Lecture Notes
Introduction to Cluster Algebra

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2 Total Positivity

The final and historically the original motivation is from the study of total positive matrices, which has a long history in classical mechanics, stochastic process, enumerative combinatorics and graph theory.

2.1 Definition

Let us consider $n \times n$ real matrices in $GL_n(\mathbb{R})$.

Definition 2.1. *Let I and J be subsets of $[1, n]$ of the same size. Then the minor $\Delta_{I,J}$ of $g \in GL_n(\mathbb{R})$ is the determinant of the submatrix of g labelled by the row index I and column index J.*

Note that there are $\begin{pmatrix} 2n \\ n \end{pmatrix} - 1$ minors!

Definition 2.2. *A matrix $g \in GL_n(\mathbb{R})$ is totally positive (resp. nonnegative) if every minor $\Delta_{I,J}(g) > 0$(resp. ≥ 0).*

Example 2.3. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$ *is totally positive if a, b, c, d, Δ. However we only need to check 4 of them since*

$$ad = \Delta + bc$$

Total positive matrix is best understood using "planar networks". Consider a graph with n sources and n sinks, such that all paths go from left to right.

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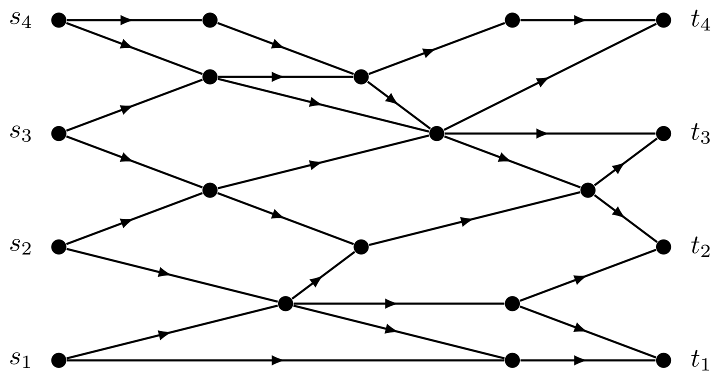


Figure 1: Planar network of order 4

Definition 2.4. *The path matrix $W = (w_{ij})$ of a planar network G is given by*

$$w_{ij} = \#\ \textit{paths from } s_i \textit{ to } t_j \textit{ in } G$$

For example, the path matrix of the graph above is

$$W = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 2 & 4 & 4 & 1 \\ 0 & 4 & 7 & 4 \\ 0 & 3 & 6 & 5 \end{pmatrix}$$

and one can check that this matrix is totally non-negative!

2.2 Properties and Decomposition

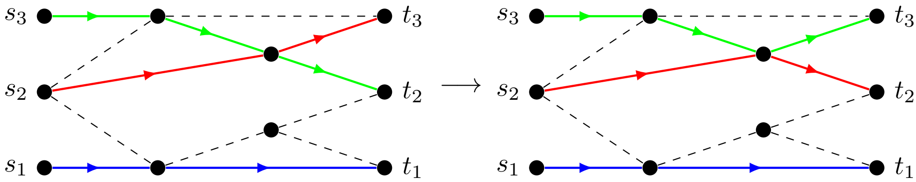
Theorem 2.5 (Lindström's Lemma)**.** *The path matrix is totally nonnegative. Each minor $\Delta_{I,J}$ equals to the number of families of non-intersecting paths from sources indexed by I to targets indexed by J.*

Proof. We can restrict to the size of $|I| = |J| = n$ and consider just the determinant of the whole weight matrix. Expanding the determinant, we have

$$\det = \sum_{w \in \mathcal{S}_n} \sum_{\pi \in paths} sgn(w)\omega(\pi)$$

where $\pi = (\pi_1, \pi_2, ..., \pi_n)$ such that π_i joins i with $w(i)$, and $\omega(\pi)$ is the product of w_{ij} associated to the paths.

For non-intersecting paths, we have $w = Id \in \mathcal{S}_n$, hence $sgn(w) = +1$. For all other intersecting paths, note that by swapping at the intersecting points, we obtain the same contribution but different permutation signs $sgn(w)$, hence the two terms cancel out.



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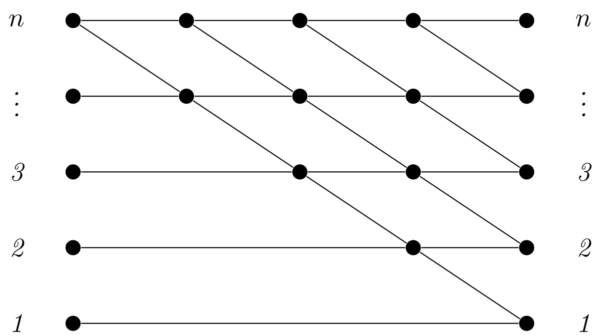
Similarly, by putting weights on the path, the same argument shows that $\Delta_{I,J}$ equals the weighted sum of families of non-intersecting paths from sources I to targets J.

