Lecture Notes
Introduction to Cluster Algebra

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6 Finite Type Classicification

Definition 6.1. *A cluster algebra is of* finite type *if it has finitely many cluster variables (hence finitely many seeds).*

Definition 6.2. *A cluster algebra is of* finite mutation type *if there are only finitely many quivers appearing in the seeds.*

Definition 6.3. *Let B be an integer square matrix.*

- *The* Cartan counterpart *of B is the matrix $A = A(B) = (a_{ij})$ defined by*

$$a_{ij} = \begin{cases} 2 & i = j \\ -|b_{ij}| & i \neq j \end{cases}$$

- *B is called sign-skew-symmetric if for every i, j, either*

$$b_{ij} = b_{ji} = 0, \qquad or \qquad b_{ij}b_{ji} < 0$$

- *B is called* 2-finite *if any matrix B' mutation equivalent to B is sign-skew-symmetric and $|b'_{ij}b'_{ji}| \leq 3$*

Theorem 6.4 (Classification)**.** *Finite type cluster algebra can be classified by the* classical Dynkin diagrams. *More precisely, for a cluster algebra $\mathcal{A}$, the following are equivalent:*

(1) $\mathcal{A}$ is of finite type.

(2) For every seed $(\mathbf{x}, \boldsymbol{y}, B)$ in $\mathcal{A}$, the entries of $B = (b_{ij})$ satisfy $|b_{ij}b_{ji}| \leq 3$

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(3) $\mathcal{A} = \mathcal{A}(B_0, \boldsymbol{y})$ *where* B_0 *is sign-skew-symmetric matrix such that* $A = A(B_0)$ *is a Cartan matrix of finite type, and* $b_{ij}b_{jk} \geq 0$ *for all* i, j, k*.*

If B, B' *are sign-skew-symmetric matrices such that* $A(B), A(B')$ *are Cartan matrices of finite type, then* $\mathcal{A}(B)$ *and* $\mathcal{A}(B')$ *are strongly isomorphic (i.e.* $B \sim B'$*) iff* $A(B)$ *and* $A(B')$ *are of the same Cartan-Killing type.*

Outline of proof. (1) $\longrightarrow$(2): Reduce to rank 2 case and show that the corresponding exchange matrix $\begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$ must be 2-finite.

(2) $\longrightarrow$(3): A diagram is not 2-finite if it has a subdiagram which is not 2-finite. Eliminate all the cases that are mutation equivalent to these diagrams.

(3) $\longrightarrow$(1): Construct a polytope $\Delta(\Phi)$ from the root system with Cartan matrix A, and show that $\Delta(\Phi)$ is isomorphic to the cluster complex (vertex = cluster variable, maximal simplex = cluster).

Finally, If $A(B) = A(B')$, can assume $\Gamma(B) = \Gamma(B')$ then show $B = B'$. Conversely, if $\mathcal{A}(B) \simeq \mathcal{A}(B')$, the polytope $\Delta(\Phi) \simeq \Delta(\Phi') \implies \Phi$ and Φ' has same rank and cardinality. Only case is (B_n, C_n) and (B_6, C_6, E_6) but the symmetrizing matrix D is different for all of them. □

In particular, the proof of (3)$\longrightarrow$(1) gives us an explicit description of the clusters.

Definition 6.5. *The* almost positive roots *are defined by* $\Phi_{\geq -1} = (-\Pi) \amalg \Phi_+$*.*

We write $x^\alpha := \prod_{i\in I} x_i^{a_i}$ if $\alpha = \sum_{i\in I} a_i\alpha_i$.

Theorem 6.6. *There is a bijection*

$$\alpha \mapsto x[\alpha]$$

between $\Phi_{\geq -1}$ *and cluster variables of* $\mathcal{A}$ *such that* $x[\alpha]$ *is expressed as Laurent polynomial in the initial cluster* $\mathbf{x}_0$*:*

$$x[\alpha] := \frac{P_\alpha(\mathbf{x}_0)}{\mathbf{x}_0^\alpha}$$

for some polynomial P_α *over* $\mathbb{ZP}$ *with nonzero constant term. Under this bijection,* $x[-\alpha_i] := x_i$*.*

