Lecture Notes
Introduction to Cluster Algebra

Ivan C.H. Ip*

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7 Generalized Associahedron

To complete the proof of the classification theorem, we want to construct a polytope $\Delta(\Phi)$ from the root system Φ such that it is isomorphic to the cluster complex $\Delta(\mathcal{A})$. We first establish a general fact.

7.1 Construction of Polytope $\Delta(\mathcal{A})$

Let Ψ be a finite set ("ground set"). Let Δ be a simple convex polytope in $\mathbb{R}^n$. Assume

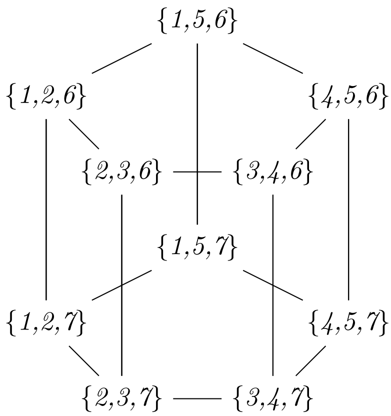
- Each vertex $v \in \Delta$ are labeled by n-tuple of elements $[v] \subset \Psi$.
- i-dimensional faces of Δ corresponds bijectively to maximal subsets of vertices $v \in \Delta$ such that the labels have exactly $n-i$ elements in common.
- A sign-skew-symmetric $n \times n$ matrix B_v is attached to each vertex $v \in \Delta$, with label $[v]$.
- For every edge $(v, \overline{v})$ of Δ, with $[\overline{v}] = [v] - \{\gamma\} \cup \{\overline{\gamma}\}$, $B_{\overline{v}}$ is obtained from B_v by matrix mutation at γ and relabeling it by $\overline{\gamma}$.

For any 2-dimensional face $F \in \Delta$, for a vertex $v \in F$, there exists two elements α, β that are not common to the labels of all other vertices in F. Define the type of $F = |b_{\alpha\beta} b_{\beta\alpha}|$, which does not depend on choice of v.

Proposition 7.1. *Assume 2-dimensional face of Δ are 4,5,6,8-gons of types 0,1,2,3 respectively. Then the cluster algebra $\mathcal{A} = \mathcal{A}(B, \boldsymbol{y})$ is of finite type if $B = B_v$ for some $v \in \Delta$.*

*Center for the Promotion of Interdisciplinary Education and Research/
Department of Mathematics, Graduate School of Science, Kyoto University, Japan
Email: ivan.ip@math.kyoto-u.ac.jp

Example 7.2. *This is an example for $\Psi = \{1,2,3,4,5,6,7\}$ and $n = 3$:*



Definition 7.3. *Let $\Sigma = (\mathbf{x}, \boldsymbol{y}, B)$ be a seed of $\mathcal{A}$.*

- *A* seed attachment *of Σ at v is a bijection between labels at v and cluster variables of $\mathbf{x}$, and identifying B and B_v.*
- *A* transport *of seed attachment along an edge $(v, \overline{v})$: the seed $\overline{\Sigma}$ attached at $\overline{v}$ is obtained from Σ by mutation in direction $x(\gamma)$ where $[\overline{v}] = [v] - \{\gamma\} \cup \{\overline{\gamma}\}$.*

Proof. Start with an initial seed attachment at v_0. We can transport to other vertex v' along a path from v to v'. It does not depend on choice of path, i.e. transport of seed attachment along a loop brings it back unchanged. By our assumption of the faces of Δ, this follows from cluster algebra of rank 2.

Any sequence of mutations of seed is uniquely lifted to a path on Δ, and transporting the initial seed attachment along the path produces the chosen sequence of mutation. Hence we have a map from vertices of Δ surjective onto the set of all seeds of $\mathcal{A}$.

In fact we have a stronger result, that there is a surjection from Ψ to the set of all cluser variables of $\mathcal{A}$. Let $v', v'' \in \Delta$ with $[v'] \cap [v''] = \{\alpha\}$. They can be joined by a path $v_1 = v', v_2, ..., v_l = v''$ such that $\alpha \in [v_i]$ for all I. Hence $x'(\alpha) = ... = x''(\alpha)$ and the seed attachement does not depend on the choice of vertex. Hence the attachment of cluster variables to the ground set is a surjection from Ψ to set of all cluster variables of $\mathcal{A}$. Since Ψ is finite, $\mathcal{A}$ is finite type. □

The proof then requires the construction of the polytope Δ for each Cartan-Killing type, which can be described combinatorically by the root systems.

