Lecture Notes
Introduction to Cluster Algebra

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8 Upper Bounds and Lower Bounds

In this section, we study the properties of upper bound and lower bound of cluster algebra. We will then identify the coordinate rings of double Bruhat cells with the upper cluster algebra in the next section.

We will use the previous notion of cluster algebra with coefficients in $\mathbb{P}$. $\mathbb{P}$ is a semifield. The ambient field $\mathcal{F} = \mathbb{QP}(u_1, ..., u_n)$, a seed is a triple $\mathcal{S} = (\mathbf{x}, \mathbf{y}, B)$ such that $\mathbf{x} = \{x_1, ..., x_n\}$ are the cluster variables, $\mathbf{y} = (y_1, ..., y_n)$ are elements of $\mathbb{P}$, and $B = (b_{ij})$ is skew-symmetrizable $n \times n$ integer matrix.

Remark 8.1. *(1) In the original paper, the exchange matrix B is allowed to be only sign-skew-symmetric (i.e. $b_{ij} > 0 \Longleftrightarrow b_{ji} < 0$ and $b_{ij} = 0 \Longleftrightarrow b_{ji} = 0$). However for simplicity, we will assume B is skew-symmetrizable below.*

(2) In the original paper, it allows more general coefficients $p_1^{\pm}, ..., p_n^{\pm} \in \mathbb{P}$ such that $y_i = \frac{p_i^+}{p_i^-}$, and forgets the semifield operation $\oplus$.

8.1 Upper bound

Fix an initial seed $\mathcal{S} = (\mathbf{x}, \mathbf{y}, B)$. Let $\mathbf{x}_j$ be the adjancent cluster to $\mathbf{x}$ defined by

$$\mathbf{x}_j = \mathbf{x} - \{x_j\} \cup \{x'_j\}$$

Recall that the exchange relation is of the form

$$x_j x'_j = P_j(\mathbf{x})$$

for some polynomial P_j which does not depend on x_j. Let

$$\mathbb{ZP}[\mathbf{x}] = \mathbb{ZP}[x_1, ..., x_n]$$

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$$\mathbb{ZP}[\mathbf{x}^{\pm}] = \mathbb{ZP}[x_1^{\pm}, ..., x_n^{\pm}]$$

be the ring of polynomials and Laurent polynomials in the cluster variables with coefficients in $\mathbb{ZP}$.

Definition 8.2. *The* upper bound *associated with the seed* $\mathcal{S}$ *is*

$$\mathcal{U}(\mathcal{S}) = \mathbb{ZP}[\mathbf{x}^{\pm}] \cap \mathbb{ZP}[\mathbf{x}_1^{\pm}] \cap \cdots \cap \mathbb{ZP}[\mathbf{x}_n^{\pm}]$$

Definition 8.3. *A seed* $\mathcal{S}$ *is called* coprime *if* $P_1, ..., P_n$ *is pairwise coprime in* $\mathbb{ZP}[\mathbf{x}]$*. i.e., any common divisor of* P_i *and* P_j *belongs to* $\mathbb{P}$*.*

Theorem 8.4. *Assume* $\mathcal{S}$ *and* $\mathcal{S}'$ *are related by seed mutation and are both coprime. Then*

$$\mathcal{U}(\mathcal{S}) = \mathcal{U}(\mathcal{S}')$$

Proof. See the supplementary notes. □

Definition 8.5. *Fix an initial seed* $\mathcal{S}_0$*. The upper cluster algbera* $\overline{\mathcal{A}} = \overline{\mathcal{A}}(\mathcal{S}_0) \subset \mathcal{F}$ *is defined by*

$$\overline{\mathcal{A}}(\mathcal{S}_0) := \bigcap_{\mathcal{S} \sim \mathcal{S}_0} \mathcal{U}(\mathcal{S})$$

Corollary 8.6. *If all seeds mutation equivalent to* $\mathcal{S}_0$ *are coprime, then* $\mathcal{U}(\mathcal{S})$ *is independent of choice of* $\mathcal{S}$*, and*

$$\mathcal{U}(\mathcal{S}) = \mathcal{U}(\mathcal{S}_0) = \overline{\mathcal{A}}(\mathcal{S}_0)$$

Proposition 8.7. *For cluster algebra of geometric type, if the* $m \times n$ *matrix* $\widetilde{B}$ *of a seed* $\mathcal{S}$ *has full rank, then all seeds mutation equivalent to* $\mathcal{S}$ *are coprime.*

Proof. It follows from two Lemma:

Lemma 8.8. *A seed of geometric type is coprime iff no two columns of* $\widetilde{B}$ *is proportional to each other with coefficient being a ratio of two odd integers.*

Proof. Let $\widetilde{B}_j$ be the j-th column of $\widetilde{B}$.

