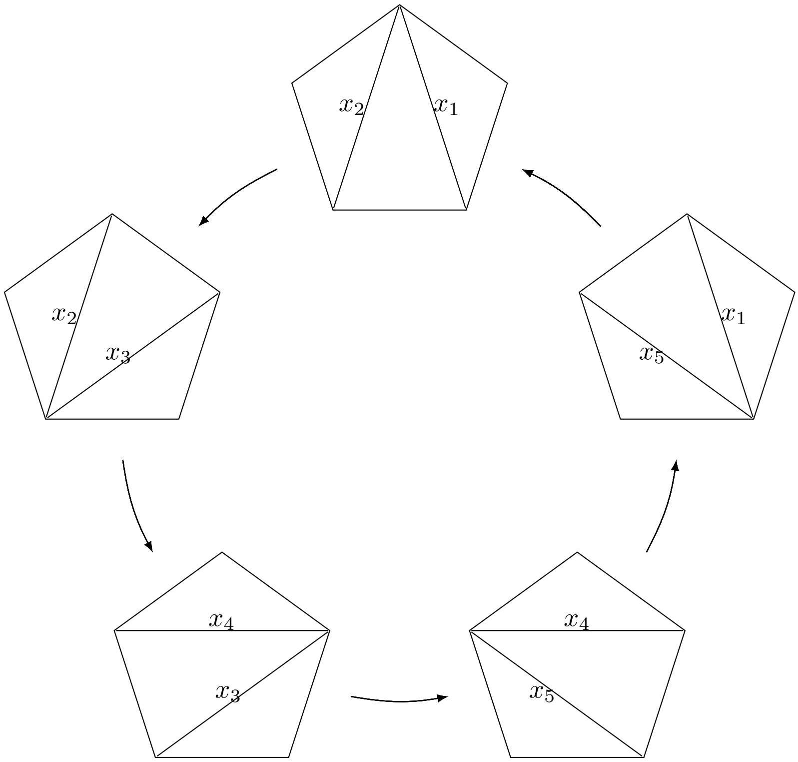
The rules for the original frieze pattern is closely related to an ancient Greek result about Euclidean geometry known as the Ptolemy's Theorem:

Theorem 1.6 (Ptolemy)**.** *If $ABCD$ is inscribed in a circle, then*

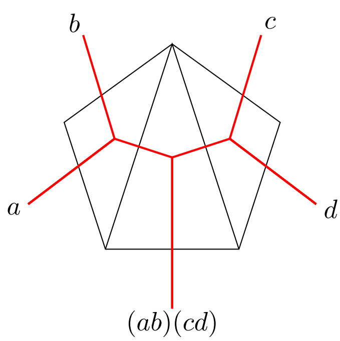
$$\overline{AC} \cdot \overline{BD} = \overline{AB} \cdot \overline{CD} + \overline{AD} \cdot \overline{BC}$$



If we consider a regular pentagon with side 1 on the circle, then the diagonals of different triangulations will satisfy the relation same as before! (Of course on Euclidean circle, x_1 and x_2 will already be fixed... but we can leviate this freedom by considering certain hyperbolic models to be discussed in later lectures.)



The relations among different triangulations here is known as the "pentagon relation", where different triangulations are related by "flip" of triangles, i.e. changing the diagonals. The pentagon, on the other hand, is also known as the (type A_2) associahedron because it is related to the usual associativity of products in 4 variables, and one can find a one-to-one correspondence between the two.



Then each flip of triangulation is related by the associativity relation $(ab)c = a(bc)$

$$\begin{tikzcd}
 & (ab)(cd) \arrow[dl] & \\
((ab)c)d \arrow[d] & & a(b(cd)) \arrow[ul] \\
(a(bc))d \arrow[r] & a((bc)d) \arrow[ur] &
\end{tikzcd}$$

For example, when $n = 3$, we obtain the following associahedron, where each vertex is 3-regular.

