Lecture Notes
Introduction to Cluster Algebra

Ivan C.H. Ip*

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7.2 Properties of Exchangeable roots

The notion of exchangeable is explained as follows

Proposition 7.20. *If C and $C' = C - \{\beta\} \cup \{\beta'\}$ are two adjacent clusters, then β, β' are exchangeable*

Proof.

- We have single linear relation

$$m_\beta \beta + m_{\beta'} \beta' = \sum_{\gamma \in C \cap C'} m_\gamma \gamma, \qquad m_\beta, m_{\beta'}, m_\gamma \in \mathbb{Z}_{\geq 0}$$

- Since cluster is $\mathbb{Z}$ basis, $m_\beta = m_{\beta'} = 1$.
- Use τ to bring $\beta = -\alpha_i$.
- Then $\gamma \in C \cap C'$ has no α_i components since it is compatible with $-\alpha_i$.
- Hence $(\beta || \beta') =_\tau (-\alpha_i || \beta'') = 1$

□

The exchange relation of two exchangeable roots in the clusters C, C' with $C' = C - \{\beta\} \cup \{\beta'\}$ are of the form

$$x[\beta]x[\beta'] = pX[\gamma] + p'X[\gamma']$$

for some polynomials in $\mathbf{x}(C \cap C')$ and some coefficients (depending on β, β', C) $p, p' \in \mathbb{P}$. We can think of it as represented by some elements in the root lattice Q:

$$X[\gamma] = x_1^{m_1} ... x_k^{m_k} \mapsto \gamma = m_1\gamma_1 + ... + m_k\gamma_k$$

where γ_i are the elements of $C \cap C'$ and $m_i \geq 0$. It turns out that one can describe γ and γ' explicitly.

*Center for the Promotion of Interdisciplinary Education and Research/
Department of Mathematics, Graduate School of Science, Kyoto University, Japan
Email: ivan.ip@math.kyoto-u.ac.jp

Theorem 7.21. *Let β, β' be exchangeable. Define*

$$\beta +_\sigma \beta' := \sigma^{-1}(\sigma(\beta) + \sigma(\beta'))$$

If Φ is not of type A_1, then the set

$$E(\beta, \beta') := \{\beta +_\sigma \beta' : \sigma \in \mathcal{D}\}$$

consists of two elements $\{\beta + \beta', \beta \uplus \beta'\}$. If $\Phi = A_1$, $E(-\alpha, \alpha) = \{0\}$.

Lemma 7.22. *If β, β' are both positive, or $\tau_\epsilon(\beta), \tau_\epsilon(\beta')$ are both positive, then*

$$\beta +_{\tau_\epsilon} \beta' = \beta + \beta'$$

since $\tau_\epsilon = t_\epsilon$ is linear when the root is positive.

Example 7.23. *In the special case where $\beta' = -\alpha_j$,*

$$(-\alpha_j) \uplus \beta = \beta - \alpha_j + \sum_{i \neq j} a_{ij} \alpha_i$$

Using this Lemma, one can determine when $\beta +_\sigma \beta'$ is $\beta + \beta'$ or $\beta \uplus \beta'$.
We list some properties of $\beta + \beta'$ and $\beta \uplus \beta'$:

Lemma 7.24. *Let β, β' be exchangeable.*

(1) No negative simple root can be cluster component of $\beta + \beta'$.

(2) The vectors $\beta + \beta'$ and $\beta \uplus \beta'$ has no common cluster components

(3) If $[\beta \uplus \beta' : \alpha_i] > 0$, then $[\beta + \beta' : \alpha_i] > 0$.

(4) All cluster components of $\beta + \beta'$ and $\beta \uplus \beta'$ are compatible with both β and β'

(5) A root $\alpha \neq \beta, \beta'$ is compatible with both β, β' iff compatible with all cluster components of $\beta + \beta'$ and $\beta \uplus \beta'$.