Example 6.7. *Looking at the quiver defining the cluster algbera, we have:*

- $\mathbb{C}[Gr(2, n+3)]$ *is cluster algebra of type* A_n *(snake diagram)*
- $\mathbb{C}[SL_3/N]$ *is cluster algebra of type* A_1
- $\mathbb{C}[SL_4/N]$ *is cluster algebra of type* A_3
- $\mathbb{C}[SL_5/N]$ *is cluster algebra of type* D_6
- $\mathbb{C}[SL_n/N]$ *is cluster algebra of infinite type for* $n \geq 6$

6.1 2-finite matrix

We first discuss (1)$\longrightarrow$(2). Let $\mathcal{A}$ be finite type. To show $|b_{ij}b_{ji}| \leq 3$ it suffices to restrict our attention to mutations in index i and j, i.e. rank 2 case. Hence let our mutation be μ_1 and μ_2, with exchange matrix

$$B = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}.$$

We show that if $|bc| > 3$, then we will obtain infinitely many cluster variables. Let the initial cluster variable be $\{x_1, x_2\}$ and we treat the rest of the variables as coefficients. Let $\langle m \rangle = \begin{cases} 1 & m \text{ is odd} \\ 2 & m \text{ is even} \end{cases}$, and we mutate the variables alternatively, which we know is a Laurent polynomial

$$x_m := \mu_{\langle m \rangle}(x_{m-1}) = \frac{N_m(x_1, x_2)}{x_1^{d_1(m)} x_2^{d_2(m)}}$$

Consider a lattice with bases $\{\alpha_1, \alpha_2\}$, and the Weyl group W generated by simple reflections s_1, s_2 given on the basis by

$$s_1 = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 1 & 0 \\ c & -1 \end{pmatrix}$$

One can think of this as an abstract root system Φ with (generalized) Cartan matrix $\begin{pmatrix} 2 & -b \\ -c & 2 \end{pmatrix}$ on simple roots $\{\alpha_1, \alpha_2\}$. Then each element $w \in W$ is of the form

$$w_1(m) := s_1 s_2 ... s_{\langle m \rangle}, \qquad w_2(m) := s_2 s_1 ... s_{\langle m+1 \rangle}$$

and we know that W is finite iff $|bc| \leq 3$. We let $w_1(0) = w_2(0) = e$.

Finally we let $\delta(m) := d_1(m)\alpha_1 + d_2(m)\alpha_2$.

Theorem 6.8. *We have*

$$\delta(m+3) = w_1(m) \cdot \alpha_{\langle m+1 \rangle}, \qquad \delta(-m) = w_2(m) \cdot \alpha_{\langle m+2 \rangle}$$

and they are all distinct. In particular, all x_m for $m \in \mathbb{Z}$ has different denominators $x_1^{d_1(m)} x_2^{d_2(m)}$.

We have already verified the denominators in the finite rank 2 case. This can be treated as an infinite version of the rank 2 case of Theorem 6.6.

Proof. For $m = 0$ and $m = 1$, we have

$$\begin{aligned} \delta(1) &= -\alpha_1 & \delta(2) &= -\alpha_2 \\ \delta(3) &= \alpha_1 & \delta(4) &= b\alpha_1 + \alpha_2 = s_1(\alpha_2) \\ \delta(0) &= \alpha_2 & \delta(-1) &= \alpha_1 + c\alpha_2 = s_2(\alpha_1) \end{aligned}$$

Now by induction on m, assume $m \geq 2$. From

$$x_{m+1}x_{m-1} = \begin{cases} x_m^b + 1 & m \text{ is odd} \\ x_m^c + 1 & m \text{ is even} \end{cases}$$

we have

$$\delta(m+1) + \delta(m-1) = \begin{cases} b\delta(m) & m \text{ is odd} \\ c\delta(m) & m \text{ is even} \end{cases}.$$

If m is odd, we have

$$\begin{aligned} \delta(m+3) &= b\delta(m+2) - \delta(m+1) \\ &=_{induction} bw_1(m-1)\cdot\alpha_1 - w_1(m-2)\cdot\alpha_2 \\ &= w_1(m-1)\cdot(s_1(\alpha_2)-\alpha_2) - w_1(m-2)\cdot\alpha_2 \\ &= w_1(m)\cdot\alpha_2 - w_1(m-2)\cdot(s_2(\alpha_2)+\alpha_2) \\ &= w_1(m)\cdot\alpha_2 \end{aligned}$$

Similar argument works with m even and m negative. □

Next we discuss the implication of 2-finite matrix.