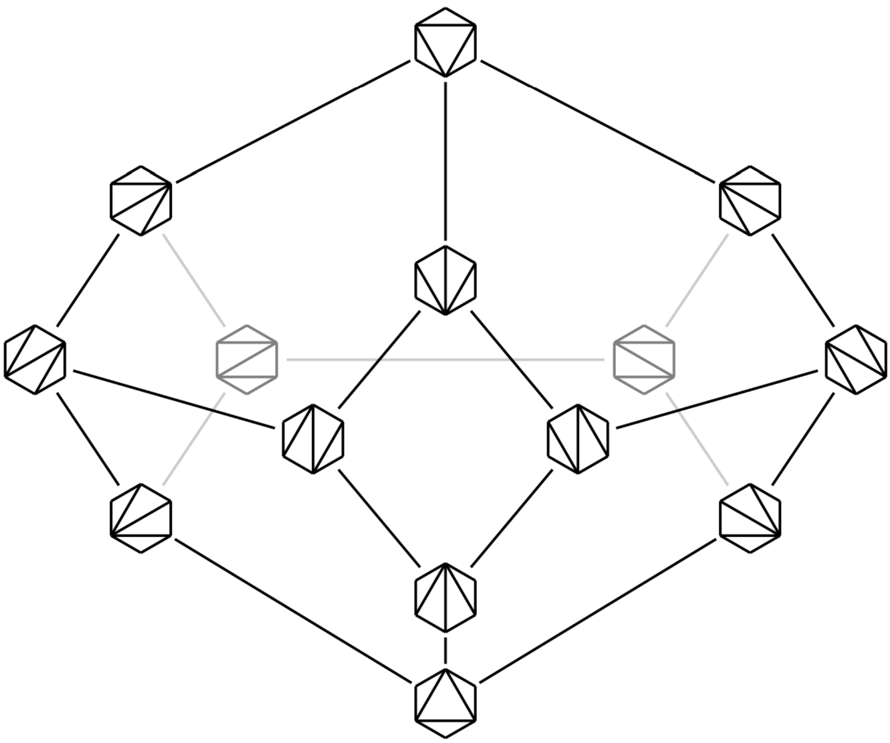


Figure 1: The A_3 associahedron

1.3 Grassmannian

Another appearance of the Ptolemy relation is the coordinate rings of the Grassmannian, which also serve a large number of exampels of cluster algebra. Let us recall the basic definition from linear algebra.

Let us consider vector spaces over $\mathbb{C}$.

Definition 1.7. *The Grassmannian variety $Gr(k,n)$ is the space of all k-dimensional linear subspace of $\mathbb{C}^n$.*

For example, $Gr(1,n) = \{\text{lines in } \mathbb{C}^n\} = \text{projective space } \mathbb{P}^n$.

In our motivation, we will stick to the example $Gr(2,n+3)$. What are the points $q \in Gr(2,n+3)$? Recall that every 2-dimensional vector subspace V can be described by a basis $v_1, v_2 \in \mathbb{C}^{n+3}$. Hence we can write q as a collection of row vectors:

$$q = \begin{pmatrix} -v_1- \\ -v_2- \end{pmatrix} \in Mat_{2,n+3}(\mathbb{C})/\{rank(q) = 2\}$$

such that $span\langle v_1, v_2\rangle = V$.

Of course, this choice is not unique: we can always choose another basis by applying a linear transform by $GL_2(\mathbb{C})$, acting on the left. i.e.,

$$\begin{aligned} & q_1 = q_2 \in Gr(2,n+3) \\ \Longleftrightarrow & q_1 = A \cdot q_2, \qquad A \in GL_2(\mathbb{C}) \end{aligned}$$

In fact, $Gr(2,n+3)$ is a projective variety, we have an embedding

$$Gr(2,n+3) \hookrightarrow \mathbb{P}(\Lambda^2\mathbb{C}^{n+3}) \simeq \mathbb{P}^N$$

given by

$$q = \begin{pmatrix} -v_1- \\ -v_2- \end{pmatrix} \mapsto [v_1 \wedge v_2]$$

where $N := \dim \Lambda^2\mathbb{C}^{n+3} = \begin{pmatrix} n+3 \\ 2 \end{pmatrix}$.

In terms of coordinate, we see that the embedding is given by "minors", i.e. determinants of 2×2 submatrix of q:

$$q = \begin{pmatrix} q_{1,1} & q_{1,2} & \cdots & q_{1,n+3} \\ q_{2,1} & q_{2,2} & \cdots & q_{2,n+3} \end{pmatrix} \mapsto [\Delta_{12}(q) : \cdots : \Delta_{n+2,n+3}(q)]$$

where $\Delta_{i,j}(q) = q_{1,i}q_{2,j} - q_{1,j}q_{2,i}$ is the 2×2 minor.