(6) If $\alpha \in -\Pi$ is compatible with all cluster components of $\beta + \beta'$, then it is compatible with all cluster components of $\beta \uplus \beta'$.

Proof. (1)
- Assume $\beta + \beta'$ has cluster expansion in the cluster C. If $-\alpha_i \in C$, all other elements of C has no α_i components.
- If $[\beta + \beta' : -\alpha_i]_C > 0$, then $[\beta + \beta' : \alpha_i] < 0$.
- This means β or $\beta' = -\alpha_i$. Let $\beta = -\alpha_i$.
- Then $(\beta || \beta') = 1 \Longrightarrow [\beta + \beta' : \alpha_i] = 0$ (!)

(2)
- Let α be the common cluster component. Apply $\sigma \in \mathcal{D}$ and assume $\alpha = -\alpha_i$.
- If $\sigma(\beta + \beta') = \sigma(\beta) + \sigma(\beta')$, then by part 1 it is impossible.

- Otherwise $\sigma(\beta \uplus \beta') = \sigma(\beta) + \sigma(\beta')$, again by part 1 it is impossible.

(3) is [CFZ, Theorem 1.17]. It is proved case by case for each Dynkin types, with 8 pages of calculations...

(4)
- Let α be cluster component of $\beta + \beta'$. Apply $\sigma \in \mathcal{D}$ with $\sigma(\beta) = -\alpha_i$.
- Suffices to show $(-\alpha_i || \sigma(\alpha)) = 0$.
- $\sigma(\beta) = -\alpha_i$ and $1 = (\beta || \beta') = (\sigma(\beta) || \sigma(\beta')) = (-\alpha_i || \sigma(\beta')) = [\sigma(\beta') : \alpha_i]$ $\Longrightarrow [\sigma(\beta) + \sigma(\beta') : \alpha_i] = 0$
- By (3) $\Longrightarrow [\sigma(\beta) \uplus \sigma(\beta') : \alpha_i] \leq 0$
- In either case, $[\sigma(\beta + \beta') : \alpha_i] \leq 0$. $\sigma(\alpha)$ is cluster component of $\sigma(\beta + \beta')$.
- Hence $[\sigma(\alpha) : \alpha_i] \leq 0$, hence $(-\alpha_i || \sigma(\alpha)) = 0$
- $\sigma(\beta \uplus \beta') = \sigma(\beta) + \sigma(\beta')$ for some $\sigma \in \mathcal{D}$.
- α is cluster component of $\beta \uplus \beta' \Longrightarrow \sigma(\alpha)$ is cluster component of $\sigma(\beta) + \sigma(\beta')$
- Hence $\sigma(\alpha)$ is compatible with $\sigma(\beta)$ and $\sigma(\beta')$, hence α compatible with β and β'.

(5) (if) Similar argument to (4).

(6) follows from (3)

□

Proposition 7.25. *If β, β' are exchangeable, then there exists two adjacent clusters C and $C' = C - \{\beta\} \cup \{\beta'\}$.*

Proof. Follows from Lemma 7.5 (4), (5).

- The set consisting of β and all cluster components of $\beta + \beta'$ and $\beta \uplus \beta'$ is compatible. Hence there exists a cluster C containing this set.
- Every element of $C - \{\beta\}$ is compatible with β' hence $C - \{\beta\} \cup \{\beta'\}$ is a cluster.

□

7.3 Exchange matrix $B(C)$

Finally we define the exchange matrix $B(C)$. Let us start with the initial seed where $A = A(B_0)$ with B_0 giving an alternate orientation. Then the mutations $\mu_\pm = \prod_{i \in I_\pm} \mu_i$ gives $\mu_\pm(B_0) = -B_0$. From this we can determine the signs of $B(C)$. To summarize:

Lemma 7.26. *There exists unique sign function $\epsilon(\beta, \beta')$ on pair of exchangeable roots*

$$\begin{aligned} &\epsilon(-\alpha_j, \beta') = -\epsilon(j) \\ &\epsilon(\tau\beta, \tau\beta') = -\epsilon(\beta, \beta'), \qquad \beta, \beta' \notin \{-\alpha_j : \tau(-\alpha_j) = -\alpha_j\} \end{aligned}$$

