Lecture Notes
Introduction to Cluster Algebra

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4 Laurent phenomenon

We now state the *Laurent phenomenon* of cluster algebra with coefficients in the semifield $\mathbb{P}$.

Theorem 4.1. *Any cluster variable can be expressed in terms of any given cluster as a Laurent polynomial with coefficients in the group ring* $\mathbb{ZP}$, *i.e.*

$$\mathcal{A}(\mathbf{x}, \boldsymbol{y}, B) \subset \mathbb{ZP}[\mathbf{x}^{\pm 1}]$$

The proof follows from the Key Lemma and the Caterpillar Lemma. Let us introduce some notation.

Consider the seed pattern $\mathbf{x}(t) = (x_1(t), ..., x_n(t))$ associated to $\mathbb{T}_n$. Generalizing the exchange relation, we consider the more general mutation in direction k.

$$\begin{aligned} x_i(t) &= x_i(t') \qquad i \neq k \\ x_k(t)x_k(t') &= P(\mathbf{x}(t)) \end{aligned}$$

where P is a polynomial in n variables, and $P(\mathbf{x})$ does not contain the variable x_k. We write this as

$$\bullet \xrightarrow[P]{k} \bullet$$

Note that in the case of cluster algebra, P is of the form

$$P(\mathbf{x}) = M_1(\mathbf{x}) + M_2(\mathbf{x})$$

for some monomials M_1 and M_2 of $\mathbf{x}$ that does not contain x_k and does not share a common cluster variable, and satisfies some other axioms.

Let $\mathbb{T}_{n,m}$ be the tree with m spine vertices, each with degree n:

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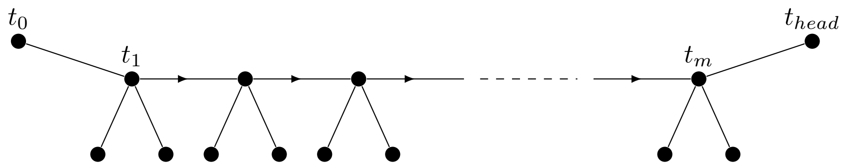


Figure 1: The caterpillar tree $\mathbb{T}_{n,m}$ for $n = 4$

Lemma 4.2 (The Caterpillar Lemma)**.** *Let $\mathbb{A}$ be a UFD (i.e. with gcd, e.g. $\mathbb{Z}$), and $\mathcal{L}_0 = \mathbb{A}[\mathbf{x}(t_0)^{\pm}]$ be Laurent polynomial in $\mathbf{x}(t_0)$ with coefficients in $\mathbb{A}$. Assume the exchange pattern $\mathbb{T}_{n,m}$ satisfies*

(i) For any edge $\bullet \xrightarrow[P]{k} \bullet$, the polynomial P does not depend on x_k, and is not divisible by any x_i.

(ii) For any two edges $\bullet \xrightarrow[P]{i} \bullet \xrightarrow[Q]{j} \bullet$, the polynomial P and $Q_0 = Q|_{x_i=0}$ is coprime in $\mathcal{L}_0$.

(iii) For any three edges $\bullet \xrightarrow[P]{i} \bullet \xrightarrow[Q]{j} \bullet \xrightarrow[R]{i} \bullet$, we have

$$L \cdot Q_0^b \cdot P = R|_{x_j \leftarrow \frac{Q_0}{x_j}}$$

where $b \in \mathbb{Z}_{\geq 0}$, L is a Laurent monomial and is coprime with P.

Then each element $x_i(t)$ is a Laurent polynomial in $x_1(t_0), ..., x_n(t_0)$ with coefficients in $\mathbb{A}$.

Lemma 4.3 (Key Lemma)**.** *Assume (i)-(iii) hold. Then $\mathbf{x}(t_1), \mathbf{x}(t_2), \mathbf{x}(t_3)$ are contained in $\mathcal{L}_0$. Furthermore, in the case $\bullet \xrightarrow[P]{i} \bullet \xrightarrow[Q]{j} \bullet \xrightarrow[R]{i} \bullet$, we have*

$$gcd(x_i(t_3), x_i(t_1)) = gcd(x_j(t_2), x_i(t_1)) = 1$$

as elements of $\mathcal{L}_0$.

Proof. It is clear that all elements in $\mathbf{x}(t_1), \mathbf{x}(t_2), \mathbf{x}(t_3)$ are in $\mathcal{L}_0$, except $x_i(t_3)$ from the case

$$\bullet_{t_0} \xrightarrow[P]{i} \bullet_{t_1} \xrightarrow[Q]{j} \bullet_{t_2} \xrightarrow[R]{i} \bullet_{t_3}$$

To simplify notation, let us write

$$\begin{aligned} x &:= x_i(t_0), \\ y &:= x_j(t_0), \\ z &:= x_i(t_1) = x_i(t_2), \\ u &:= x_j(t_2) = x_j(t_3), \\ v &:= x_i(t_3) \end{aligned}$$

such that $\mathcal{L}_0 = \mathbb{A}[x_k(t_0)^{\pm}]_{k\neq i,j}[x^{\pm}, y^{\pm}]$. Note that the variables x_k for $k \notin \{i,j\}$ do not change in all four clusters. Then we have to show

$$v \in \mathcal{L}_0 \tag{4.1}$$

$$gcd(z,u) = 1 \tag{4.2}$$

$$gcd(z,v) = 1 \tag{4.3}$$

Also recall that P, R only depend on x_j and do not depend on x_i, while Q only depends on x_i and does not depend on x_j. So let us write it as polynomial in one variable (where we treat the rest of the common variables as coefficients).

