Recall from lecture 1 that we have an identification of
\[ \mathbb{C}[\text{Gr}(2, n + 3)] := \mathbb{C}[\Delta_{ij}] / (\Delta_{ik}\Delta_{il} = \Delta_{ij}\Delta_{kl} + \Delta_{il}\Delta_{jk}) \]
with a cluster algebra \( \mathcal{A} \) of type \( A_n \), given by identification with triangulations of an \((n + 3)\)-gon:

\[
\begin{array}{|c|c|c|}
\hline
\mathbb{C}[\text{Gr}(2, n + 3)] & \mathcal{A} \text{ of type } A_n & (n + 3) - \text{gon} \\
\hline
\Delta_{ij}, |i - j| \neq 1 & \text{cluster variables} & \text{diagonals} \\
\Delta_{i,i+1} & \text{frozen variables} & \text{sides} \\
\text{Plücker relations} & \text{Exchange relations} & \text{Flipping of diagonals/Ptolemy’s Theorem} \\
\hline
\end{array}
\]

In this lecture, we will look at the general Grassmannian \( \text{Gr}(k, n) \), and show that it also possesses a cluster algebra structure of geometric type. We will follow [Scott].

Remark 10.1. The cluster algebra structure of the Grassmannian leads to the notion of “Positive Grassmannian”, where each cluster gives a “total positivity criterion” on the positive variety of \( \text{Gr}(k, n) \), generalizing the results for total positivity of matrices. Recently this has been utilized to study scattering amplitudes problems in the popular paper by Akarni-Hamed et al. [ABCGPT].
vectors, up to a change of basis:

$$\begin{pmatrix} -v_1 \\ -v_2 \\ \vdots \\ -v_k \end{pmatrix} \in \text{Mat}_{k,n}(\mathbb{C}) : \text{rank}(q) = k \} / GL_k(\mathbb{C})$$

**Definition 10.2.** We have the Plücker embedding:

$$\text{Gr}(k, n) \hookrightarrow \mathbb{P}(\Lambda^k \mathbb{C}^n) \simeq \mathbb{P}^N, \quad q \mapsto [v_1 \wedge v_2 \wedge \cdots \wedge v_k]$$

where \( N = \binom{n}{k} - 1 \).

In terms of coordinate, they are given by minors

$$q \mapsto [\Delta^A]$$

where \( A \subset [1, n] \) is a \( k \)-element subset, and \( \Delta^A = \Delta_{[1,k],A} \) is the minor of \( q \).

Let us also define \( \Delta^{i_1, \ldots, i_k} = \text{sgn}(\pi) \Delta^{j_1, \ldots, j_k} \) if \( \pi = \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} \) is a permutation, and let \( \Delta^{i_1, \ldots, i_k} = 0 \) if some indices are the same.

**Lemma 10.3.** The Plücker relation is given by

$$\sum_{r=0}^{k} (-1)^r \Delta^{i_1, i_2, \ldots, i_{k-1}, j_r} \Delta^{i_0, \ldots, \hat{j_r}, \ldots, j_k} = 0$$

for any tuples \( i_1 < i_2 < \cdots < i_{k-1} \) and \( j_0 < j_1 < \cdots < j_k \), and we omit the hat indices.

A consequence of the Plücker relations is the short Plücker relations given by

$$\Delta^{i_{ij}} \Delta^{I_{st}} = \Delta^{i_{is}} \Delta^{I_{jt}} + \Delta^{i_{js}} \Delta^{I_{it}}$$

where \( I \) is a subset of size \( k - 2 \) disjoint from \( \{i, j, s, t\} \), and \( \{i, j\} \) and \( \{s, t\} \) are crossing (if we label a \( n \)-gon clockwise by \( 1, \ldots, n \), then the chord \([ij]\) and \([st]\) cross each other). It can be obtained from the Lemma by letting \( I = (i_1, \ldots, i_{k-2}) = (j_1, \ldots, j_{k-2}) \), and using the fact that \( \Delta^{i_1, \ldots, i_k} = 0 \) if some indices are the same.

**Definition 10.4.** The homogeneous coordinate ring of \( \text{Gr}(k, n) \) is defined by

$$\mathcal{A} := \mathbb{C}[\text{Gr}(k, n)] := \mathbb{C}[\Delta^A]|_{A|=k}/\langle \text{Plücker relations} \rangle$$

where we recall that for a projective variety \( X \in \mathbb{P}^N \), its affine cone is defined as \( \widehat{X} := \pi^{-1}(X) \cup \{0\} \subset \mathbb{C}^{N+1} \) where \( \pi : \mathbb{C}^{N+1} \longrightarrow \mathbb{P}^N \) is the projection.