Again let the Dynkin diagram be bipartite into two parts I_+, I_-. Let Π be simple roots, Φ_+ be positive roots, $\Phi_{\geq -1} := \Phi_+ \cup (-\Pi)$ be almost positive roots, $Q = \mathbb{Z}\Pi$ the root lattice, $Q_+ = \mathbb{Z}_{\geq 0}\Pi_+$ the positive root lattice. Let $A = (a_{ij})$ be the Cartan matrix.

Definition 7.4. *Sign function* $\epsilon : I \longrightarrow \{+,-\}$

$$\epsilon(i) = \begin{cases} + & i \in I_+ \\ - & i \in I_- \end{cases}.$$

Denote by $[\gamma : \alpha_i]$ *the coefficient of* α_i *in the expansion of* $\gamma \in Q$ *in the basis* Π.

Definition 7.5. *Piecewise-linear reflection* $\tau_\pm : \Phi_{\geq -1} \longrightarrow \Phi_{\geq -1}$*:*

$$\tau_\epsilon(\alpha) := \begin{cases} \alpha & \alpha = -\alpha_i, i \in I_{-\epsilon} \\ t_\epsilon(\alpha) & \textit{otherwise} \end{cases}$$

for $\epsilon \in \{+,-\}$, *where we recall*

$$t_\epsilon = \prod_{i \in I_\epsilon} s_i.$$

This can be extended to the whole $\gamma \in Q$ *as follows: if we let*

$$\gamma_\epsilon := \sum_{\substack{i \in I_{-\epsilon} \\ k_i < 0}} k_i \alpha_i, \qquad \gamma'_\epsilon := \gamma - \gamma_\epsilon,$$

then

$$\tau_\epsilon(\gamma) = \gamma_\epsilon + t_\epsilon(\gamma'_\epsilon).$$

Equivalently, in terms of coordinate,

$$[\tau_\epsilon(\gamma) : \alpha_i] = \begin{cases} -[\gamma : \alpha_i] - \sum_{i \neq j} a_{ij} [\gamma : \alpha_j]_+ & i \in I_\epsilon \\ [\gamma : \alpha_i] & i \in I_{-\epsilon} \end{cases}.$$

Let $\mathcal{D}$ *denote the group generated by* $\langle \tau_+, \tau_- \rangle$.

Theorem 7.6. *(1)* $\tau_\pm$ *is an involution and preserve* $\Phi_{\geq -1}$

(2) $\tau_\pm(\alpha) = t_\pm(\alpha)$ *for any* $\alpha \in Q_+$

(3) The order of $\tau_-\tau_+$ *equals* $\frac{h+2}{2}$ *if* $w_0 = -1$, *and* $h+2$ *otherwise. Hence* $\mathcal{D}$ *is a dihedral group*

(4) Every $\mathcal{D}$*-orbit in* $\Phi_{\geq -1}$ *has nonempty intersection with* $-\Pi$. *There is a bijection between* $\mathcal{D}$*-orbits in* $\Phi_{\geq -1}$ *and the* $\langle -w_0 \rangle$*-orbits in* $(-\Pi)$

Proof. (1) and (2) follows from definition. (3) and (4) see [FZ-YSystem, Theorem 2.6], which describe the orbits explicitly by some patterns. □

Example 7.7. *One orbit for* A_2*:*

$$\tau_- \circlearrowright -\alpha_1 \xleftrightarrow{\tau_+} \alpha_1 \xleftrightarrow{\tau_-} \alpha_1 + \alpha_2 \xleftrightarrow{\tau_+} \alpha_2 \xleftrightarrow{\tau_-} -\alpha_2 \circlearrowright \tau_+$$

Example 7.8. *Two orbits for A_3: We have $\pm\alpha_1 \longleftrightarrow \mp\alpha_3$ and $\alpha_2 \longleftrightarrow -\alpha_2$ in the w_0 orbit.*

$$\tau_- \circlearrowright -\alpha_1 \overset{\tau_+}{\longleftrightarrow} \alpha_1 \overset{\tau_-}{\longleftrightarrow} \alpha_1+\alpha_2 \overset{\tau_+}{\longleftrightarrow} \alpha_2+\alpha_3 \overset{\tau_-}{\longleftrightarrow} \alpha_3 \overset{\tau_+}{\longleftrightarrow} -\alpha_3 \circlearrowright \tau_-$$

$$\tau_+ \circlearrowright -\alpha_2 \overset{\tau_-}{\longleftrightarrow} \alpha_2 \overset{\tau_+}{\longleftrightarrow} \alpha_1+\alpha_2+\alpha_3 \circlearrowright \tau_-$$

The following result is the most important tools in this section.

