Lecture Notes
Introduction to Cluster Algebra

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8.6 Proof of Theorem 8.4

Theorem . *Assume $\mathcal{S}$ and $\mathcal{S}'$ are related by seed mutation and are both coprime. Then*

$$\mathcal{U}(\mathcal{S}) = \mathcal{U}(\mathcal{S}')$$

Proof. It follows from several Lemma. The main technique is to restrict attention to just x_1 and x_2, treating other variables as "coefficients".

Lemma 8.30. *For arbitrary seed:*

$$\mathcal{U}(\mathcal{S}) = \bigcap_{j=1}^{n} \mathbb{ZP}[x_1^{\pm}, ..., x_{j-1}^{\pm}, x_j, x_j', x_{j+1}^{\pm}, ..., x_n^{\pm}] \tag{8.1}$$

$\Longleftrightarrow$ *Enought to show* $\mathbb{ZP}[\mathbf{x}^{\pm}] \cap \mathbb{ZP}[\mathbf{x}_1^{\pm}] = \mathbb{ZP}[x_1, x_1', x_2^{\pm}, ..., x_n^{\pm}]$.

Proof.
- $\supset$ obvious.
- For $y \in \mathbb{ZP}[\mathbf{x}^{\pm}]$, $y = \sum_{m=-N}^{N} c_m x_1^m$, $c_m \in \mathbb{ZP}[x_2^{\pm}, ..., x_n^{\pm}]$.
- $y = \sum_{m=0}^{N} c_m P_1^m {x_1'}^{-m} + \sum_{m=1}^{N} \frac{c_{-m}}{P_1^m} {x_1'}^m$
- If $y \in Z\mathbb{P}[\mathbf{x}_1^{\pm}]$ then $\frac{c_{-m}}{P_1^m} \in \mathbb{ZP}[x_2^{\pm}, ..., x_n^{\pm}]$.

□

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Lemma 8.31. *If P_1 is coprime with P_j, $j = 2, ..., n$:*

$$\mathcal{U}(\mathcal{S}) = \bigcap_{j=2}^{n} \mathbb{ZP}[x_1, x_1', x_2^{\pm}, ..., x_{j-1}^{\pm}, x_j, x_j', x_{j+1}^{\pm}, ..., x_n^{\pm}] \tag{8.2}$$

$\Longleftrightarrow$ *By Lemma 8.30, enough to show* $\mathbb{ZP}[x_1, x_1', x_2^{\pm}] \cap \mathbb{ZP}[x_1^{\pm}, x_2, x_2'] = \mathbb{ZP}[x_1, x_1', x_2, x_2']$

Proof. If $b_{12} = b_{21} = 0$, then $x_1 x_1' = P_1, x_2 x_2' = P_2$:

$$y = \sum c_{m_1,m_2} x_1^{m_1} x_2^{m_2} \in \mathbb{ZP}[x_1, x_1', x_2, x_2']$$

- c_{m_1,m_2} is divisible by $P_1^{-m_1}$ for $m_1 < 0 \leq m_2$,
- c_{m_1,m_2} is divisible by $P_2^{-m_2}$ for $m_2 < 0 \leq m_1$,
- c_{m_1,m_2} is divisible by $P_1^{-m_1} P_2^{-m_2}$ for $m_1, m_2 < 0$
- From Lemma 8.30, this is the same description of $\mathbb{ZP}[x_1, x_1', x_2^{\pm}] \cap \mathbb{ZP}[x_1^{\pm}, x_2, x_2']$ if P_1 and P_2 are coprime.

If $b_{12} \neq 0$, then exchange relation is

$$x_1 x_1' = P_1 = q_2 x_2^c + r_2,$$
$$x_2 x_2' = P_2 = q_1 x_1^b + r_1$$

Show that

$$\mathbb{ZP}[x_1, x_2^{\pm}] \cap \mathbb{ZP}[x_1^{\pm}, x_2, x_2'] = \mathbb{ZP}[x_1, x_2, x_2'] \tag{8.3}$$

- $\supset$ obvious.
- For $y \in \mathbb{ZP}[x_1^{\pm}, x_2, x_2']$, write

  $$y = \sum_{m \in \mathbb{Z}} x_1^m (c_m + c_m'(x_2) + c_m''(x_2'))$$

  where c_m', c_m'' are $\mathbb{ZP}$-polynomials without constant term.
- By substituting x_2', show that if $y \in \mathbb{ZP}[x_1, x_2^{\pm}]$ also, then smallest degree of x_1 must be ≥ 0