Also note that \( \text{Gr}(k, n) \simeq \text{Gr}(n - k, n) \) hence we will only consider the case when \( 2 \leq k \leq \frac{n}{2} \).
10.2 Postnikov diagram

To parametrize the clusters of Plücker coordinates, we introduce the Postnikov diagram:

**Definition 10.5.** Let $\pi \in S_n$ be a permutation. Label a $2n$-gon clockwise by $1', 1, 2', 2, ..., n', n$. A Postnikov diagram for $\pi$, or $\pi$-diagram, is a collection of $n$ oriented paths, joining $i$ and $\pi(i)'$, such that

1. No path intersects itself
2. All path intersections are transversal
3. Each intersection alternates in orientation from: left, right, left, ..., finally from right.

4. We do not allow such configuration:

5. Postnikov diagrams are identified up to isotopy, and we also identify this local untwisting:

**Proposition 10.6.** By Property (3), Postnikov diagram cuts the $2n$-gon into regions of two types:

- even region: the boundary alternate in orientation
• odd region: the boundary is oriented

We will sometimes shade the odd regions for clarity.

We label the even region with index $i$ if the region lies to the left of the $i$-th wire joining $i \rightarrow \pi(i)$.

**Definition 10.7.** We let $\pi_{k,n}$ be the permutation $i \mapsto i + k \mod n$.

**Example 10.8.** An example of $\pi_{3,7}$-diagram. We shade the odd regions and label the even regions.
Figure 1: Postnikov diagram for \( \pi_{3,7} \).

Note that any \( \pi_{k,n} \)-diagram has \( n \) boundary cells labeled by \([1,k], [2,k+1], \ldots, [n,k-1]\) (cyclically).

**Proposition 10.9** (Postnikov).  
(1) Number of even regions in a Postnikov diagram for \( \pi_{k,n} \) is \( k(n-k) + 1 \)

(2) Each even region is labeled by \( k \)-subset of \([1,n]\)

(3) Every \( k \)-subset occurs as labeling in some \( \pi_{k,n} \)-diagram.

**Definition 10.10.** A geometric exchange on a \( \pi \)-diagram is a local move (i.e. nothing appear inside this configuration) which gives a new \( \pi \)-diagram with new labelling. Here \( I \) is a common labelling of all the regions.
Proposition 10.11 (Postnikov). Any $\pi_{k,n}$-diagram can be transformed to one another by a sequence of geometric exchange.

Example 10.12. Geometric exchange corresponds to flipping of triangles when $k = 2$. Given a triangulation of a polygon, we associate the corresponding Postnikov diagram as follows:

Then a single flip of triangulation corresponds to the geometric exchange.
Theorem 10.13. For $4 \leq k + 2 \leq n$, there exists a special $\pi_{k,n}$-diagram $A_{k,n}$ where each even region is a 4-gon.

Example 10.14. An example of the special diagram $A_{3,8}$. Note that all even regions are quadrilateral.
The labels of the interior even regions are given by disjoint union of two intervals $I \sqcup I'$, such that if $I = [i, j]$ and $I' = [i', j']$, then $[ii']$ forms the triangulations of the snake diagram of an $n$-gon.

In general, the construction of $A_{k,n}$ looks something like this, by stacking “bands” on top one by one. (In this example, $k$ is odd).
Figure 4: $A_{k,k+2}$ diagram

Figure 5: $A_{k,k+3}$ diagram

Figure 6: $A_{k,k+4}$ diagram
10.3 Cluster algebra structure

Fix a Postnikov diagram $D$. We define a quiver $Q$ from the Postnikov diagram by:

- The nodes of $Q$ correspond to even regions
- We assign the arrows of $Q$ (in red) by the configuration

Let $\tilde{B}(D)$ be the corresponding exchange matrix to the quiver $Q$.

- The ambient field $\mathcal{F}$ = rational functions generated by $x_K$ where $K$ are the label of the even regions.
- Cluster variables $x(D)$ = set of indeterminates corresponding to interior labels
- $c = \{x[1,k], x[2,k+1], \ldots, x[n,k-1]\}$ = set of indeterminates corresponding to boundary labels.
- $\mathcal{A}_{k,n}$ be the cluster algebra generated by the seed $(x(D), c, \tilde{B}(D))$ inside $\mathcal{F}$

**Theorem 10.15.** Each $\pi_{k,n}$-diagram give rise to a seed of $\mathcal{A}_{k,n}$. If $D'$ is obtained from $D$ by a single geometric exchange through a 4-sided cell with label $K$, then $\mu_K(\tilde{B}(D)) = \tilde{B}(D')$.