Proposition 7.9. *There is a unique function called the* compatibility degree

$$\begin{aligned} \Phi_{\geq -1} \times \Phi_{\geq -1} &\longrightarrow \mathbb{Z}_{\geq 0} \\ (\alpha, \beta) &\mapsto (\alpha || \beta) \end{aligned}$$

such that

$$(-\alpha_i || \alpha) = max([\alpha : \alpha_i], 0)$$

$$(\tau_\epsilon \alpha || \tau_\epsilon \beta) = (\alpha || \beta)$$

Remark 7.10. *This proposition is important since most of the proofs below uses the following strategy:*

- *$\mathcal{D}$-orbit does not change $(\cdot||\cdot)$.*
- *Can always apply some element from $\mathcal{D}$ to go back to negative roots.*
- *We can study the case with negative root explicitly, or we can remove the negative roots and use induction on the positive part.*

Definition 7.11.

- *α and β are compatible if $(\alpha||\beta) = (\beta||\alpha) = 0$.*
- *α and β are exchangeable if $(\alpha||\beta) = (\beta||\alpha) = 1$.*
- *Let $\Delta(\Phi)$ be the simplicial complex on the ground set $\Phi_{\geq -1}$ whose simplices are mutually compatible roots. The maximal simplices are called* clusters.

Theorem 7.12.

(1) Each cluster in $\Delta(\Phi)$ is a $\mathbb{Z}$-basis for Q. In particular all clusters are of the same size n.

(2) Every element of the root lattice has a unique cluster expansion (i.e. linear combinations of mutually compatible roots with nonnegative coefficients).

(3) Let $[\gamma : \alpha]_{clus}$ denote the coefficient of α in the cluster expansion. Then the coefficient is invariant under $\sigma \in \mathcal{D}$,

$$[\sigma(\gamma) : \sigma(\alpha)]_{clus} = [\gamma : \alpha]_{clus}$$

(4) The cones $\mathcal{C}(\Delta(\Phi))$ spanned by the simplices in $\Delta(\Phi)$ form a complete simplicial fan in $Q_\mathbb{R}$

Please see Example 7.16 to understand the statements.

Proof. (1) is [FZ-YSystem, Theorem 1.8] By induction.

- For every $i \in I$, if C is cluster for Φ that contains $-\alpha_i$, then $C - \{-\alpha_i\}$ is cluster for $\Phi(I - \{i\})$. (all other elements in C does not have component of α_i by definition of compatibility.)
- Hence C is a $\mathbb{Z}$ basis for Q_Φ iff $C - \{-\alpha_i\}$ is a $\mathbb{Z}$ basis for $Q_{\Phi(I-i)}$.
- Need to consider the case when C are all positive roots only.
- Since $\tau_\pm(C) = t_\pm(C)$ are also clusters, we can arrive at a cluster C' which is no longer positive.
- Then we can remove the negative root and apply induction.

(2) is [FZ-YSystem, Theorem 3.11] Again by induction.

- Let $S_+(\gamma) = \{i \in I : [\gamma : \alpha_i] > 0\}$ be the positive support.
- If $\alpha \in \Phi_+$ occurs in expansion of γ, then $S_+(\alpha) \subset S_+(\gamma)$.
- If $\alpha \in -\Pi$ occurs in expansion of γ, then $[\gamma : \alpha] < 0$.
- Let $\gamma^{(+)} := \sum_{i \in S_+(\gamma)} [\gamma : \alpha_i]\alpha_i$.
- Then γ has unique cluster expansion in Φ iff $\gamma^{(+)}$ has unique cluster expansion in $\Phi(S_+(\gamma))$.
- Need to consider the case when $\gamma \in Q_+$ only. Then γ has unique cluster expansion iff $t_\epsilon(\gamma) = \tau_\epsilon(\gamma)$ has unique cluster expansion (all components are $+$, and t_ϵ is linear).
- Move γ outside Q_+ by $t_\pm$, and then can apply induction.

(3): $\tau_\pm$ is a linear map in each cluster cone.
(4) follows from (2). □

Definition 7.13. *A normal fan $\mathcal{N}(\mathcal{P})$ of a simple convex polytope is a simplicial fan where each maximal cone correspond to a vertex $\phi \in P$ by*

$$C_\phi := \{\gamma \in V^* \simeq \mathbb{R}^n : \max_{\psi \in P} \langle \gamma, \psi \rangle = \langle \gamma, \phi \rangle\}$$

Theorem 7.14. *The simplicial fan $\mathcal{C}(\Delta(\Phi))$ is the normal fan of a simple n-dimensional convex polytope, the generalized associahedron.*

Hence the generalized associahedron P is the dual complex of the cluster complex. In particular, the exchange graph is the 1-skeleton of P, and the maximal simplex of P is labeled by $\Phi_{\geq -1}$. In particular the cluster complex is topologically homeomorphic to an n dimensional sphere.