Proposition 6.9. *Every 2-finite matrix is skew-symmetrizable*

Definition 6.10. *A diagram of a sign-skew-symmetric matrix B is the weighted directed graph $\Gamma(B)$ such that we have weighted arrows $i \xrightarrow{|b_{ij}b_{ji}|} j$ if $b_{ij} > 0$.*

Lemma 6.11. *Let B be 2-finite matrix. Then*

(1) The edges of every triangle in $\Gamma(B)$ are oriented in a cyclic way,

(2) The edge weights are either {1,1,1} or {2,2,1}

Proof. Suppose $b_{ij}, b_{ik}, b_{kj} > 0$. Then in $B' = \mu_k(B)$, we have $b'_{ij} = b_{ij} + b_{ik}b_{kj} \geq 2$ and $b'_{ji} = b_{ji} - b_{jk}b_{ki} \leq -2$ violating 2-finiteness. Hence the triangle are oriented in a cylic way. Hence we can assume

$$B = \begin{pmatrix} 0 & a_1 & -c_1 \\ -a_2 & 0 & b_1 \\ c_2 & -b_2 & 0 \end{pmatrix}$$

all variables positive integers. Let $-c_1$ has maximal absolute value. Then

$$\mu_2(B) = \begin{pmatrix} 0 & -a_1 & a_1b_1 - c_1 \\ a_2 & 0 & -b_1 \\ -a_2b_2 + c_2 & b_2 & 0 \end{pmatrix}$$

and by (1) we have

$$a_1b_1 - c_1 \geq 0, a_2b_2 - c_2 \geq 0$$

We show that the inequalities must be equal: Otherwise

$$a_2b_2 > c_2 \geq 1 \Longrightarrow max(a_2, b_2) \geq 2$$
$$a_1b_1 > c_1 \geq max(a_1, b_1) \Longrightarrow a_1, b_1 \geq 2$$
$$max(a_1a_2, b_1b_2) \geq 4(!)$$

Hence we have $c_1 = a_1b_1$ and $c_2 = a_2b_2$. The only choices for $\{a_1a_2, b_1b_2, c_1c_2\}$ are $\{1,1,1\}, \{2,2,1\}$ and

$$\{3,1,3\} = \{3 \cdot 1, 1 \cdot 1, 3 \cdot 1\}$$

or

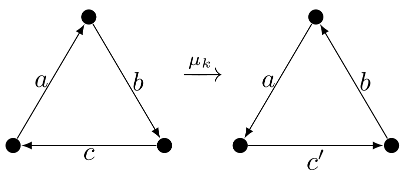
$$\{1,3,3\} = \{1 \cdot 1, 3 \cdot 1, 3 \cdot 1\}.$$

In the case $\{3,1,3\}$, $B' = \mu_1(B)$ has $|b'_{23}b'_{32}| = 4$ violating 2-finiteness. Similarly for $\{1,3,3\}$. □

6.2 Diagram mutation

Lemma 6.12. *Let B be 2-finite with diagram $\Gamma(B)$. The diagram $\Gamma' = \Gamma(\mu_k(B))$ can be obtained by*

- *The orientation of all edges incident to k are reversed, with same weights*
- *For any path $i \longrightarrow k \longrightarrow j$, the diagram changes as*



where

$$\sqrt{c} + \sqrt{c'} = \sqrt{ab}$$

- *All the rest of edges and weights remain unchanged.*

Observation. Any subdiagram of a 2-finite diagram is 2-finite. Equivalently, any diagram that has a 2-infinite subdiagram is 2-infinite.

6.2.1 Tree diagrams

Proposition 6.13. *Let T be a subdiagram of a diagram Γ such that*

(1) T is a tree

(2) T is attached to Γ by a single vertex $v \in T$, i.e. $\Gamma - \{v\}$ and $T - \{v\}$ are disjoint.

Then any two orientations on T is mutation equivalent.