**Proof.** Explicitly checking local configurations, which reduces to two cases up to symmetry. See [Scott].

Hence cluster algebra mutation generalize the geometric exchange to other configurations.

**Theorem 10.16.** There is an isomorphism $\phi : \mathcal{A}_{k,n} \rightarrow \mathbb{C}[Gr(k,n)]$ of $\mathbb{C}[c]$-algebra such that

$$x_K \mapsto \Delta^K$$

for every $k$-subsets $K \subset [1,n]$.

**Proof.** The main ingredient is to show that

1. $\dim(Gr(k,n)) = \text{rank}(\mathcal{A}_{k,n}) + |c|$
\( \phi(x_K) \mapsto \Delta^K \) is consistent with the exchange relation and Plücker relations

(3) \( \phi(x) \) is a regular function in \( \mathbb{C}[\hat{\text{Gr}}(k,n)] \) for any cluster variables \( x \in \mathcal{A} \).

(1) follows from our construction directly. We know that the dimension of \( \text{Gr}(k,n) \) is \( k(n-k) \), hence dimension of the affine cone \( \hat{\text{Gr}}(k,n) \) is \( k(n-k) + 1 \), which is the number of even regions in a Postnikov diagram.

(2) Any \( \pi_{k,n} \)-diagram can be obtained from one another by sequence of geometric exchanges. Then the correspondence between cluster variables \( x_K \) and minors \( \Delta^K \) implies that cluster mutation corresponds to the short Plücker relation.

\[
x_{Ist}x_{Iij} = x_{Iit}x_{Ijs} + x_{Ijt}x_{Iis} \\
\iff \Delta_{Ist} \Delta_{Iij} = \Delta_{Iit} \Delta_{Ijs} + \Delta_{Ijt} \Delta_{Iis}
\]

(3) Finally, by considering the special diagram \( \mathbf{A}_{k,n} \) (so that each interior region can be mutated by just the geometric exchange above) and using the Laurent Phenomenon, one shows that \( \phi(x) \) is regular on an open subset \( U \subset \hat{\text{Gr}}(k,n) \) where \( \hat{\text{Gr}}(k,n) \setminus U \) has codimension at least 2, which implies that \( \phi(x) \) is regular everywhere. This is almost the same proof giving the upper cluster algebra structure of Double Bruhat cells. This also implies that the cluster algebra and the upper cluster algebra coincides.

\[\square\]

**Example 10.17.** The quiver configuration in \( k = 2 \) recovers the case we have seen
Theorem 10.18. Using the special diagram $A_{k,n}$, the cluster type of $\mathbb{C}[\hat{\text{Gr}}(k,k + l)]$ is described by the quiver $A_{k-1} \times A_{l-1}$ where all the arrows are alternating.

In particular we have the following cluster algebra type for $\mathbb{C}[\hat{\text{Gr}}(k,n)]$ with $2 \leq k \leq \frac{n}{2}$:

<table>
<thead>
<tr>
<th>$\text{Gr}$</th>
<th>$\text{Gr}(2, n + 3)$</th>
<th>$\text{Gr}(3, 6)$</th>
<th>$\text{Gr}(3, 7)$</th>
<th>$\text{Gr}(3, 8)$</th>
<th>$\text{Gr}(3, 9)$</th>
<th>$\text{Gr}(4, 8)$</th>
<th>other</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type of $\mathbb{C}[\text{Gr}]$</td>
<td>$A_n$</td>
<td>$D_4$</td>
<td>$E_6$</td>
<td>$E_8$</td>
<td>$E_8^{(1,1)}$</td>
<td>$E_7^{(1,1)}$</td>
<td>infinite</td>
</tr>
</tbody>
</table>

where $E_7^{(1,1)}$ and $E_8^{(1,1)}$ are finite mutation type (see the classification in Lecture 6 Theorem 6.20)

Example 10.19. For $\text{Gr}(3,8)$, from Figure 3, we can see that the quiver (only the non-frozen part) associated to $A_{3,8}$ is given by $A_2 \times A_4$, which is mutation equivalent to $E_8$. 
Figure 7: The quiver corresponding to $A_{3,8}$ diagram

References