The construction of generalized associahedron is given by:

Theorem 7.15. *[CFZ, Theorem 1.4,1.5] Let $F : -\Pi \longrightarrow \mathbb{R}$ satisfies*

$$\sum_{i \in I} a_{ij} F(-\alpha_j) > 0 \qquad \forall j \in I$$

and extends uniquely to $\langle \tau_+, \tau_- \rangle$-invariant function on $\Phi_{\geq -1}$. Then the generalized associahedron $\mathcal{P}$ is given by

$$\langle z, \alpha \rangle \leq F(\alpha), \qquad \forall \alpha \in \Phi_{\geq -1}$$

with normal fan $\mathcal{C}(\Delta(\Phi))$.

Example 7.16. *[Type A_2]Both exchange graph and cluster complex are pentagons.*

$$\max(-z_1, -z_2, z_1, z_2, z_1 + z_2) \leq c$$

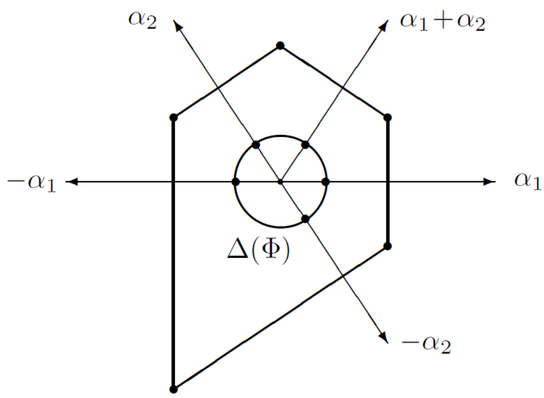


Figure 1: Type A_2 Associahedron

Example 7.17 (Type A_3)**.** *The Associahedron is also called Stasheff polytope. There are two $\mathcal{D}$-orbits:*

$$\{-\alpha_1, -\alpha_3, \alpha_1, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3\} \text{ and } \{-\alpha_2, \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\}$$

Then for $0 < c_1 < c_2 < 2c_1$ we have

$$\max(-z_1, -z_3, z_1, z_3, z_1 + z_2, z_2 + z_3) \leq c_1$$
$$\max(-z_2, z_2, z_1 + z_2 + z_3) \leq c_2$$