Then for example the assumption is

$$R\left(\frac{Q(0)}{y}\right) = L(y)Q(0)^b P(y)$$

We have

$$z = \frac{P(y)}{x}$$

$$u = \frac{Q(z)}{y} = \frac{Q\left(\frac{P(y)}{x}\right)}{y}$$

$$v = \frac{R(u)}{z} = \frac{R\left(\frac{Q(z)}{y}\right)}{z} = \frac{R\left(\frac{Q(z)}{y}\right) - R\left(\frac{Q(0)}{y}\right)}{z} + \frac{R\left(\frac{Q(0)}{y}\right)}{z}$$

Since

$$\frac{R\left(\frac{Q(z)}{y}\right) - R\left(\frac{Q(0)}{y}\right)}{z} \in \mathcal{L}_0$$

and

$$\frac{R\left(\frac{Q(0)}{y}\right)}{z} = \frac{L(y)Q(0)^b P(y)}{z} = L(y)Q(0)^b x \in \mathcal{L}_0$$

(4.1) follows. Next we have

$$u = \frac{Q(z)}{y} \equiv \frac{Q(0)}{y} \mod z$$

Since $z = \frac{P(y)}{x}$, and x, y are units in $\mathcal{L}_0$, we have $gcd(z,u) = gcd(P(y), Q(0)) = 1$ proving (4.2). Finally, let $f(z) = R\left(\frac{Q(z)}{y}\right)$. Then

$$v = \frac{f(z) - f(0)}{z} + L(y)Q(0)^b x$$

We have

$$\frac{f(z) - f(0)}{z} \equiv f'(0) = R'\left(\frac{Q(0)}{y}\right) \cdot \frac{Q'(0)}{y} \mod z$$

Hence

$$v \equiv R'\left(\frac{Q(0)}{y}\right) \cdot \frac{Q'(0)}{y} + L(y)Q(0)^b x \mod z.$$

This is a linear polynomial in x, with coefficients in the rest of the variables of the cluster $\mathbf{x}(t_0)$. Hence (4.3) follows from $gcd(L(y)Q(0)^b, P(y)) = 1$. $\square$

Proof of Caterpilla Lemma. We use induction on m. The case of $m = 2$ is trivial, while we have shown $m = 3$ from the Key Lemma. Let t_3 be the vertex such that

$$t_0 \xrightarrow{i} t_1 \xrightarrow{j} t_2 \xrightarrow{i} t_3$$

Then the path $t_3 \longrightarrow ... \longrightarrow t_{head}$ and $t_1 \longrightarrow ... \longrightarrow t_{head}$ is shorter than the path $t_0 \longrightarrow ... \longrightarrow t_{head}$, hence by induction we have

$$\mathbf{x}(t_{head}) \in \mathcal{L}(\mathbf{x}(t_1))$$
$$\mathbf{x}(t_{head}) \in \mathcal{L}(\mathbf{x}(t_3))$$

where we denote $\mathcal{L}(\mathbf{x}) = \mathbb{A}[\mathbf{x}^{\pm}]$. Now

$$\mathbf{x}(t_1) = \{x_1(t_0), ..., x_n(t_0)\} \setminus \{x_i(t_0)\} \cup \{x_i(t_1)\}$$
$$\mathbf{x}(t_3) = \{x_1(t_0), ..., x_n(t_0)\} \setminus \{x_i(t_0), x_j(t_0)\} \cup \{x_j(t_2), x_i(t_3)\}$$

Let $x'_k := x_k(t_{head})$, then

$$x'_k = \frac{f}{x_i(t_1)^a} = \frac{g}{x_j(t_2)^b x_i(t_3)^c}$$

for $f, g \in \mathcal{L}_0$, $a, b, c \in \mathbb{Z}_{\geq 0}$ such that $gcd(f, x_i(t_1)) = gcd(g, x_j(t_2)x_i(t_3)) = 1$. Hence

$$(x_j(t_2)^b x_i(t_3)^c) f = (x_i(t_1)^a) g \in \mathcal{L}_0$$

and by the Key Lemma, we must have $a = b = c = 0$, hence $x'_k \in \mathcal{L}_0$ for all k. $\square$

Proof of Laurent Phenomenon for cluster algebra. We want to show that the exchange relations of the cluster algebra with coefficients in $\mathbb{A} = \mathbb{ZP}$ satisfy the conditions of the Caterpillar Lemma. Note that in the Caterpillar Lemma, we can assume i and j are connected (i.e. $b_{ij} \neq 0$), since otherwise we have $\mu_i \circ \mu_j \circ \mu_i = \mu_i \circ \mu_i \circ \mu_j = \mu_j$ and we can reduce the situation by induction. Hence we assume $b_{ij} = b$ and $b_{ji} = -c$ for some integers $b, c \in \mathbb{Z}_{\neq 0}$.