- If $\widetilde{B}_k = \pm\frac{b}{a}\widetilde{B}_j$ for some odd and coprime integers a, b, then $P_j = L^a + M^a$, $P_k = L^b + M^b$ and $L + M$ is a common factor.
- Use Newton polytope. Assume P_j and P_k has a common factor. Then $N(P_j)$ and $N(P_k)$ are parallel, and we must have the form $P_j = L^a + M^a, P_k = L^b + M^b$ for some monomial L, M and some coprime integers a, b. Then it follows from that fact that $t^a + 1$ and $t^b + 1$ have a common factor iff a, b are odd.

□

Lemma 8.9. *Matrix mutations preserve the rank of* $\widetilde{B}$

Proof. Homework □

□

Hence Theorem 8.4 and Corollary 8.6 are satisfied for cluster algbra of geometric type if $\widetilde{B}$ has full rank. An important class of examples is given by the coordinate rings of double Bruhat cells $G^{u,v}$ in a complex semisimple Lie group G.

8.2 Lower bound

Definition 8.10. *Lower bound* $\mathcal{L}(\mathcal{S})$ *is given by*

$$\mathcal{L}(\mathcal{S}) = \mathbb{ZP}[x_1, x_1', ..., x_n, x_n'] = \mathbb{ZP}[\mathbf{x}, \mathbf{x}_1, ..., \mathbf{x}_n]$$

The cluster algebra $\mathcal{A} = \mathcal{A}(\mathcal{S}_0)$ *is the union of all lower bounds*

$$\mathcal{A}(\mathcal{S}_0) = \bigcup_{\mathcal{S}\sim\mathcal{S}_0} \mathcal{L}(\mathcal{S})$$

Corollary 8.11. *We have for any seed* $\mathcal{S} \sim \mathcal{S}_0$*:*

$$\mathcal{L}(\mathcal{S}) \subset \mathcal{A}(\mathcal{S}_0) \subset \overline{\mathcal{A}}(\mathcal{S}_0) \subset \mathcal{U}(\mathcal{S})$$

Note that the middle inclusion $\mathcal{A}(\mathcal{S}_0) \subset \overline{\mathcal{A}}(\mathcal{S}_0)$ is just the Laurent Phenomenon!

8.3 Acyclic case: Properties of $\mathcal{L}(\mathcal{S}) \subset \mathcal{U}(\mathcal{S})$

We don't need acyclicity for double Bruhat cell. So just skim through the easy proofs!

Definition 8.12. *Let* $\Gamma(\mathcal{S})$ *be the quiver (or diagram) corresponding to the seed* $\mathcal{S}$*. A seed* $\mathcal{S}$ *is called acyclic if* $\Gamma(\mathcal{S})$ *has no oriented cycles.*

Fact 8.13. *For an acyclic seed, one can reindex such that* $b_{ij} \geq 0$ *for all* $i > j$*. (i.e. "flowing" through the quiver)*

Definition 8.14. *A monomial in* $x_1, x_1', ..., x_n, x_n'$ *is called standard if it contains no product of the form* $x_j x_j'$*. They span* $\mathcal{L}(\mathcal{S})$ *as* $\mathbb{ZP}$*-module.*

Theorem 8.15. *The standard monomials in* $x_1, x_1', ..., x_n, x_n'$ *are linear independent over* $\mathbb{ZP}$ *iff* $\mathcal{S}$ *is acyclic. In particular,* $x_j x_j' - P_j(\mathbf{x})$ *generates all the relations among* $x_1, x_1', ..., x_n, x_n'$*.*

Proof. If:

- Let $\mathbf{m} \in \mathbb{Z}^n$, $\mathbf{x}^{\langle \mathbf{m} \rangle} := x_1^{\langle m_1 \rangle} ... x_n^{\langle m_n \rangle}$ where

$$x_j^{\langle m_j \rangle} := \begin{cases} x_j^{m_j} & m_j \geq 0 \\ (x_j')^{-m_j} & m_j < 0 \end{cases}$$

  It is Laurent polynomial in $x_1, ..., x_n$.

