11 Cluster Algebra from Surfaces

In this lecture, we will define and give a quick overview of some properties of cluster algebra from surfaces. We will follow [Surface-I] and [Surface-II].

11.1 Bordered Surfaces with Marked Points

Definition 11.1. A bordered surface with marked points is a pair \((S, M)\) where

- \(S\) a connected oriented 2-dimensional Riemann surface (possibly with boundary \(\partial S\)).
- \(M \subset S\) a non-empty finite set of marked points in \(S\)
- \(m \in M\) in the interior \(S - \partial S\) are called punctures
- Each connected boundary component has at least one marked point
- We require \(S\) to have at least one triangulation (see below). Hence \(S\) is NOT a:
  - sphere with 1 or 2 punctures
  - 1-gon (monogon) with 0 or 1 puncture
  - 2-gon (digon) with 0 punctures
  - 3-gon (triangle) with 0 punctures

\((n\text{-gon means a disk with } n\text{ marked points})\) We also exclude sphere with 3 punctures.

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Example 11.2. An example:

Figure 1: $S$ is a torus with 2 punctures, 1 boundary components, and 5 marked points

We also have some restrictions on the arcs of triangulations:

Definition 11.3. An arc $\gamma$ in $(S, M)$ is a curve up to isotopy, such that

- Endpoints of $\gamma$ lie in $M$
- $\gamma$ does not self-intersect outside endpoints
- $\gamma$ is disjoint from $M$ and $\partial S$ outside endpoints
- $\gamma$ does not cut out an 1-gon or 2-gon with no punctures.

Two arcs are compatible with they do not intersect in the interior of $S$. The set of all arcs is denoted by $A^0(S, M)$.

Definition 11.4. An ideal triangulation is a maximal collection of distinct pairwise compatible arcs. The arcs cut $S$ into ideal triangulations. (We allow self-folded triangles.)

Each ideal triangulation consists of $n = 6g + 3b + 3p + c - 6$ arcs, where

- $g$ = genus
- $b$ = # boundary components
- $p$ = # punctures
Definition 11.5. We can flip the triangles as usual. But edges inside a self-folded triangle cannot be flipped!

Definition 11.6. The arc complex $\Delta^0(S,M)$ is a simplicial complex with

- vertex = the arcs $\in A^0(S,M)$,
- simplex = compatible arcs
- maximal simplices = ideal triangulations

The dual graph is $E^0(S,M)$: vertices = triangulations, edges = flips
Example 11.7. Once-punctured triangle. We see that $\Delta^0(S, M)$ has a boundary, and $E^0(S, M)$ is not 3-regular, since some edges cannot be flipped.

Figure 5: $\Delta^0(S, M)$ and $E^0(S, M)$

Proposition 11.8. 
- $\Delta^0(S, M)$ is contractible unless $(S, M)$ is a polygon without punctures, then it is $\simeq$ sphere $S^n$.
- $E^0(S, M)$ is connected. Hence we can obtain any triangulations by flipping.
- $\pi_1$ of $E^0$ is generated by 4 and 5-cycles.

Theorem 11.9. There exists a triangulation for $(S, M)$ with no self-folded triangles.

Hence we can always start with a nice triangulation, and obtain any other triangulations by flipping.

11.2 Cluster Algebra

Recall that given a triangulation (without self-folded triangles), we can associate to it a quiver $Q$, such that a flip correspond to quiver mutation:
Then for example, a change of triangulation will induce a change of quivers, called the “quiver mutation”, that will be describe in more detail later.

If $\tilde{B}$ is the adjancency matrix of the quiver $Q$, then we can define a cluster algebra of geometric type $A$ of rank $n$ with initial seed $(\tilde{x}, \tilde{B})$ where $x = (x_1, ..., x_n, x_{n+1}, ..., x_m)$

- $n=$number of arcs of the triangulation
- cluster variables $x = (x_1, ..., x_n)$ labeled by the arcs,
- frozen variables $(x_{n+1}, ..., x_m)$ labeled by the sides of $S$ (connected components of $\partial S - M$)

**Remark 11.10.** In this quick overview, for simplicity, we will ignore the sides and only consider the principal part $B$ of $\tilde{B}$. The cluster algebra structures (cluster complex and exchange graph) are mostly independent of the coefficients (frozen variables). However, a major part of this theory deals with the properties and combinatorics of general coefficients (in $\mathbb{P}$) that is very interesting on its own.