It is skew-symmetric

$$\epsilon(\beta', \beta) = -\epsilon(\beta, \beta')$$

Pictorially it is defined by

$$\begin{aligned} -\alpha_i \cdots\cdots\cdots \xrightarrow{\tau_{-\epsilon}} \beta \underbrace{\xrightarrow{\tau_\epsilon} \cdots \xrightarrow{\tau_\pm}}_{k_\epsilon(\beta)} -\alpha_j \\ -\alpha_i' \cdots \xrightarrow{\tau_{-\epsilon}} \beta' \underbrace{\xrightarrow{\tau_\epsilon} \cdots\cdots\cdots \xrightarrow{\tau_\pm}}_{k_\epsilon(\beta')} -\alpha_j' \end{aligned}$$

Then $k_\epsilon(\beta) < k_\epsilon(\beta') \implies \epsilon(\beta, \beta') := \epsilon$.

Since τ_+, τ_- covers all the roots of $\Phi_{\geq -1}$, it also gives a combinatorial description of $B(C)$ for any C. The explicit expression for $B(C)$ is given as follows:

Definition 7.27. *Let $C' = C - \{\beta\} \cup \{\beta'\}$ be an adjancent cluster of C by exchanging β. Define the matrix $B(C)$ for each cluster of $\Delta(\Phi)$ as*

$$\begin{aligned} b_{\alpha\beta}(C) &= \epsilon(\beta, \beta') \cdot [(\beta + \beta') - (\beta \uplus \beta') : \alpha]_C \\ &= \epsilon(\beta, \beta') \cdot ([\beta + \beta' : \alpha]_{clus} - [\beta \uplus \beta' : \alpha]_{clus}) \end{aligned}$$

Hence the exchange relation is of the form

$$x_\beta x_{\beta'} = p(C) x_{\beta+\beta'} + p'(C) x_{\beta \uplus \beta'}$$

for some coefficients $p(C), p'(C) \in \mathbb{P}$

Let $C_0 = \{-\alpha_1, ..., -\alpha_n\}$ be the initial seed.

Lemma 7.28. *The exchange matrix defined above satisfies*

$$b_{-\alpha_i, -\alpha_j}(C_0) = \begin{cases} 0 & i = j \\ \epsilon(j) a_{ij} & i \neq j \end{cases}$$

$$b_{\tau\alpha, \tau\beta}(\tau C) = -b_{\alpha\beta}(C)$$

In particular, $A(B(C_0))$ is a Cartan matrix.

Proof. Let us prove the first statement. By definition,

$$\begin{aligned} \epsilon(-\alpha_j, \beta) &= -\epsilon(j) \\ -\alpha_j \uplus \beta &= -\alpha_j + \beta + \sum_{k \neq j} a_{kj} \alpha_k \end{aligned}$$

Hence

$$\begin{aligned} b_{-\alpha_i,-\alpha_j}(C_0) &= \epsilon(-\alpha_j,\alpha_j)\cdot[(-\alpha_j+\beta)-(-\alpha_j \uplus \beta) : -\alpha_i]_{C_0} \\ &= -\epsilon(j)\cdot[-\sum_{k\neq j} a_{kj}\alpha_k : -\alpha_i]_{C_0} \\ &= \begin{cases} 0 & i=j \\ -\epsilon(j)a_{ij} & i\neq j \end{cases} \end{aligned}$$

□

Theorem 7.29. *$B(C)$ gives a seed attachment for (the dual complex of) $\Delta(\Phi)$. i.e.*

(1) $B(C)$ is sign-skew-symmetric:

$$b_{\alpha\beta}b_{\beta\alpha} < 0 \text{ or } b_{\alpha\beta} = b_{\beta\alpha} = 0$$

(2) If $C' = C - \{\gamma\} \cup \{\gamma'\}$ is an adjancent cluster, then $B(C')$ is obtained from $B(C)$ by matrix mutation

$$b_{\alpha\beta}(C') = \mu_\gamma(b_{\alpha\beta}(C))$$

(3) The dual graph of $\Delta(\Phi)$ has 2-dimensional face given by 4,5,6,8-gon, with the corresponding $B(C)$ matrix having type $0, 1, 2, 3$.