- Order $\mathbf{m}$ lexicographically. $((4,3,2) > (4,3,1) > (3,3,1)$ etc.)
- Since $b_{ij} \geq 0$ for $i > j$, the "lexicographically smallest" Laurent monomial that appears in $x_j^{\langle m_j \rangle}$ equals to $x_j^{m_j}$ times some monomial in $x_{j+1}, ..., x_n$.
- Hence if $\mathbf{m} < \mathbf{m}'$, then first monomial in $\mathbf{x}^{\langle \mathbf{m} \rangle}$ precedes the one in $\mathbf{x}^{\langle \mathbf{m}' \rangle}$, i.e. they have different "leading term"
- $\Longrightarrow$ Linear independence.

Only if: [Technical, see [ClusterIII]]. If $1 \longrightarrow 2 \longrightarrow ... \longrightarrow \ell \longrightarrow 1$ is oriented cycle, then

$$x_1' ... x_\ell' = \sum_{K \subsetneq \{1,...,\ell\}} f_K(x_1, ..., x_n) \prod_{k \in K} x_k'$$

for some polynomials $f_K \in \mathbb{ZP}[\mathbf{x}]$. By reducing $x_i x_i'$ on right hand side, we get linear dependence of standard monomials. □

Theorem 8.16. *If $\mathcal{S}$ is coprime and acyclic, then*

$$\mathcal{L}(\mathcal{S}) = \mathcal{U}(\mathcal{S})$$

Proof. See [ClusterIII, Section 6] for a quite technical 7-page proof. □

Corollary 8.17. *If a cluster algebra possesses a coprime and acyclic seed, then it coincides with the upper cluster algebra.*

Theorem 8.18. *$\mathcal{A}(\mathcal{S}) = \mathcal{L}(\mathcal{S})$ iff $\mathcal{S}$ is acyclic. In particular this applies to cluster algberas of finite type. If $\mathcal{S}$ is coprime, then we have $\mathcal{A} = \overline{\mathcal{A}}$*

Hence to summarize, we have

coprime	*acyclic*	
$\times$	$\times$	$\mathcal{L}(\mathcal{S}) \subset \mathcal{A} \quad \subset \quad \overline{\mathcal{A}} \subset \mathcal{U}(\mathcal{S})$
$\checkmark \forall \mathcal{S}$	$\times$	$\overline{\mathcal{A}} = \mathcal{U}(\mathcal{S})$
$\times$	$\checkmark$	$\mathcal{L}(\mathcal{S}) = \mathcal{A}$
$\checkmark$	$\checkmark$	$\mathcal{L}(\mathcal{S}) = \mathcal{A} \quad = \quad \overline{\mathcal{A}} = \mathcal{U}(\mathcal{S})$

8.4 Examples

Example 8.19. *All rank $n \le 2$ cluster algebras is finitely generated, because all seeds are acyclic, hence $\mathcal{A} = \mathcal{L}(\mathcal{S})$ is finitely generated by x_1, x_1', x_2, x_2'.*

Example 8.20. *A skew-symmetrizable cluster algebra of rank 3 is finitely generated iff it has an acyclic seed.*

Proof. If $\mathcal{A}$ has no acyclic seed, one construct a tropical valuation such that for fixed $t_0 \in \mathbf{T}_3$, for every $r \in \mathbb{Z}_{>0}$:

- $\nu_i(t) \ge 0$ for all i and all $t \in \mathbf{T}_3$ with $d(t_0, t) \le r$
- $\nu_i(t) < 0$ for some i and some $t \in \mathbf{T}_3$ with $d(t_0, t) = r + 1$

where $\nu_i(t) := \nu(x_i(t))$ and $d(t, t')$ is the distance between t and t' in $\mathbf{T}_3$. □

Example 8.21. *For $\mathcal{S}$ corresponding to the Markov quiver, $\mathcal{A}(\mathcal{S}) \ne \overline{\mathcal{A}}(\mathcal{S})$.*

Proof. The exchange relation is of the form:

$$x_k x_k' = p_k^- x_i^2 + p_k^+ x_j^2, \qquad (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$$

