Lecture Notes
Introduction to Cluster Algebra

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11.6 Lambda lengths

In the last section, we describe the geometric realization of the cluster variables. They are given by "lambda lengths" introduced by R. Penner in the study of decorated Teichmüller spaces.

Here is one definition of Teichmüller space:

Definition 11.32. *Teichmüller space $\mathcal{T}(\boldsymbol{S}, \boldsymbol{M})$ is the moduli space of hyperbolic structures with constant curvature -1 on $\boldsymbol{S} \setminus \boldsymbol{M}$ and geodesic boundary $\partial \boldsymbol{S} \setminus \boldsymbol{M}$, modulo diffeomorphisms of $\boldsymbol{S}$ fixing $\boldsymbol{M}$ that are homotopic to identity.*

$\mathcal{T}(\boldsymbol{S}, \boldsymbol{M})$ is a manifold of dimension $n - p = 6g + 3b + 2p + c - 6$.

Our choice of bordered surface $(\mathbf{S}, \mathbf{M})$ gauranteed a hyperbolic structure. Recall that we can view them using Poincaré's uniformization.

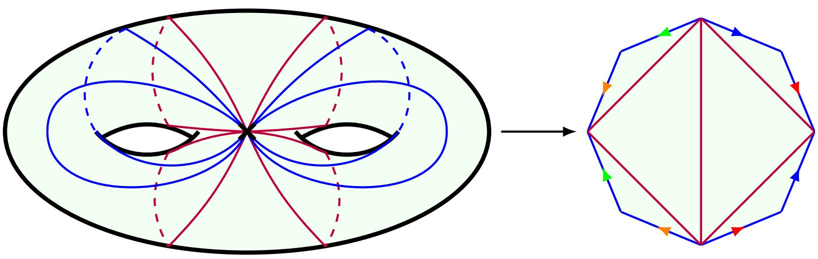

Figure 1: Viewing the 2-torus with 1 puncture by identifying sides of a polygons. An ideal triangulation is just a triangulation of the polygon.

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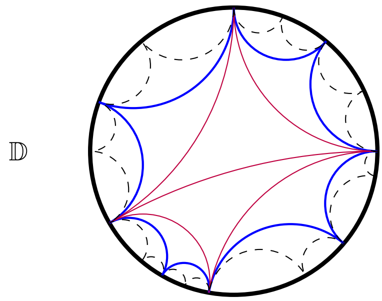


Figure 2: The universal cover will be the Poincaré disk $\mathbb{D}$, and the surface can be recovered by $\mathbb{D}/\Gamma$ for some discrete subgroup $\pi_1(\mathbf{S},\mathbf{M}) \simeq \Gamma \subset PSL(2,\mathbb{R})$.

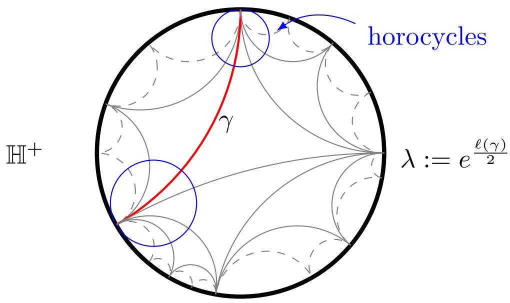
Definition 11.33. *Decorated Teichmüller space* $\widetilde{\mathcal{T}}(\boldsymbol{S},\boldsymbol{M})$ *is* $\mathcal{T}(\boldsymbol{S},\boldsymbol{M})$ *together with a collection of horocycles* h_p *around each* $p \in \boldsymbol{M}$*. Horocycle is a curve perpendicular to every geodesics from* p*.*

Remark 11.34. *In the realization of the hyperbolic space as a Poincaré unit disk, all* $p \in \boldsymbol{M}$ *lies on the boundary, and* h_p *can be realized as circles tangent to* p*.*

