Lecture Notes
Introduction to Cluster Algebra

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3.4 Semifields and coefficients

The mutation does not really use the frozen variables. So let us treat them as "coefficients", which leads to a more general notion of cluster algebra with semifields as coefficients.

Let us denote

$$y_j := \prod_{i=n+1}^{m} x_i^{b_{ij}}, \qquad j = 1, ..., n.$$

Then $y_1, ..., y_n$ encodes the same information as the lower $(n-m) \times n$ submatrix of $\widetilde{B}$. Hence a labeled seed can equivalently be presented as triples $(\mathbf{x}, \mathbf{y}, B)$ where $\mathbf{x} = (x_1, ..., x_n), \mathbf{y} = (y_1, ..., y_n)$.

Now the mutation of x_k becomes:

$$\begin{aligned} x_k x_k' &= \prod_{i=n+1}^{m} x_i^{[b_{ik}]_+} \prod_{i=1}^{n} x_i^{[b_{ik}]_+} + \prod_{i=n+1}^{m} x_i^{[-b_{ik}]_+} \prod_{i=1}^{n} x_i^{[-b_{ik}]_+} \\ &= \frac{y_k}{y_k \oplus 1} \prod_{i=1}^{n} x_i^{[b_{ik}]_+} + \frac{1}{y_k \oplus 1} \prod_{i=1}^{n} x_i^{[-b_{ik}]_+} \end{aligned}$$

where the *semifield addition* is defined by

$$\prod_i x_i^{a_i} \oplus \prod_i x_i^{b_i} := \prod_i x_i^{\min(a_i, b_i)}$$

in particular,

$$1 \oplus \prod_i x_i^{b_i} := \prod_i x_i^{-[-b_i]_+}$$

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The mutation of the frozen variables $x_{n+1}, ..., x_m$ also induces the mutation of the coefficient y-variables:

$$(y'_1, ..., y'_n) := \mu_k(y_1, ..., y_n)$$

$$y'_j := \begin{cases} y_k^{-1} & i = k \\ y_j(y_k \oplus 1)^{-b_{kj}} & \text{if } j \neq k \text{ and } b_{kj} \leq 0 \\ y_j(y_k^{-1} \oplus 1)^{-b_{kj}} & \text{if } j \neq k \text{ and } b_{kj} \geq 0 \end{cases}$$

This is called *tropical Y-seed mutation rule* This is general, we can use any semifield!

Definition 3.1. *A semifield $(\mathbb{P}, \circ, \oplus)$ is an abelian group $(\mathbb{P}, \circ)$ (written multiplicatively) together with a binary operator $\oplus$ such that*

$$\oplus : \mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{P}$$
$$(p, q) \mapsto p \oplus q$$

is commutative, associative, and distributive:

$$p \circ (q \oplus r) = p \circ q \oplus p \circ r$$

Note: $\oplus$ may not be invertible!

Example 3.2. *Examples of semifield $(\mathbb{P}, \circ, \oplus)$:*

- $(\mathbb{R}_{>0}, \times, +)$
- $(\mathbb{R}, +, \min)$
- $(\mathbb{Q}_{sf}(u_1, ..., u_m), \cdot, +)$ *"subtraction-free" rational functions*
- $(Trop(y_1, ..., y_m), \cdot, \oplus)$ *Laurent monomials with usual multiplication and*

$$\prod_i x_i^{a_i} \oplus \prod_i x_i^{b_i} := \prod_i x_i^{\min(a_i, b_i)}$$

$\mathbb{P}$ is called the *coefficient group* of our cluster algbera.

Proposition 3.3. *If $(\mathbb{P}, \circ, \oplus)$ is a semifield, then*

- $(\mathbb{P}, \circ)$ *is torsion free (if there exists p, m such that $p^m = 1$, then $p = 1$).*
- *Let $\mathbb{ZP}$ be the group ring of $(\mathbb{P}, \circ)$. Then it is a domain ($p \circ q = 0 \Longrightarrow p = 0$ or $q = 0$).*
- *Can define field of fractions $\mathbb{QP}$ of $\mathbb{ZP}$*

Proof. (1) If $p^m = 1$, then note that $1 \oplus p \oplus ... \oplus p^{m-1} \in \mathbb{P}$ but $0 \notin \mathbb{P}$, we can write

$$p = p\frac{1 \oplus p \oplus ... \oplus p^{m-1}}{1 \oplus p \oplus ... \oplus p^{m-1}} = \frac{p \oplus p^2 \oplus ... \oplus p^m}{1 \oplus p \oplus ... \oplus p^{m-1}} = 1$$

(2) Let $p, q \in \mathbb{ZP}$ with $p \circ q = 0$. Then p and q are contained in $\mathbb{Z}H$ for some finitely generated subgroup H of $\mathbb{P}$. Since $H \subset \mathbb{P}$ is abelian, $H \simeq \mathbb{Z}^n$, hence $\mathbb{Z}H \subset \mathbb{Z}(x_1, ..., x_n)$ consists of all Laurent polynomials in x_i. In particular $\mathbb{Z}H$ is an integral domain, and hence $p = 0$ or $q = 0$.