Proof. Induction on size of T. ∎

Definition 6.14. *A diagram Γ is called extended Dynkin tree diagram if*

- Γ *is a tree diagram with edge weights* ≤ 3
- Γ *is not Dynkin diagram*
- Γ *every connected subdiagram of* Γ *is a Dynkin diagram*

The complete list gives the Dynkin diagrams associated with untwisted affine Lie algebras *except* $A_n^{(1)}$ *which is a loop.*

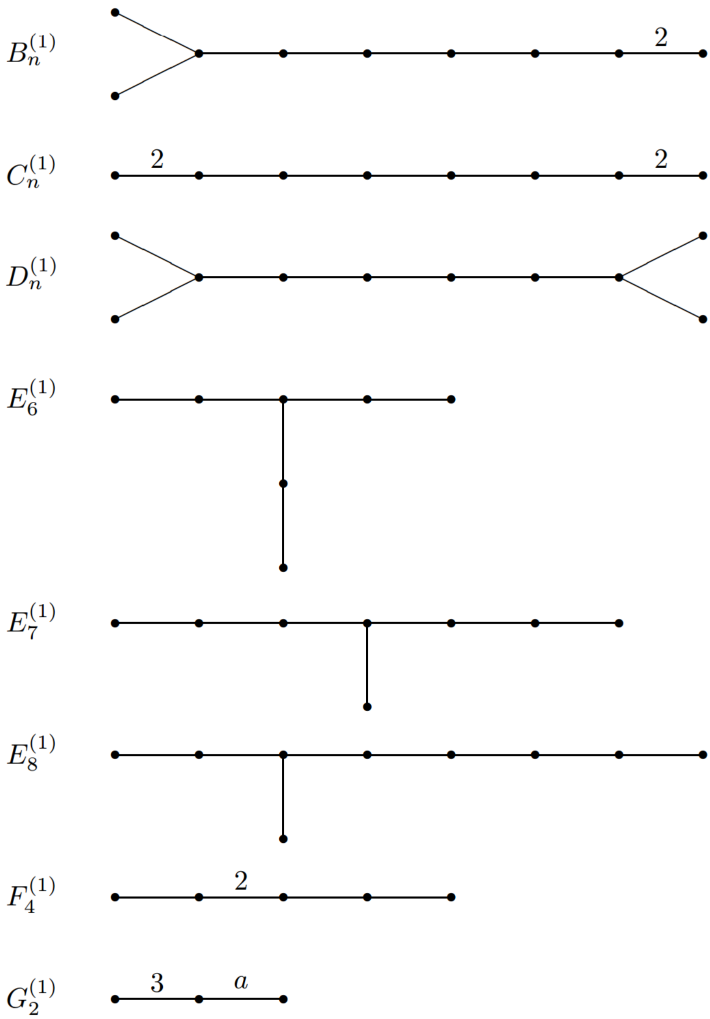


Figure 1: Extended Dynkin tree diagrams

Proposition 6.15. *Any 2-finite tree diagram is an orientation of a Dynkin diagram*

Proof. Let us show that if it is not Dynkin, then it is not 2-finite. Hence it suffices to take the minimal subdiagram that is not Dynkin, therefore it is enough to consider the extended Dynkin tree diagrams, with arbitrary orientations.

For $X_n^{(1)}$ with $X = B, C, D$, orient the tree from left to right. Mutate at second vertex from left and consider subdiagram: $X_n^{(1)} \longrightarrow X_{n-1}^{(1)}$. Hence only need to consider $D_4^{(1)}, B_3^{(1)}, C_2^{(1)}$.

- $C_2^{(1)} \sim$ triangle with weights (2,2,2). (!)
- $B_3^{(1)}$Mutate at branch: subdiagram C_2^1. (!)
- (HW) $D_4^{(1)} \sim$ contain 2-infinite triangle (!)
- (HW) $G_2^{(1)} \sim$ 2-infinite triangle (!)
- (HW) $F_4^{(1)} \sim$ subdiagram C_2^1. (!)