Definition 11.35. *Let* $\gamma \in \boldsymbol{A}^0(\boldsymbol{S},\boldsymbol{M})$ *or a boundary segment be a geodesic. Let* $\ell(\gamma)$ *be the signed distance between the horocycles at either end of* γ *(positive if 2 horocycles do not intersect, negative if intersect).*

The lambda length *is defined by*

$$\lambda(\gamma) := e^{\frac{\ell(\gamma)}{2}}$$



Theorem 11.36. *For any triangulation* T *of* $(\boldsymbol{S},\boldsymbol{M})$*, the map*

$$\prod_{\gamma \in T \cup \{\textit{boundary segments}\}} \lambda(\gamma) : \widetilde{\mathcal{T}}(\boldsymbol{S},\boldsymbol{M}) \longrightarrow \mathbb{R}^{n+c}_{>0}$$

is a homeomorphism.

Remark 11.37. *One can obtain a coordinate system on $\mathcal{T}(\boldsymbol{S}, \boldsymbol{M})$ which does not depend on the choice of horocycles, by taking the "cross-ratios" of lambda length around each quadrilateral, which is called the "shear coordinate", and satisfy certain linear relations around punctures.*

Remark 11.38. *Another model for lambda length is the Minkowski space. Consider $M = \mathbb{R}^{2,1}$ with quadratic form $\langle u, u\rangle = -x^2 - y^2 + z^2$. Then the hyperboloid model is given by*

$$\mathbb{H}_{>0} := \{u \in M : \langle u, u\rangle = 1, z > 0\}$$

The steoreographic projection from

$$\begin{aligned} \mathbb{H}_{>0} &\longrightarrow \mathbb{D} = \{(x, y, 0) \in \mathbb{R}^{2,1} : x^2 + y^2 = 1\} \\ x &\mapsto \overline{x} \end{aligned}$$

is given by joining $x \in \mathbb{H}_{>0}$ with $(0, 0, -1)$ and intersecting at $\mathbb{D}$.

If $x, y \in \mathbb{H}_{>0}$, then the hyperbolic distance $d = d(\overline{x}, \overline{y})$ satisfies

$$\cosh^2 d = \langle x, y\rangle^2$$

Define the positive light cone is

$$L^+ := \{u \in M : \langle u, u\rangle = 0, z > 0\}$$

Then it correponds to horocycles:

$$u \in L^+ \longleftrightarrow h(u) := \{w \in \mathbb{H} : \langle w, u\rangle = 1\}$$

The lambda length between the two horocycles $h(u), h(v)$ is then given by

$$\lambda_{uv}^2 = \langle u, v\rangle$$

Proposition 11.39. *If $\alpha, \beta, \gamma, \delta \in \boldsymbol{A}^0(\boldsymbol{S}, \boldsymbol{M})$ or boundary segments, and cut out a quadrilateral in $\boldsymbol{S}$ as in the picture with η, θ the diagonals, then we have Ptolemy relation*

$$\lambda(\eta)\lambda(\theta) = \lambda(\alpha)\lambda(\gamma) + \lambda(\beta)\lambda(\delta)$$

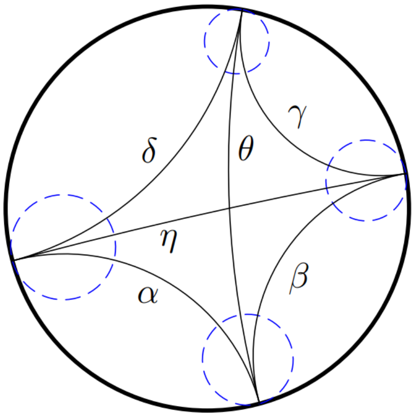


Figure 3: Ptolemy relation

Proof. It follows from the classical Ptolemy's relation! Let us rewrite

$$\lambda_{ij}^2 = \langle u_i, u_j \rangle$$

Then the Ptolemy's relation can be rewritten as

$$\lambda_{13}\lambda_{24} = \lambda_{12}\lambda_{34} + \lambda_{14}\lambda_{23}$$