Hence $\mathcal{A}$ is of finite type by Proposition **??**.

Proof. Mostly using the $\tau \in \mathcal{D}$ invariance of b and ϵ to reduce to checking the case for $\alpha = -\alpha_i$.

(3) is proved by induction on the rank of the root system:

- Rank 2 is known (4,5,6,8-gon correspond to type $A_1 \times A_1, A_2, B_2, G_2$ respectively). Assume $n \geq 3$.
- If L is a loop, all the vertices share $n-2$ common elements.
- Use τ to bring one of them to $-\alpha_i$. The type does not change by τ-invariance of b.
- Since remaining elements are compatible with $-\alpha_i$, they do not have α_i components. Hence one can remove α_i and consider a lower rank root system with the same loop L.

□

7.4 Denominator Theorem

We have established a surjection $\alpha \mapsto x[\alpha]$ from vertex of $\Phi_{\geq -1}$ to the cluster variables by the previous Theorem. The denominator Theorem tells us that in fact this is a bijection, where each $x[\alpha]$ has different denominators. Here $-\alpha_i$ correspond to x_i of the initial seed $\mathbf{x}_0$. We now proceed to prove the denominator theorem.

Proof of Denominator Theorem. We will prove that

$$x[\alpha] := \frac{P_\alpha(\mathbf{x}_0)}{\mathbf{x}_0^\alpha}$$

By induction on

$$k(\alpha) = min(k_+(\alpha), k_-(\alpha)) \geq 0$$

- If $k(\alpha) = 0$, α is negative root.
- Assume $k(\alpha) = k \geq 1$ and the theorem holds for all roots α' with $k(\alpha') < k$.
- We have

$$\alpha = \tau_{\epsilon(j)}^{(k)}(-\alpha_j) = \tau_{-\epsilon(j)}^{(k-1)}(\alpha_j)$$

  for some $j \in I$. Since α_j and $-\alpha_j$ are exchangeable, so are $\alpha, \tau(-\alpha_j)$ where $\tau := \tau_{-\epsilon(j)}^{(k-1)}$.
- Then we have the exchange relation

$$x[\alpha]x[\tau(-\alpha_j)] = q\prod_{i\neq j} x[\tau(-\alpha_j)]^{-a_{ij}} + r$$

  for some $q, r \in \mathbb{P}$.
- For $k = 1$ we have $\alpha = \alpha_j$ and

$$x[\alpha_j] = \frac{q\prod_{i\neq j} x_i^{-a_{ij}} + r}{x_j}$$

- For $k \geq 2$, all roots appearing above has $k(\alpha') < k$, hence by induction we have

$$x[\alpha] = x^{\tau(-\alpha_j)-\gamma} \cdot \frac{q\prod_{i\neq j} P_{\tau(-\alpha_i)}^{-a_{ij}} + rx^\gamma}{P_{\tau(-\alpha_j)}}$$

  where $P_{\alpha'}$ are polynomials in $\mathbf{x}_0$, and we denote

$$\gamma = \sum_{i\neq j}(-a_{ij}) \cdot \tau(-\alpha_i)$$

- Since $x[\alpha]$ is a Laurent polynomial, the fraction is actually a polynomial. We have

$$\gamma = \tau(\sum_{i\neq j} a_{ij}\alpha_i) = \tau(\alpha_j \uplus (-\alpha_j)) = \tau(\alpha_j) + \tau(-\alpha_j) = \alpha + \tau(-\alpha_j)$$

□

References

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