For any valuation, we have

$$\nu(x_k) + \nu(x_k') = \min(2\nu(x_i), 2\nu(x_j))$$

Hence one can construct a tropical valuation such that $\nu(x) = 1$ for any cluster variable x. In particular $\mathcal{A}$ becomes a graded algebra under ν with zero-degree component $\mathbb{ZP}$. Then one construct a nonconstant element $y \in \mathcal{U}(\mathcal{S})$ with $\nu(y) = 0$ by

$$\begin{aligned} y := & \frac{p_1^+ p_2^+ x_1^2 + p_1^- p_2^- x_2^2 + p_1^+ p_2^- x_3^2}{x_1 x_2} \in \mathbb{ZP}[\mathbf{x}^{\pm}] \cap \mathbb{ZP}[\mathbf{x}_3^{\pm}] \\ = & \frac{p_1^+ p_2^+ x_1 + p_2^- x_1'}{x_2} \in \mathbb{ZP}[\mathbf{x}_1^{\pm}] \\ = & \frac{p_1^- p_2^- x_2 + p_1^+ x_2'}{x_1} \in \mathbb{ZP}[\mathbf{x}_2^{\pm}] \end{aligned}$$

Hence $y \in \mathcal{U}(\mathcal{S})$ with $\nu(y) = 0$ and $y \notin \mathcal{A}$. □

Example 8.22 (D. Speyer)**.** *For $\mathcal{S}$ corresponding to*

$$B = \begin{pmatrix} 0 & 3 & -3 \\ -3 & 0 & 3 \\ 3 & -3 & 0 \end{pmatrix}$$

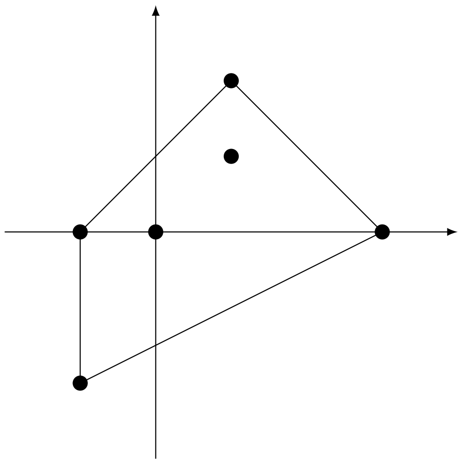
the upper cluster algebra is not finitely generated.

8.5 Newton Polytope and Tropical Valuation

Most part of the proof uses the notion of tropical valuation, which is some kind of "degree" on $\overline{\mathcal{A}}$. In particular, if one can find a valuation ν such that $\nu(x) \geq 0$ for all $x \in S \subset \overline{\mathcal{A}}$ but $\nu(y) < 0$ for some $y \in \overline{\mathcal{A}}$, then y is not generated by S (polynomially).

Definition 8.23. *Let $y = y(x_1, ..., x_n)$ be a Laurent polynomial. The* Newton polytope *$N(y)$ of y is the convex hull in $\mathbb{R}^n$ of all lattice points $\boldsymbol{m} = (m_1, ..., m_n)$ such that the coefficients $\mathbf{x}^{\boldsymbol{m}} = x_1^{m_1}...x_n^{m_n}$ is nonzero in y.*

Example 8.24. *Let $y = x_1^3 + 1 + x_1x_2^2 + x_1x_2 + \frac{1}{x_1} + \frac{1}{x_1x_2^2}$. Then $N(y)$ looks like:*



We have some easy facts about Newton polytope:

Proposition 8.25. *Let P, Q be two Laurent polynomials. Then*

(1) $N(PQ) =$ Minkowski sum $N(P)+N(Q) := \{x+y \in \mathbb{R}^n | x \in N(P), y \in N(Q)\}$

(2) $N(P+Q) \subset$ Convex Hull of $N(P) \cup N(Q)$. Equality holds if $P, Q \in Q_{sf}(x_1, ..., x_n)$ are subtraction-free rational functions.