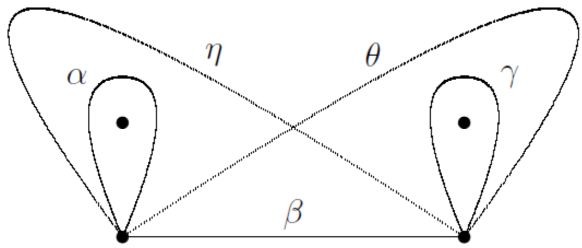
Note that it is invariant under rescaling of u_i. Hence we can scale u_i such that they are at the same height on L^+. But this means that they lie on a circle.

The Euclidean distance $d(u_i, u_j)$ is

$$\begin{aligned} d(u_i, u_j)^2 &= (x_i - x_j)^2 + (y_i - y_j)^2 \\ &= x_i^2 + y_i^2 + x_j^2 + y_j^2 - 2x_i x_j - 2y_i y_j \\ &= 2(z^2 - x_i x_j - y_i y_j) \\ &= 2\langle u_i, u_j \rangle = 2\lambda_{ij}^2 \end{aligned}$$

since $x_i^2 + y_i^2 = z^2$ as $u_i \in L^+$. Hence the Ptolemy's relation follows from the classical version. $\square$

Example 11.40. *The Ptolemy's relation also works when some edges are identified:*



$$\lambda(\eta)\lambda(\theta) = \lambda(\alpha)\lambda(\gamma) + \lambda(\beta)^2$$

The Ptolemy's relation gives us all the realization of exchange relations of the cluster variables when $(\mathbf{S}, \mathbf{M})$ has no punctures. The case when $\mathbf{S}$ has punctures is more complicated, where the Ptolemy relation for digon flip cannot give exchange relation since the two terms on the right hand side has common factors!

$$\lambda(\eta)\lambda(\theta) = \lambda(\alpha)\lambda(\gamma) + \lambda(\beta)\lambda(\gamma).$$

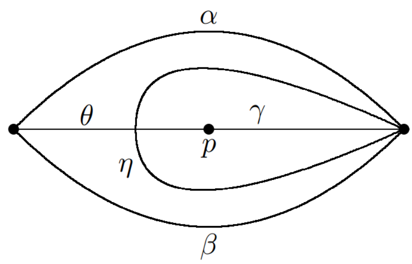


Figure 4: 2-gon flip

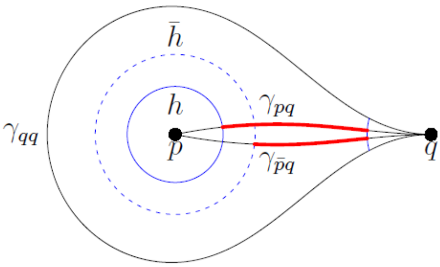
Hence one need to modify the definition of lambda length in the punctured case. We need a notion of lambda length for tagged arcs.

Definition 11.41. *A conjugated horocycle $\overline{h}$ of h at $p \in \boldsymbol{M}$ is a horocycle such that*

$$L(h)L(\overline{h}) = 1$$

where $L(h)$ is the hyperbolic length of the horocycle h.

Lemma 11.42. *Given the configuration of a 1-gon with 1 puncture, horocycle h around the puncture p and its conjugated horocycle $\overline{h}$*



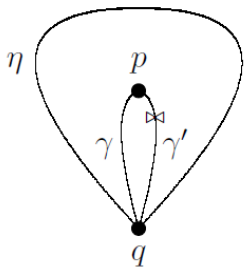
We have

$$\lambda_{qq} = \lambda_{qp}\lambda_{\bar{p}q}$$

where λ_γ is the geodesic length of γ.