Show that

$$\mathbb{ZP}[x_1, x_1', x_2^{\pm}] = \mathbb{ZP}[x_1, x_1', x_2, x_2'] + \mathbb{ZP}[x_1, x_2^{\pm}] \tag{8.4}$$

$\supset$ obvious.
$\subset$ Enough to show that ${x_1'}^N x_2^{-M} \in \mathbb{ZP}[x_1, x_1', x_2, x_2'] + \mathbb{ZP}[x_1, x_2^{\pm}]$ for $N, M > 0$

- Let $p = -\frac{q_1}{r_1}$. Rewrite $x_2^{-1} = px_1^b x_2^{-1} + r_1^{-1}x_2' \equiv px_1^b x_2^{-1} \mod \mathbb{ZP}[x_1, x_2']$
- Conclude $x_2^{-1} = p^N {x_1}^{Nb} x_2^{-1} \mod \mathbb{ZP}[x_1, x_2']$
- Conclude $x_2^{-M} \in \mathbb{ZP}[x_1, x_2'] + x_1^N \mathbb{ZP}[x_1, x_2^{-1}]$

Obvious:

$$\mathbb{ZP}[x_1, x_1', x_2, x_2'] \subset \mathbb{ZP}[x_1^{\pm}, x_2, x_2'] \tag{8.5}$$

(8.3), (8.4), (8.5) implies

$$\begin{aligned}
&\mathbb{ZP}[x_1, x_1', x_2^{\pm}] \cap \mathbb{ZP}[x_1^{\pm}, x_2, x_2'] \\
&= (\mathbb{ZP}[x_1, x_1', x_2, x_2'] + \mathbb{ZP}[x_1, x_2^{\pm}]) \cap \mathbb{ZP}[x_1^{\pm}, x_2, x_2'] \\
&= \mathbb{ZP}[x_1, x_1', x_2, x_2'] + (\mathbb{ZP}[x_1, x_2^{\pm}] \cap \mathbb{ZP}[x_1^{\pm}, x_2, x_2']) \\
&= \mathbb{ZP}[x_1, x_1', x_2, x_2'] + \mathbb{ZP}[x_1, x_2, x_2'] \\
&= \mathbb{ZP}[x_1, x_1', x_2, x_2']
\end{aligned}$$

□

Lemma 8.32. *Let $\mathcal{S}' = \mu_1(\mathcal{S})$. Let $x_2'' = \mu_2(x_2(\mathcal{S}'))$. Then*

$$\mathbb{ZP}[x_1, x_1', x_2, x_2', x_3^{\pm}, ..., x_n^{\pm}] = \mathbb{ZP}[x_1, x_1', x_2, x_2'', x_3^{\pm}, ..., x_n^{\pm}] \tag{8.6}$$

$\iff$ *Enough to show* $\mathbb{ZP}[x_1, x_1', x_2, x_2'] = \mathbb{ZP}[x_1, x_1', x_2, x_2'']$
$\iff$ *Enough to show* $x_2'' \in \mathbb{ZP}[x_1, x_1', x_2, x_2']$ *by symmetry.*

- *if* $b_{12} = 0$, $x_2'' = px_2'$ *for some* $p \in \mathbb{P}$. *Obvious.*
- *Otherwise* $x_2 x_2'' = q_3 {x_1'}^b + r_3$. *Mutation rule implies* $r_1 r_3 = q_1 q_3 r_2^b$, *and calculate directly*

$$x_2'' = q_3 r_1^{-1} (x_1')^b x_2' - q_1 q_3 r_1^{-1} \frac{(q_2 x_2^c + r_2)^b - r_2^b}{x_2} \in \mathbb{ZP}[x_1, x_1', x_2, x_2']$$

Combine all 3 Lemmas to give the proof of Theorem 8.4. □

References

[ClusterIII] A. Berenstein, S. Fomin, A. Zelevinsky, *Cluster Algebras III: Upper bounds and double Bruhat cells*, Duke Mathematical Journal, **126** (1), (2005): 1-51