□

Then we can set our ambient field to be $\mathcal{F} := \mathbb{QP}(u_1, ..., u_n)$. Now we can rewrite previous definitions and results:

Definition 3.4. *A labeled seed in $\mathcal{F}$ is $(\mathbf{x}, \boldsymbol{y}, B)$ with*

- $\mathbf{x} = \{x_1, ..., x_n\}$ *free generating set of* $\mathcal{F}$
- $\boldsymbol{y} = \{y_1, ..., y_n\} \subset \mathbb{P}$ *any elements*
- $B = n \times n$ *skew-symmetrizable* $\mathbb{Z}$*-matrix*

We have mutations for all $\mathbf{x}, \boldsymbol{y}$ and B, together with the exchange patterns.

A cluster algebra with coefficients $\mathbb{P}$ is then

$$\mathcal{A}(\mathbf{x}, \boldsymbol{y}, B) := \mathbb{ZP}\left[\bigcup_{t \in \mathbb{T}_n} \mathbf{x}(t)\right]$$

Example 3.5. *For rank $n = 2$ we have in the most general case:*

t	B_t	$\boldsymbol{y}_t$		$\mathbf{x}_t$	
0	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	y_1	y_2	x_1	x_2
1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$y_1(y_2 \oplus 1)$	$\frac{1}{y_2}$	x_1	$\frac{x_1 y_2 + 1}{x_2(y_2 \oplus 1)}$
2	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\frac{1}{y_1(y_2 \oplus 1)}$	$\frac{y_1 y_2 \oplus y_1 \oplus 1}{y_2}$	$\frac{x_1 y_1 y_2 + y_1 + y_2}{(y_1 y_2 \oplus y_1 \oplus 1) x_1 x_2}$	$\frac{x_1 y_2 + 1}{x_2(y_2 \oplus 1)}$
3	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\frac{y_1 \oplus 1}{y_1 y_2}$	$\frac{y_2}{y_1 y_2 \oplus y_1 \oplus 1}$	$\frac{x_1 y_1 y_2 + y_1 + x_2}{(y_1 y_2 \oplus y_1 \oplus 1) x_1 x_2}$	$\frac{y_1 + x_2}{x_1(y_1 \oplus 1)}$
4	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\frac{y_1 y_2}{y_1 \oplus 1}$	$\frac{1}{y_1}$	x_2	$\frac{y_1 + x_2}{x_1(y_1 \oplus 1)}$
5	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	y_2	y_1	x_2	x_1

Remark 3.6. *The prevoius cluster algebra of geometric type = cluster algebra with coefficients $\mathbb{P} = Trop(x_{n+1}, ..., x_m)$.*

- We only need the $n \times n$ matrix B
- There are no frozen variables
- Mutation of $\mathbf{y}$ only involve two variables

- But we need to mutate all $\mathbf{y}$ variables, there are usually more $\mathbf{y}$ variables than cluster variables
- Y-pattern do not in general exhibit Laurent phenomenon

Definition 3.7. *Two cluster algebra $\mathcal{A}(\mathcal{S})$ and $\mathcal{A}(\mathcal{S}')$ are called* strongly isomorphic *if there exists a $\mathbb{Z}\mathbb{P}$-algebra isomorphism $\mathcal{F} \longrightarrow \mathcal{F}'$ sending some seed in $\mathcal{S}$ into a seed in $\mathcal{S}'$, thus inducing a bijection $\mathcal{S} \longrightarrow \mathcal{S}'$ of seeds and an algebra isomorphism $\mathcal{A}(\mathcal{S}) \longrightarrow \mathcal{A}(\mathcal{S}')$*

Any cluster algebra $\mathcal{A}$ is uniquely determined by any single seed $(\mathbf{x}, \mathbf{y}, B)$. Hence $\mathcal{A}$ is determined by B and $\mathbf{y}$ up to strong isomoprhism, and we can write $\mathcal{A} = \mathcal{A}(B, \mathbf{y})$.