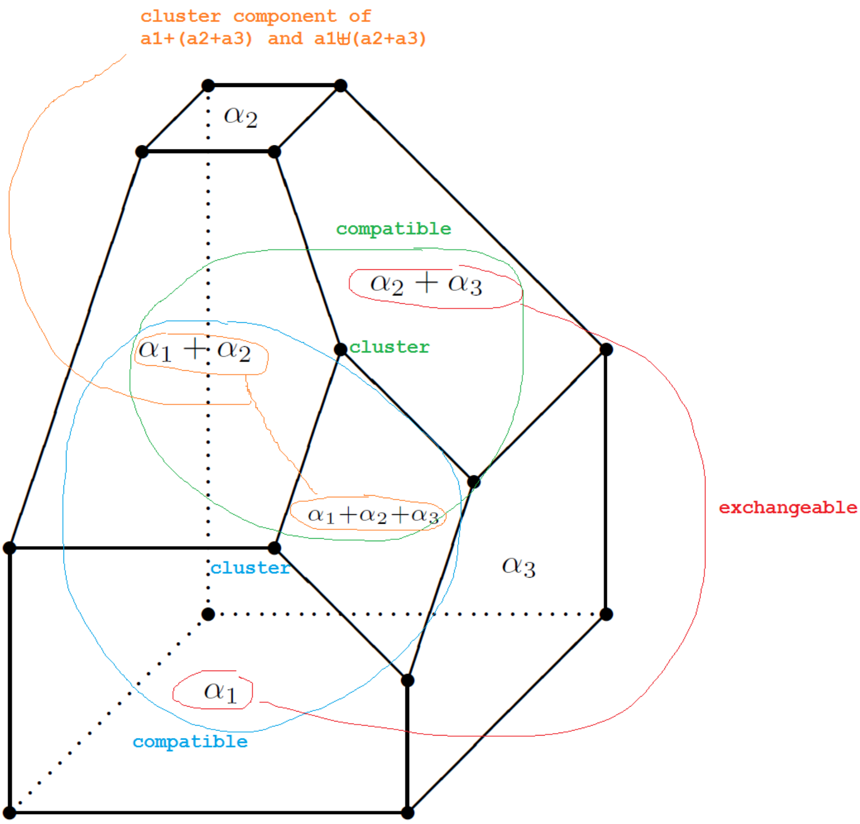


Figure 2: Type A_3 Associahedron, $(c_1, c_2) = (\frac{3}{2}, 2)$

Example 7.18 (Type C_2). *It is a hexagon, given by*

$$\max(-z_1, z_1, z_1 + z_2) \leq c_1$$
$$\max(-z_2, z_2, 2z_1 + z_2) \leq c_2$$

for $0 < c_1 < c_2 < 2c_1$.

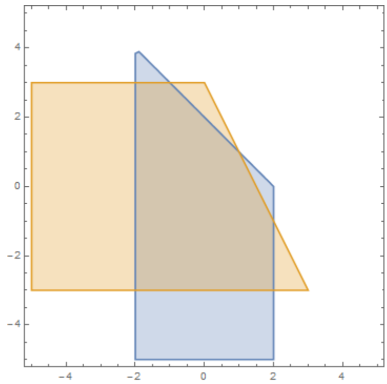


Figure 3: Type C_2 Associahedron (the intersection), $(c_1, c_2) = (2, 3)$

Example 7.19 (Type C_3)**.** *Cyclohedron, also known as Bott-Taubes polytope. There are 3 $\mathcal{D}$-orbits. Then we have*

$$\begin{aligned}
\max(-z_1, z_1, z_1 + z_2, z_2 + z_3) &\leq c_1 \\
\max(-z_2, z_2, z_1 + z_2 + z_3, z_1 + 2z_2 + z_3) &\leq c_2 \\
\max(-z_3, z_3, 2z_2 + z_3, 2z_1 + 2z_2 + z_3) &\leq c_3
\end{aligned}$$

for $c_2 < 2c_1, c_1 + c_3 < 2c_2, c_2 < c_3$.

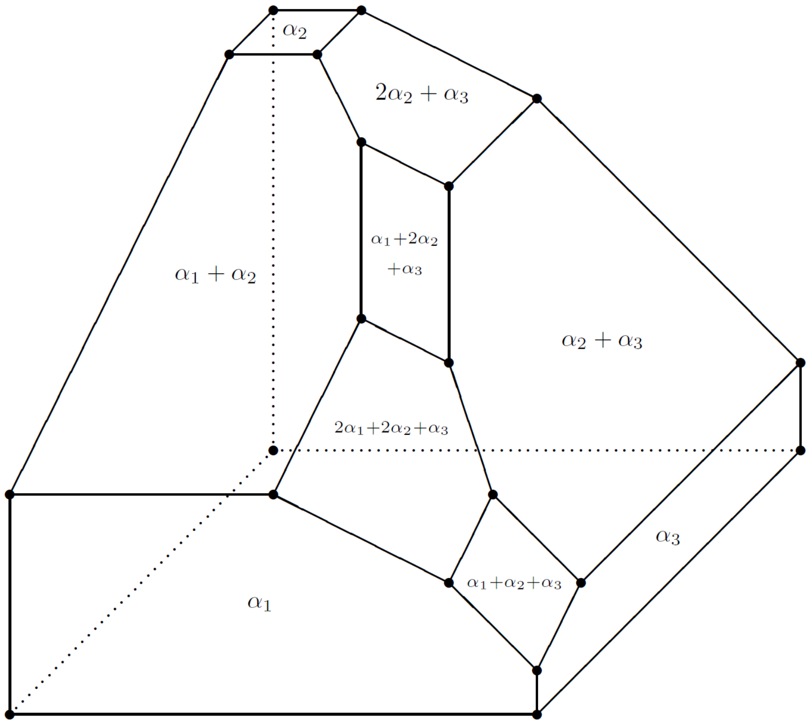


Figure 4: Type C_3 Generalized Associahedron, $(c_1, c_2, c_3) = (\frac{5}{2}, 4, \frac{9}{2})$

References

[CFZ] F. Chapoton, S. Fomin, A. Zelevinsky, *Polytopal Realization of Generalized Associahedra*, Canadian Mathematical Bulletin, 45(4), 537-566.

[FZ-ClusterII] S. Fomin, A. Zelevinsky, *Cluster Algebras II: Finite Type Classification*, Inventiones mathematicae 154.1 (2003): 63-121.

[FZ-YSystem] S. Fomin, A. Zelevinsky, *Y System and Generalized Associahedra*, Annals of Mathematics 158.3 (2003): 977-1018.