Definition 6.16. *For $p, q, r \in \mathbb{Z}_{\geq 0}$ define*

- *Tree $T_{p,q,r}$: A_p, A_q, A_r join at one extra vertex.*
- *$S^s_{p,q,r}$: $A_{p-1}, A_{q-1}, A_{r-1}$ joined to 3 consecutive vertices on a cyclically oriented $s+3$ cycle.*

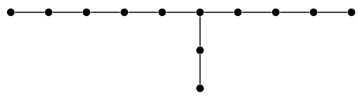


Figure 2: Tree diagram $T_{5,4,2}$

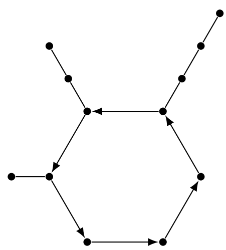


Figure 3: Diagram $S^3_{4,3,2}$

Lemma 6.17. $S^s_{p,q,r} \sim T_{p+r-1,q,s}$

Proof. Remove A_q, we get A_{p+s+r}. Reorient the tree. Mutate at $\mu_1 \circ ... \circ_{s+r}$. □

- $E_6^{(1)} = T_{2,2,2} \sim S^2_{2,2,1} \supset D_5^{(1)}$
- $E_7^{(1)} = T_{3,1,3} \sim S^3_{3,1,1} \supset E_6^{(1)}$
- $E_8^{(1)} = T_{2,1,5} \sim S^5_{2,1,1} \supset E_7^{(1)}$

□

6.2.2 Cycles

Proposition 6.18. *Each 2-finite cycle of length n is cyclically oriented, and is either:*

(a) oriented cycle with weights =1: $\sim D_n$, or

(b) oriented triangle with weights $(2,2,1)$: $\sim B_3$, or

(c) oreinted 4-cycle with weights $(2,1,2,1)$: $\sim F_4$.

Proof.

- Cyclic oriented: Induction on n. Let v has one incoming and one outgoing edge. $\Gamma' = \mu_v(\Gamma)$ has an $n-1$ cycle Γ'' as subdiagram, which is cyclic by induction $\Longrightarrow \Gamma$ is cyclic.
- Γ'' has same edge weights product π as Γ ($c = 0 \Longrightarrow ab = c'$). Hence $\pi = 1$ or 4 (since triangle is (1,1,1) or (2,2,1)).
- If $\pi = 1$, $\Gamma = S^{n-3}_{1,1,1} \sim T_{1,1,n-3} = D_n$
- If $\pi = 4$, it has 2 edges with weight 2. If it is not (b) or (c), it contains $C^{(1)}_m$.

□

6.2.3 Remaining cases

$n \leq 3$ either tree or cycle, already done. By induction: pick vertex $v \in \Gamma$ so that $\Gamma' = \Gamma - \{v\}$ is connected. We know Γ' is 2-finite, hence $\Gamma' \sim X_n$. Now do the mutation and assume $\Gamma' \simeq X_n$, and see how v is connected to Γ'.

Case 1: Γ' no branch point: A_n, B_n, F_4, G_2.

- v connect to one vertex: tree.
- v connect to > 2 vertices: wrong cycles (!)
- v connect to 2 vertices v_1, v_2: Γ has a cycle $\mathcal{C}$ which is of the three types (a)-(c) by Prop 6.18.
  - (a)
    - · If Γ has weight ≥ 2: it contains $B^{(1)}_m$ or $G^{(1)}_2$ (!) unless $\mathcal{C}$ is 3-cycle, then $\mu_v(\Gamma) \simeq B_{n+1}$.
    - · If Γ has unit weight: $\Gamma \sim S^s_{p,0,r}$
  - (b)
    - · If one of $(v, v_1), (v, v_2)$ has weight 1, $\mu_v(\Gamma) \sim$ tree.
    - · If both $(v, v_1), (v, v_2)$ has weight 2, and $\Gamma - \mathcal{C}$ has weight ≥ 2, then $\Gamma \supset C^{(1)}_m$ or $\Gamma \supset G^{(1)}_2$ (!)
    - · Otherwise $\Gamma \sim B_{n+1}$.
  - (c) Any diagram $\{v'\} \cup \mathcal{C}$ is 2-infinite. Extra edge weight = 1,2,3: contain $B^{(1)}_3$, $C^{(1)}_2$, $G^{(1)}_2$ respectively.

Case 2: $\Gamma' \sim D_n \sim n$-cycle with unit weights.