Corollary 2.6. *If the planar network has non-negative real weights, then its weight matrix is totally nonnegative.*

Example 2.7. *Many combinatorial matrices can be shown to be totally nonnegative. For example, the network*



gives the "Pascal triangle"

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

which is totally nonnegative by the Corollary.

Lemma 2.8 (Cryer)**.** *An invertible square matrix g is totally nonnegative iff it has a Gaussian decomposition*

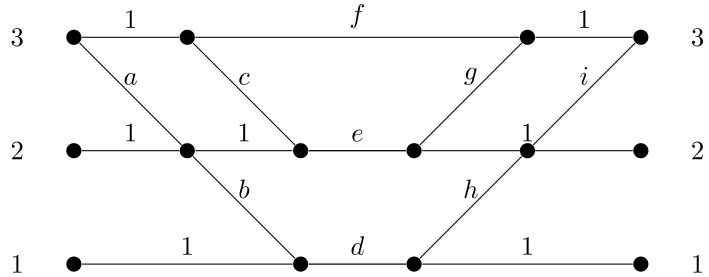
$$g = g_- g_0 g_+$$

where g_- is lower-triangular unipotent, g_0 is diagonal, g_+ is upper-triangular unipotent, and all three factors are totally nonnegative.

Example 2.9. *Any 3×3 totally positive matrix is given by*

$$\begin{pmatrix} d & dh & dhi \\ bd & bdh + e & bdhi + e(g+i) \\ abd & abdh + (a+c)e & abdhi + (a+c)e(g+i) + f \end{pmatrix}$$

which can be represented by the following planar networks,

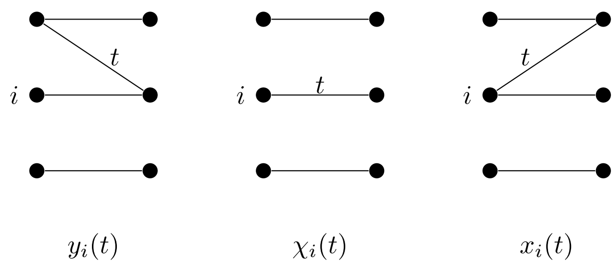


and is composed of the three elementary network correpsonding to the elementary Jacobi matrices

$$x_i(t) = I + tE_{i,i+1}$$

$$y_i(t) = I + tE_{i+1,i}$$

$$\chi_i(t) = I + (t-1)E_{i,i}$$



Theorem 2.10 (Loewner-Whitney)**.** *Any invertible totally non-negative matrix is a product of elementary Jacobi matrices with non-negative matrix entries. In particular we only need to check n^2 minors to determine total positivity.*

Lusztig generalize this idea to the notion of total positivity to other semiseimple groups G, by defining $G_{\geq 0}$ as the semigroup generated by the Chevalley generators. He showed that $G_{\geq 0}$ can be described by inequalities of the form $\Delta(x) \geq 0$ where Δ ranges over the appropriate *dual canonical basis*, but this set is infinite and very hard to understand. This leads to another motivation for Fomin and Zelevinksy to introduce the cluster algebra, where this set can be replaced by a much simpler and finite set called the *generalized minors*.

For $G = SL_n$, $G_{\geq 0}$ forms a multiplicative semigroup, and the study of $G_{\geq 0}$ can be reduced to the investigation of its sub-semigroup $N_{\geq 0} \subset G_{\geq 0}$ of upper-triangular unipotent totally nonnegative matrices.

Remark 2.11. *Strata of $N_{\geq 0}$ is ismorphic to Bruhat order of S_n.*

Example 2.12. $N_{\geq 0} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ *with $x, y, z, xy - z \geq 0$. $N_{\geq 0}$ is a cone with base $M_{\geq 0} := \{z \leq x(1-x)\}$:*