Example 8.26. *Let $P = L^{\boldsymbol{m}} + M^{\boldsymbol{m}'}$ be a sum of two Laurent monomials. Then $N(P)$ is a straightline. If P' is a factor of P, then $N(P')$ is also a straightline parallel to $N(P)$.*

Definition 8.27. *Let $\overline{\mathcal{A}}_{sf} \subset \overline{\mathcal{A}} - \{0\}$ be the set of nonzero elements of $\overline{\mathcal{A}}$ that can be written as subtraction-free rational expressions (i.e. $\frac{P(\mathbf{x})}{Q(\mathbf{x})}$ where P and Q has coefficients in $\mathbb{Z}_{\geq 0}\mathbb{P}$).*

Definition 8.28. *A* tropical vaulation *on $\overline{\mathcal{A}}$ is a map*

$$\nu : \overline{\mathcal{A}} - \{0\} \longrightarrow \mathbb{R}$$

satisfying:

$$\nu(p) = 0 \qquad p \in \mathbb{ZP} \tag{8.1}$$

$$\nu(xy) = \nu(x) + \nu(y) \qquad x, y \in \overline{\mathcal{A}} - \{0\} \tag{8.2}$$

$$\nu(x+y) \geq \min(\nu(x), \nu(y)) \qquad x, y, x+y \in \overline{\mathcal{A}} - \{0\} \tag{8.3}$$

$$\nu(x+y) = \min(\nu(x), \nu(y)) \qquad x, y \in \overline{\mathcal{A}}_{sf} \tag{8.4}$$

Lemma 8.29. *For any cluster* $\mathbf{x} = \{x_1, ..., x_n\}$ *of* $\mathcal{A}$ *and any* $(\nu_1, ..., \nu_n) \in \mathbb{R}^n$*, there is a tropical valuation* ν *on* $\overline{\mathcal{A}}$ *such that* $\nu(x_i) = \nu_i$ *for* $i = 1, ..., n$*.*

Proof. Let $y \in \overline{\mathcal{A}} - \{0\}$. Then the tropical valuation is defined by

$$\nu(y) = \min_{\mathbf{m}=(m_1,...,m_n)\in N(y)} (m_1\nu_1 + ... + m_n\nu_n)$$

When $n = 1$, ν looks like the "lowest degree" of a Laurent polynomial. In general, it is the "lowest degree in the direction of $(\nu_1, ..., \nu_n)$". □

Proof of Theorem 8.18. Theorem 8.16 shows that if $\mathcal{S}$ is acyclic and coprime, then $\mathcal{L}(\mathcal{S}) = \mathcal{A}(\mathcal{S})$. The coprimality condition can be lifted by using "universal coefficients", i.e. considering generic coefficients $\mathbb{P}$, i.e. extending $\mathbb{ZP}$ to a bigger coefficient group $\mathbb{A}$, conclude that $\mathcal{L}(\mathcal{S}) = \mathcal{A}(\mathcal{S})$ in $\mathbb{A}$, and restrict the formula back to the case $\mathbb{ZP}$.

Only if: Show that if a seed $\mathcal{S}$ is not acyclic, then $\mathcal{L}(\mathcal{S}) \neq \mathcal{A}$. More precisely, if a seed contains

$$1 \longrightarrow 2 \longrightarrow ... \longrightarrow \ell \longrightarrow 1$$

let $\mathcal{S}^{(0)}, \mathcal{S}^{(1)}, ..., \mathcal{S}^{(\ell-1)}$ be the sequence of seeds such that $\mathcal{S}^{(0)} = \mathcal{S}$ and $\mathcal{S}^{(k)}$ is mutation of $\mathcal{S}^{(k-1)}$ in the direction of k. Then the cluster variable y such that $\{y\} = \mathbf{x}^{\ell-1} - \mathbf{x}^{\ell-2}$ does not belong to $\mathcal{L}(\mathcal{S})$.

- Construct a valuation ν such that $\nu(x) \geq 0$ for $x \in \mathcal{L}(\mathcal{S})$, but $\nu(y) < 0$.
- Construct by induction on ℓ.
- For $\ell = 3$, set $\nu_1 = \min(|b_{21}|, |b_{31}|), \nu_2 = \nu_3 = 1, \nu_i = 0$ for $i > 3$. Then check directly that $\nu(x_i) \geq 0, \nu(x_i') \geq 0$ but $\nu(y) = -1$ for $y = x_2''$.
- For induction, apply the construction to $\mathcal{S}^{(1)}$, which then consists of the smaller cycle $2 \longrightarrow 3 \longrightarrow ... \longrightarrow \ell \longrightarrow 2$, and check that ν still satisfies the condition we want.

□

References

[ClusterIII] A. Berenstein, S. Fomin, A. Zelevinsky, *Cluster Algebras III: Upper bounds and double Bruhat cells*, Duke Mathematical Journal, **126** (1), (2005): 1-51