- v connect to one vertex v_1
  - (v, v_1) has weight ≥ 2: $\Gamma \supset B_3^{(1)}$ or $G_2^{(1)}$.
  - (v, v_1) has weight 1: $\Gamma \sim$ tree
- v connect to two adjacent vertices:
  - $\mathcal{C}$ is (1,1,1) triangle: $\mu_v(\Gamma) = cycle \sim D_{n+1}$
  - $\mathcal{C}$ is (2,2,1) triangle: $\mu_v(\Gamma)$ has bad cycle (!)
- v connect to two non-adjacent vertices: bad cycle (!)

Case 3: $\Gamma' \sim E_n = T_{1,2,n-4} \sim S_{1,2,1}^{n-4}$ an oriented $(n-1)$ cycle $\mathcal{C}$ with one extra edge of weight 1 connecting $v_1 \notin \mathcal{C}$.

- v connect to v_1 only.
  - (v, v_1) has weight ≥ 2, then $\Gamma \supset B_3^{(1)}, G_2^{(1)}$.
  - (v, v_1) has weight 1: $\Gamma \sim$ tree.
- v connect to $v_2 \in \mathcal{C}$ only: $\Gamma \supset D_m^{(1)}, B_3^{(1)}$ or $G_2^{(1)}$.
- v connect to ≥ 2 vertices of $\mathcal{C}$: $\Gamma - \{v_1\} \sim D_n$ Reduce to case 2.
- v connect to v_1 and single vertex $v_2 \in \mathcal{C}$: Let v_0 be the only vertex on $\mathcal{C}$ adjacent to v_1 in Γ'.
  - $v_2 \neq v_0$ nor connect to v_0, bad cycles (!)
  - $v_2 = v_0$: (v, v_1, v_2) is an oriented triangle. If (v, v_1) has weight 1, $\mu_{v_1}(\Gamma) \sim$ tree. Otherwise $\mu_{v_1}(\Gamma)$ contains $B_3^{(1)}$.
  - v_2 connect to v_0: $\Gamma - \{v_0\}$ has no branch point: reduce to Case 1.

This completes the proof of the classification.

6.3 Finite mutation type

A larger class of finiteness of cluster algbera is given by finite mutation type:

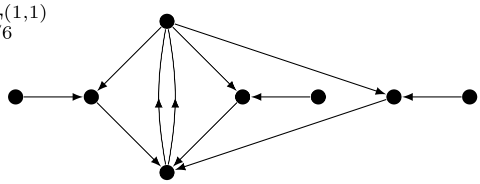
Definition 6.19. *A skew-symmetric cluster algebra (i.e. defined by a quiver) is of* finite mutation type *if the quiver mutation class is finite.*

Theorem 6.20. *Finite mutation type cluster algebra is classified by*

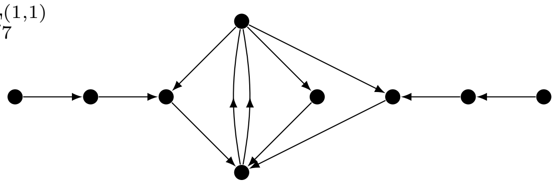
- *cluster algebra from surface (quiver arising from the triangulation of surfaces, following our previous rules)*
- *cluster algebra of rank $n = 2$*
- *cluster algebra from the E_6, E_7, E_8 and $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$ quivers (see Figure 1)*

- *cluster algebra from quivers of 5 exceptional types:*

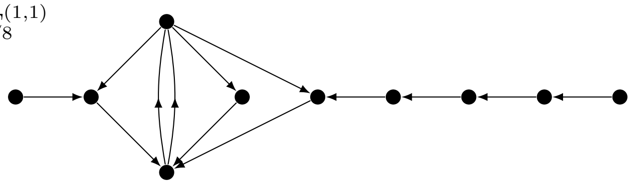
$E_6^{(1,1)}$



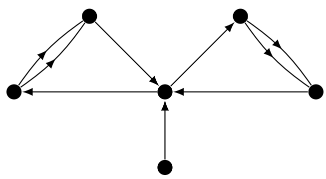
$E_7^{(1,1)}$



$E_8^{(1,1)}$



X_6



X_7