One can now define the adjancency matrix for a triangulation with self-folded triangles: they should be obtained from $B$ by appropriate mutation. Any triangulation can be glued from 3 kind of “puzzle pieces”, and $B$ is obtained by summing up each matrix matching the row and column indices.
Corollary 11.11. By construction

- $\Delta^0(S, M)$ is a subcomplex of the cluster complex of $\mathcal{A}$.
- $E^0(S, M)$ is a subgraph of the exchange graph of $\mathcal{A}$.

In order to get the full cluster complex and exchange graph, we need to extend the definition of our triangulations to include tagged arcs.

11.3 Examples

Here we consider some examples:

Example 11.12. Some examples in lower rank: (please work out the quivers...)

- 4-gon, 0 punctures (type $A_1$)
- 5-gon, 0 punctures (type $A_2$)
- 2-gon, 1 puncture (type $A_1 \times A_1$)
- 3-gon, 1 puncture (type $A_3$)
- torus, 1 puncture (Markov quiver)

Example 11.13. Polygons:

- Type $A_n$: $(n + 3)$-gon with 0 punctures, from snake diagram
- Type $D_n$: $n$-gon with 1 puncture
Also consider some special types:

**Lemma 11.14.** Let $\Gamma$ be $n$-cycle ($n \geq 3$) with $n_1$ edges in one direction and $n_2$ edges in another direction. Then the mutation equivalence class of $\Gamma$ depends only on the unordered pair $\{n_1, n_2\}$.

We call its type $\tilde{A}(n_1, n_2)$. Note that $\tilde{A}(n,0) \simeq \tilde{A}(0,n) \simeq D_n$ (See Lecture 6).

Also recall the affine diagram:

![Diagram](image)

**Example 11.15.**
- Type $\tilde{A}(n_1,n_2)$: Annulus with 0 puncture, $n_1$ marked points on one boundary, $n_2$ marked points on another.
- Type $\tilde{A}(2,2)$: 1-gon with 2 punctures
- Type $\tilde{D}_{n-1}$: $(n-3)$-gon with 2 punctures.
11.4 Tagged arcs

We introduce tagging to resolve the self-folded triangles.

**Definition 11.16.** A tagged arcs is an arc in \((S, M)\) with a tagging (plain or \(\odot\)) on each end

- Endpoint on \(\partial S\) is tagged plain
- Both ends of a loop are tagged the same way
- The arc does not cut out an 1-gon with 1 puncture

The set of all tagged arcs is \(A^\triangledown(\Sigma, M)\). We let \(\alpha_0\) to be the untagged version of the tagged arc \(\alpha\).

**Definition 11.17.** Replacement \(\tau\) of an ordinary arc \(\gamma\) is a tagged arc \(\tau(\gamma)\):

- if \(\gamma\) does not cut out an 1-gon with 1 puncture, then \(\tau(\gamma) = \gamma\) with plain tags
- Otherwise if \(\gamma\) is a loop, it is given by the following replacement:

\[ \gamma \quad \tau(\gamma) \]

Figure 9: \(\gamma\) and \(\tau(\gamma)\)
Hence there is a map from $A^0(S, M) \to A^\infty(S, M)$.

**Definition 11.18.** Two tagged arcs $\alpha, \beta \in A^\infty(S, M)$ are compatible if

- The untagged version $\alpha_0, \beta_0$ are compatible
- If untagged version of $\alpha_0, \beta_0$ are different, and they share an endpoint $a$, then they must be tagged the same way at $a$.
- If untagged version of $\alpha_0, \beta_0$ are the same, they must be tagged the same way in at least one endpoint.

**Definition 11.19.** Tagged arc complex $\Delta^\infty(S, M)$ is the simplicial complex where

- vertex = the tagged arcs $\in A^0(S, M)$,
- simplex = compatible tagged arcs
- maximal simplices = “tagged triangulations”

Also let $E^\infty(S, M)$ be the dual graph. The edges of $E^\infty(S, M)$ give us the “flipping of tagged triangulation”.