For cluster algebra, the exchange polynomial for edge k is of the form $P(\mathbf{x}) = M_1(\mathbf{x}') + M_2(\mathbf{x}')$ where $\mathbf{x}' = \mathbf{x} \setminus \{x_k\}$. In particular (i) is satisfied.

For condition (ii), we note that Q is of the form $Q = x_i^c \star + \star$, in particular, Q_0 is a monomial, hence it is coprime with P.

The exchange relation implies condition (iii). To see this, let us write

$$P = M_i(t_0) + M_i(t_1), \qquad R = M_i(t_2) + M_i(t_3)$$

where

$$M_i(t_k) = \prod_{b_{ij}(t_k)>0} x_i(t_k)^{b_{ij}(t_k)}$$

Then we see that

$$\frac{M_i(t_1)}{M_i(t_0)} = \left.\frac{M_i(t_2)}{M_i(t_3)}\right|_{x_j \leftarrow \frac{Q_0}{x_j}} \tag{4.4}$$

implies condition (iii). (Adding 1 on both sides, and simplify, with $L \cdot Q_0^b = \frac{M_i(\frac{Q_0}{x_j})}{M_i(t_0)}$. Again L is monomial, hence $gcd(P, L) = 1$.) We can show (4.4) directly by the exchange relation. Using the seed $\mathbf{x}(t_1)$ as base, (4.4) can be written as

$$\prod_{i \in I} x_i^{b_{ij}} = \left.\prod_{i \in I} x_i^{b'_{ij}}\right|_{x_j \leftarrow \frac{Q_0}{x_j}} \tag{4.5}$$

where $x_i = x_i(t_1) = x_i(t_2), b_{ij} = b_{ij}(t_1), b'_{ij} = b_{ij}(t_2)$, and

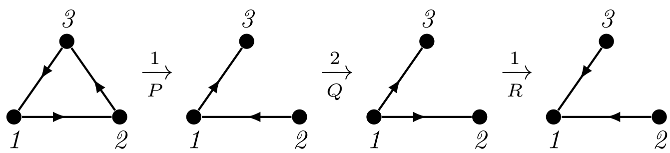
$$Q_0 = \left.\left(\prod_{k \in I} x_k^{[b_{jk}]_+} + \prod_{k \in I} x_k^{[-b_{jk}]_+}\right)\right|_{x_i=0}.$$

Then we can check case by case (assuming $b_{ji} \neq 0$) that Q_0 is a monomial given by

$$Q_0 = \prod_{k \in I, b_{ij}b_{jk}>0} x_k^{|b_{jk}|}.$$

Then it is not difficult to see that the substitution in (4.5) is equivalent to the mutation rule $B = \mu_j(B')$. □

Example 4.4. *To illustrate the proof of the Key Lemma, consider the cluster algebra defined by the leftmost quiver:*



Then we have

$$\begin{aligned}
x &= x_1 \\
y &= x_2 \\
z &= x'_1 = \frac{x_2 + x_3}{x_1} \\
u &= x'_2 = \frac{1 + x'_1}{x_2} = \frac{x_1 + x_2 + x_3}{x_1 x_2} \\
v &= x''_1 = \frac{1 + x'_2 x_3}{x'_1} = \frac{x_1 + x_3}{x_2}
\end{aligned}$$

$$gcd(z,u) = gcd(z,v) = 1$$

$$\begin{aligned} P &= x_2 + x_3, & P(\alpha) &= \alpha + x_3 & & \\ Q &= 1 + x_1, & Q(\alpha) &= 1 + \alpha, & Q(0) &= 1 \\ R &= 1 + x_2x_3, & R(\alpha) &= 1 + \alpha x_3 & & \end{aligned}$$

$$R\left(\frac{Q(0)}{y}\right) = 1 + \frac{1}{x_2}x_3 = \frac{1}{x_2} \cdot 1 \cdot P(x_2)$$

Finally, the positive conjecture is proved in most situation.

Theorem 4.5 (Lee-Schiffler (2015))**.** *For any skew-symmetric cluster algebra $\mathcal{A}$, any seed $(\mathbf{x}, \boldsymbol{y}, B)$, and any cluster variable u, the Laurent polynomial expansion of u in the cluster $\mathbf{x}$ has coefficients in $\mathbb{Z}_{>0}\mathbb{P}$.*

The case for cluster algebra coming from acyclic quiver is proven by Kimura-Qin (2014) and the case from surface by Musiker-Schiffler-Williams (2011).