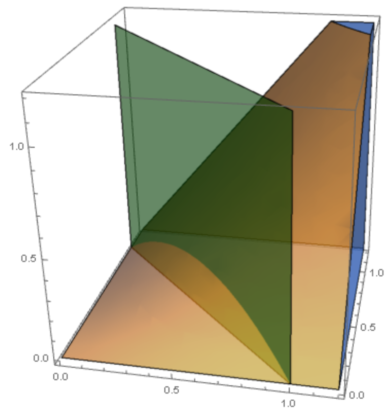


Figure 2: The cone $xy \geq z$ cut out by $x + y = 1$

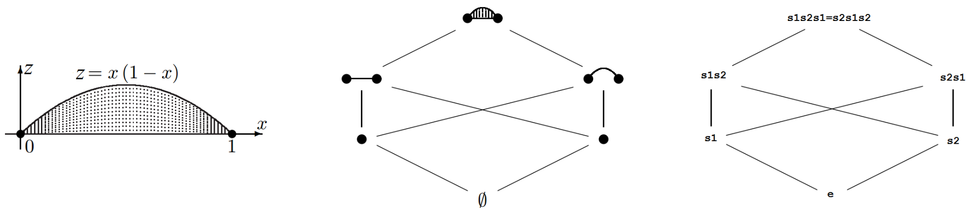


Figure 3: Relation to Bruhat order of S_3

2.3 Pseudoline arrangements of $\mathbb{C}[G/N]$

We will look into the coordinate ring of the *base/principal affine space* for $G = SL_n$. Recall the subgroup $N \subset G$ of unipotent upper-triangular matrices acts on G by right multiplication. The algebra $\mathcal{A} = \mathbb{C}[G/N]$ consists of regular functions on G

which are invariant under the action of N, i.e. certain column operations. It is known that $\mathcal{A}$ is generated by *flag minors*

$$\Delta_I(x) := \det(x_{ij} | i \in I, j \leq |I|)$$

A point $[x]$ in G/N represented by a matrix $x \in G$ is called totally positive if all flag minors $\Delta_I(x) > 0$.

By Cryer's lemma, $x \in G$ is totally positive if both $[x]$ and $[x^T]$ are totally positive in G/N.

Although there are $2^n - 2$ flag minors, we only need to test positivity of $\dim(G/N) = \frac{(n-1)(n+2)}{2}$ minors. We introduce the pseudoline arrangement.

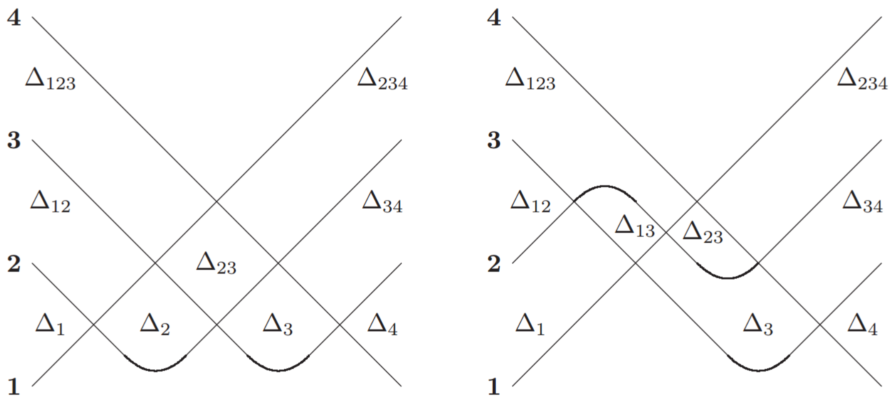


Figure 4: Two pseudoline arrangements, and associated chamber minors

To each region R, we associate *chamber minor* $\Delta_{I(R)}$ labeled by pseudolines passing below R. There are 2 types of regions, *bounded* and *unbounded* regions.

The $2(n-1)$ minors associated with unbounded regions are called *frozen*: $\Delta_1, \Delta_{12}, \Delta_{123}, \Delta_4, \Delta_{34}, \Delta_{234}$.

The $\binom{n-1}{2}$ minors associated with bounded regions form the *cluster*: $\{\Delta_2, \Delta_3, \Delta_{23}\}$ and $\{\Delta_{13}, \Delta_3, \Delta_{23}\}$ for the left and right figure respectively. The two cluster differ by one element only. In fact one do a *local move* to relate the two arrangements:

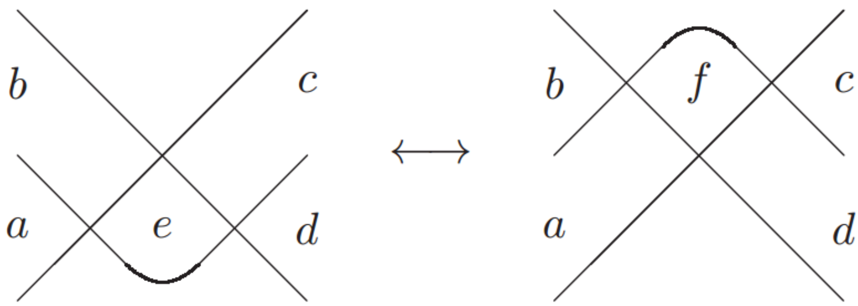


Figure 5: A local move in a pseudoline arrangement

The chamber minors associated with the regions satisfy

$$ef = ac + bd$$

which is the generalized Plücker relations, also called the *exchange relation*. For example, in the figure above we have

$$\Delta_2\Delta_{13} = \Delta_{12}\Delta_3 + \Delta_1\Delta_{23}$$

The new chamber minor is given by *subtraction-free* expression

$$f = \frac{ac + bd}{e}$$

and *all* flag minors can be obtained in this way by iterated local moves. In particular, if all initial cluster is positive at a given point in G/N, so do all other flag minors.