**Remark 11.20.** If $S$ has no punctures, $\Delta^\infty = \Delta^0$ and $E^\infty = E^0$.

We can see now that we can extend our previous complexes:
Figure 10: $\Delta^\infty(S, M)$ for 1 punctured triangle

Figure 11: $E^0(S, M)$ and $E^\infty(S, M)$ for $S=2$-gon with 1 puncture

**Proposition 11.21.** • If $(S, M)$ is not a closed surface with exactly one punct-
ture, then $E^\triangledown(S, M)$ and $\Delta^\triangledown(S, M)$ is connected.

- If $(S, M)$ is a closed surface with one puncture, then $E^\triangledown(S, M)$ and $\Delta^\triangledown(S, M)$ has two isomorphic components: one with all ends of arg tagged plain, and another with all ends of arg tagged $\triangledown$.

We arrive at the main theorem:

**Theorem 11.22.** Let $A$ be the cluster algebra corresponding to $(S, M)$. Then

- If $(S, M)$ is not a closed surface with exactly one puncture, then $\Delta(A) \simeq \Delta^\triangledown(S, M)$ and exchange graph of $A$ is $\simeq E^\triangledown(S, M)$.

- If $(S, M)$ is a closed surface with exactly one puncture, then $\Delta(A) \simeq$ a connected component of $\Delta^\triangledown(S, M)$, and the exchange graph is $\simeq$ a connected component of $E^\triangledown(S, M)$.

To understand the idea behind the proof of the Theorem, we need to describe explicitly the tagged flipping as well as the adjacency matrix associated to a tagged triangulation.

**Definition 11.23.** The undone version of a tagged triangulation $T$ is an ordinary triangulation $T^0$ where

- if all arcs from a puncture is tagged $\triangledown$, remove the tag.
- for all other puncture, undo the map $\tau$ by replacing $\gamma$ with a loop

**Proposition 11.24.** Tagged flipping has 2 types:

(D) A “digon flip” as in Figure 11.

(Q) A “quadrilateral flip”, which flips the corresponding undone version of the triangulation.

**Example 11.25.** Illustrating (Q): sphere with 4 punctures.
Proposition 11.26. The adjacency matrix of a tagged triangulation $T$ is defined as

$$B(T) := B(T^0)$$

Then $B(T)$ satisfies the same matrix mutation rule with tagged flipping.

Corollary 11.27. The cluster algebra from surface is of finite mutation type

Proof. The adjacency matrix can only take value 0, ±1, ±2.

Example 11.28. The adjacency matrix (quiver) associated with Figure 11:
11.5 Denominator Theorem

One can also describe the cluster variables explicitly, just like in the finite type. First we define the intersection number:

**Definition 11.29.** The intersection number of 2 tagged arcs $\alpha, \beta$ is

$$(\alpha|\beta) := A + B + C + D$$

where

- $A =$ number of intersection of $\alpha_0$ and $\beta_0$ outside the endpoints
- $B = 0$ unless $\alpha_0$ is a loop based at $a$, with $\beta_0$ intersect $\alpha_0$ at $\beta_1, ..., \beta_m$, then $B$ is the $(-1)\times$ number of contractible triangle formed by $\beta_i, \beta_{i+1}, a$.
- $C = \begin{cases} -1 & \alpha_0 = \beta_0 \\ 0 & \text{otherwise} \end{cases}$
- $D =$ number of ends of $\beta$ sharing an endpoint with $\alpha$ but tagged differently.

From the Laurent phenomenon of cluster algebra, we also have an expression of the denominator vectors:

**Definition 11.30.** Fix an initial seed $(x_0, B)$. Any $z \in A$ can be written as a Laurent polynomial:

$$z = \frac{P(x_0)}{\prod_{x \in x_0} x^{d(x|z)}}$$

where $P$ is a polynomial in $x_0$, and $d(x|z)$ is called the denominator vector.
Recall that each cluster variable correspond to a tagged arc $a \mapsto x[\alpha]$.

**Theorem 11.31.** For any tagged arcs $\alpha, \beta$, the denominator vector $d(x[\alpha]|x[\beta])$ equals the intersection number $(\alpha|\beta)$.

**References**