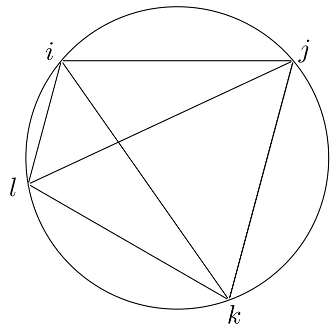
Proposition 1.8. *This map is well-defined.*

Proof. First, at least one of $\Delta_{i,j}(q) \neq 0$ because q has rank=2. Furthermore, if $q_1 = q_2 \in Gr(2,n+3)$, then $q_1 = A \cdot q_2$ for some $A \in GL_2(\mathbb{C})$, and we have $\Delta_{i,j}(q_1) = det(A)\Delta_{i,j}(q_2)$, hence all the coordinates are rescaled by the same factor $det(A) \neq 0$, hence determined the same point in $\mathbb{P}(\Lambda^2\mathbb{C}^{n+3})$. □

In fact, $Gr(2,n+3)$ is a smooth projective variety, cut out by certain quadratic equations, called the "Plücker relations"

$$\Delta_{i,k}\Delta_{j,l} = \Delta_{i,j}\Delta_{k,l} + \Delta_{i,l}\Delta_{j,k}, \qquad 1 \leq i < j < k < l \leq n+3$$

which is essentially the same as Ptolemy relation!



Hence understanding the (homogeneous coordinate) ring of functions on $Gr(2, n+3)$, is essentially the same as studying the algebra generated by the minors Δ_{ij}

$$\mathcal{A} := \mathbb{C}[\widehat{Gr(2, n+3)}] := \mathbb{C}[\Delta_{ij}]_{i \neq j}/(\text{Plücker relations})$$

where

$$\widehat{Gr(2, n+3)} := \theta^{-1}(Gr(2, n+3)) \cup \{0\} \subset \mathbb{C}^N$$

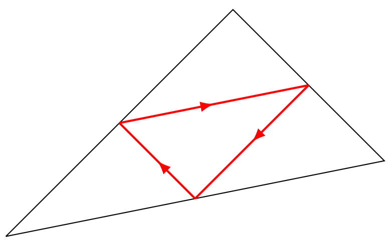
denote the *affine cone*, where $\theta : \mathbb{C}^N \longrightarrow \mathbb{P}^N$ is the projection.

Formally, one can consider the $(n+3)$-gon, and associate Δ_{ij} to the diagonals joining $i - j$. This will give us the analogue of the variables x_n discussed in the previous examples. Since there is a relation between the overlapping variables, we see that $\mathcal{A}$ will locally be described by different triangulations of the $(n+3)$-gon, where the corresponding minors are non-zero. Hence, for each triangulations, the corresponding n minors defines the cluster variables of the algebra $\mathcal{A}$. More precisely, the non-overlapping diagonals form a linear basis of $\mathcal{A}$:

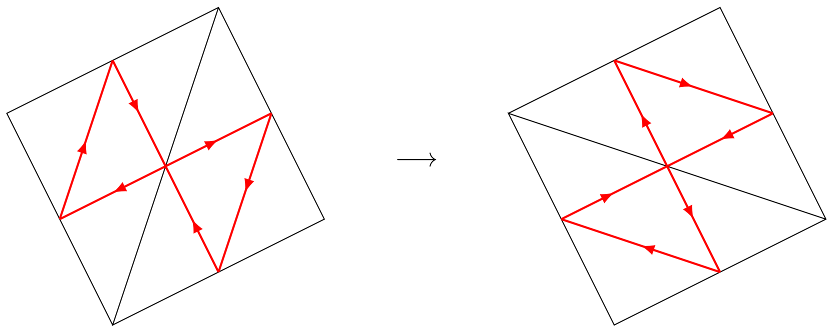
$$\mathcal{A} = \sum_{T \in triangulations} \mathbb{K}[\Delta_{ij}]_{\Delta_{ij} \text{ is diagonal of } T}$$

where $\mathbb{K} = \mathbb{C}[\Delta_{12}, \Delta_{23}, ..., \Delta_{n+3,1}]$ is generated by the sides of the $(n+3)$-gon.

As a side note, for each triangulations, one can naturally associate to it a "quiver" by putting an arrow between the edges (usually not an edge of the polygon) of the triangles in the following way:



Then for example, a change of triangulation will induce a change of quivers, called the "quiver mutaiton", that will be describe in more detail later.



One can then rewrite the Plücker relation as

$$\Delta_K \Delta'_K = \prod_{\Delta_I \longrightarrow \Delta_K} \Delta_I + \prod_{\Delta_K \longrightarrow \Delta_I} \Delta_I$$

where Δ_K and Δ'_K are the minors corresponding to the diagonal of the two pictures. We will see that this is a general way of writing mutations in cluster algebra associate with quivers.