To describe the cluster structure of $\mathcal{A}$, it turns out there is a hidden relation. How do we exchange Δ_{23} in the first diagram? We need more universal combinatorial language of quivers.

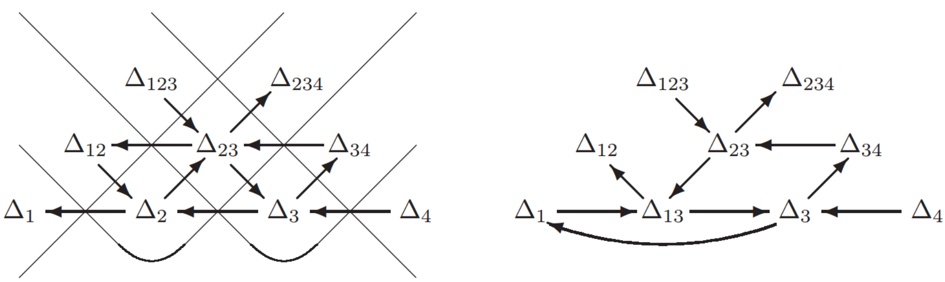


Figure 6: Quivers corresponding to pseudoline arrangements

Then an exchange relation is very simple:

$$\Delta_k \Delta'_k = \prod_{i \leftarrow k} \Delta_i + \prod_{i \rightarrow k} \Delta_i$$

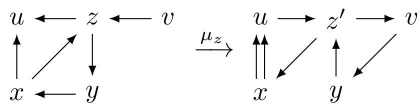
and at the same time, the quiver mutation is proceeded by three steps:

(1) For each pair of directed edges $i \longrightarrow k \longrightarrow j$, introduce a new edge $i \longrightarrow j$ (unless both i, j are frozen)

(2) Reverse direction of all edges incident to k

(3) Remove all oriented 2-cycles.

Example 2.13. *Consider the example:*



Now we can exchange Δ_{23} and define the rational function Ω with

$$\Delta_{23}\Omega = \Delta_{123}\Delta_{34}\Delta_2 + \Delta_{12}\Delta_{234}\Delta_3$$

This cluster with $\{\Delta_2, \Delta_3, \Omega\}$ is no longer associated to pseudoline arrangements. In general, repeating such mutation, the generators produced by this process become rational functions on the base affine space, and they generate the ring of all such functions, giving a cluster algebra structure in $\mathbb{C}[SL_n/N]$.

In the special case of $n = 4$, this recursive process produces a *finite* number of distinct clusters. There are 14 of them. Altogether, they contain 15 generators: $2^4 - 2 = 14$ flag minors Δ_I, and the variable

$$\Omega = \Delta_2\Delta_{134} - \Delta_1\Delta_{234}$$

and they form an exchange graph below:

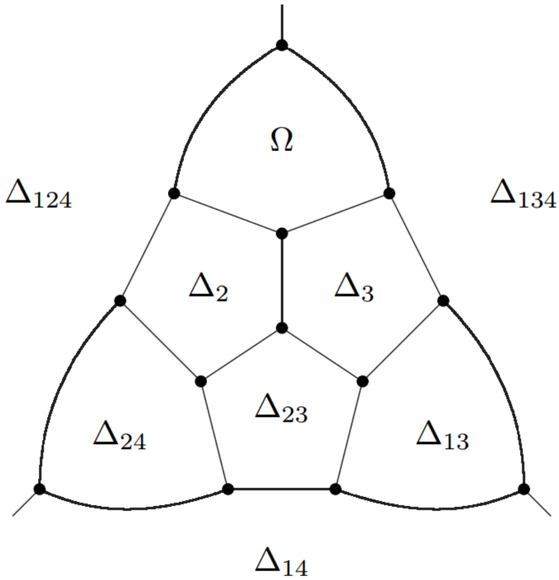


Figure 7: Exchange graph of $\mathbb{C}[SL_4/N]$

We see that this graph is identical to the triangulations of hexagon described previously!

To summarize all the examples, we notice some common features:

(1) Each "cluster" is described by finite number of variables

(2) We exchange one cluster with another by changing only one variable

(3) This exchange relation is subtraction free

(4) All other variables can be expressed as Laurent polynomials of the initial variables

(5) The Laurent polynomials have positive coefficients

This motivates the general definition of a cluster algebra.