Definition 11.43. *We define the lambda length of tagged arcs as follows: If both end of γ is plain, then $\lambda(\gamma)$ is as usual. If γ is $\bowtie$ at one end p, we use the conjugated horocycle $\overline{h_p}$ to define $\lambda(\gamma)$ instead.*

Lemma 11.44. *Let γ, γ' be tagged arc at $p, q \in \boldsymbol{M}$ such that $\gamma_0 = \gamma'_0$ and same tag at q, different tag at p.*



Then

$$\lambda(\eta) = \lambda(\gamma)\lambda(\gamma')$$

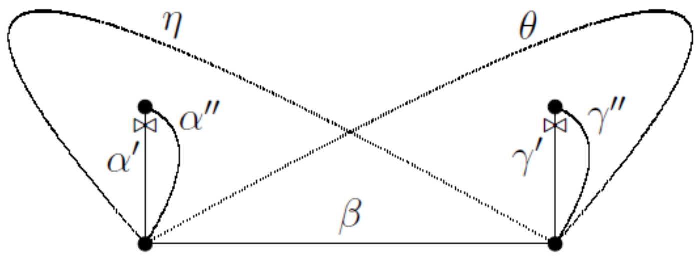
where $\lambda(\eta)$ is computed using the horocycles h_q or $\overline{h_q}$ depending on the tag of γ, γ' at q.

We can now state the exchange relations for the lambda length of tagged arcs.

Definition 11.45. *Ptolemy's relation (Q): Same as the undone version. But we replace 1-punctured 1-gon by the product of the tagged arcs.*

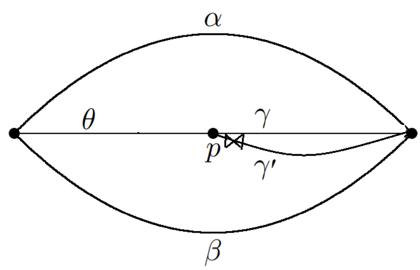
Example 11.46. *We have by Lemma 11.44*

$$\begin{aligned}\lambda(\eta)\lambda(\theta) &= \lambda(\alpha)\lambda(\gamma) + \lambda(\beta)^2 \\ &= \lambda(\alpha')\lambda(\alpha'')\lambda(\gamma')\lambda(\gamma'') + \lambda(\beta)^2\end{aligned}$$



Definition 11.47. *Digon relation (D): Again by Lemma 11.44, we can rewrite* $\lambda(\eta) = \lambda(\gamma)\lambda(\gamma')$ *and get*

$$\begin{aligned} \lambda(\eta)\lambda(\theta) &= \lambda(\alpha)\lambda(\gamma) + \lambda(\beta) + \lambda(\gamma) \\ \iff \lambda(\gamma')\lambda(\theta) &= \lambda(\alpha) + \lambda(\beta) \end{aligned}$$



Theorem 11.48. *The lambda lengths for tagged arcs form a cluster exchange pattern. In summary*

- *Coefficient semifield* $\mathbb{P}$ *is generated by lambda lengths of boundary segments*
- *Ambient field cF is generated over* $\mathbb{P}$ *by lambda lengths of a given triangulation* T_0 *with no self-folded triangles*
- *The exchange graph* $\boldsymbol{E}(\boldsymbol{S}, \boldsymbol{M})$ *is the exchange graph of tagged triangulations*
- *The seeds are labeled by vertices of* $\boldsymbol{E}(\boldsymbol{S}, \boldsymbol{M})$
- *Each cluster* $\mathbf{x}(T)$ *consists of lambda lengths of tagged arcs in* T
- *Each exchange matrix* $B(T)$ *is the signed adjacency matrix of* T
- *The exchange relation out of each seed is given by the Ptolemy relations (Q) and digon relations (D).*

The exhcnage pattern does not depend on the choice of initial triangulation.

References

[Surface-I] S. Fomin, M. Shapiro, and D. Thurston, *Cluster algebras and triangulated surfaces. Part I: Cluster complexes*, Acta Math. **201** (2008), 83146.

[Surface-II] S. Fomin, D. Thurston, *Cluster algebras and triangulated surfaces. Part II: Lambda lengths*, Memoirs of the American Mathematical Society (to appear), arXiv:1210.5569