9 Double Bruhat Cells

In this section, we describe the identification between double Bruhat cells and certain upper cluster algebra.

9.1 Notation and Definitions

Let \([1, r] := \{1, 2, 3, \ldots, r\}\).

9.1.1 Lie Theory

- \(G\) a simply-connected, connected, semisimple complex algebraic group of rank \(r\)
- \(B, B_-\) the opposite Borel subgroups
- \(N, N_-\) the unipotent radicals
- \(H = B \cap B_-\) a maximal torus
- \(W = \text{Norm}_G(H)/H\) the Weyl group
- \(\mathfrak{g} = \text{Lie}(G)\) the Lie algebra
- \(\mathfrak{h} = \text{Lie}(H)\) the Cartan subalgebra
- A root system \(\Phi\) and decomposition of root space: \(\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha\)
- \(\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset \mathfrak{h}^*\) the simple roots, such that \(\mathfrak{g}_{\alpha_i} \subset \text{Lie}(N)\).
- \(\alpha_i^\vee \in \mathfrak{h}\) simple coroot, such that Cartan matrix is given by \(c_{ij} = \alpha_j(\alpha_i^\vee)\)
• \( \phi_i : SL_2 \rightarrow G \) the embedding corresponding to \( \mathfrak{sl}_2 \simeq \langle g_{\alpha_i}, g_{-\alpha_i} \rangle \hookrightarrow \mathfrak{g} \)

• The root subgroups are defined for \( t \in \mathbb{C} \) by

\[

d_i(t) := \phi_i \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in N, \quad h_i(t) := \phi_i \begin{pmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H, \quad x_{-i}(t) := \phi_i \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix} \in N_-
\]

9.1.2 Weyl group

Weyl group is generated by simple reflections \( s_i \), and they can be represented by \( s_i = \overline{s}_i H \) where

\[
\overline{s}_i = \phi_i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \text{Norm}_G(H)
\]

Can also be written as

\[
\overline{s}_i = d_i(-1)x_{-i}(1)x_i(-1)
\]

• In general for \( w \in W \), we write

\[
\overline{w} \in \text{Norm}_G(H) \subset G
\]

for its representative in \( G \) given by products of \( \overline{s}_i \).

• \( i = (i_1, \ldots, i_l) \) is called a reduced word for \( w \in W \) if \( w = s_{i_1} \ldots s_{i_l} \) is a reduced expression.

In type \( A_r \), \( W \simeq S_{r+1} \), and \( \overline{s}_i \) is the permutation matrix (with appropriate signs).

9.1.3 \( W \times W \)

Now consider \( W \times W \).

• We use \(-1, \ldots, -r\) for the simple reflections on first copy of \( W \), and \( 1, \ldots, r \) for second copy.

• A reduced word for \((u, v) \in W \times W\) is an arbitrary shuffle of reduced word for \( u \) written in \([-1, r]\) and reduced word for \( v \) written in \([1, r]\).

• Let \( \epsilon(i) \) denote the sign of \( i \in \pm[1, r] \).

• Let \( \text{Supp}(u, v) = \{ i : (u, v) \) contains either \( i \) or \(-i\} \subset [1, r] \) be the support.

9.1.4 Bruhat decompositions

**Theorem 9.1.** The group \( G \) has Bruhat decompositions

\[
G = \coprod_{u \in W} BuB = \coprod_{v \in W} B_- v B_-\]

The double Bruhat cells are

\[
G^{u, v} = BuB \cap B_- v B_-
\]

Hence \( G \) is disjoint union of the double Bruhat cells.
Example 9.2. When $G = SL_{r+1}(\mathbb{C})$, $B$ (resp. $B_-$) can be chosen to be the upper (resp. lower) triangular matrix. Then Bruhat decompositions say that any element in $G$ can be reduced to a permutation matrix after certain row and column operations.

There is an explicit description of the Bruhat cells for type $A_r$:

**Proposition 9.3.** For $G = SL_{r+1}(\mathbb{C})$, $x \in G$ belongs to $BwB$ iff

- $\Delta_w([1,i],[1,i])(x) \neq 0$ for $i = 1, \ldots, r$
- $\Delta_w([1,i-1]\cup\{j\},[1,i])(x) = 0$ for $i < j$ and $w(i) < w(j)$

This follows from the explicit description below in Prop 9.19.

Example 9.4. In particular for $G = SL_3(\mathbb{C})$, let $u = v = w_0 = s_1s_2s_1 = s_2s_1s_2$. Then $G^{w_0,w_0} \subset SL_3(\mathbb{C})$ consists of $3 \times 3$ matrices $x$ with $\det(x) = 1$ such that

\[ x_{13} \neq 0, \quad x_{31} \neq 0, \quad \det\begin{pmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{pmatrix} \neq 0, \quad \det\begin{pmatrix} x_{12} & x_{22} \\ x_{31} & x_{32} \end{pmatrix} \neq 0 \]

The following Theorem gives important structural properties of the double Bruhat cells:

**Theorem 9.5.**

- $G^{w,v}$ is biregularly isomorphic to a Zariski open subset of an affine space of dimension $r + l(u) + l(v)$.
- Let $i = (i_1, \ldots, i_N)$ be a reduced word for $(u,v) \in W \times W$. Define the map $x_1 : H \times \mathbb{C}^N \to G$ by

  \[ x_1(a;t_1,\ldots,t_N) = ax_{i_1}(t_1)\cdots x_{i_N}(t_N) \]

  Then the map $x_1$ restricts to a biregular isomorphism between $H \times \mathbb{C}^N_{\neq 0}$ and a Zariski open subset of the double Bruhat cell $G^{w,v}$.

**Proof.** Let us show that $x_1(H \times \mathbb{C}^N_{\neq 0}) \subset B_-vB_-$. Consider the part of the words $(i_{k_1}, \ldots, i_{k_l})$ that form a reduced word for $v$ (i.e. all those with $\epsilon = +$). Note that we have

\[ x_i(t) \in B_- s_i B_-, \quad x_{-i}(t) \in B_- \]

Hence

\[ x_1(a;t_1,\ldots,t_N) \in B_- \cdot B_- s_{ik_1} B_- \cdot B_- \cdots B_- \cdot B_- s_{ik_l} B_- \cdot B_- = B_-vB_- \]

since it is well-known that

\[ B_- w' B_- \cdot B_- w'' B_- = B_- w' w'' B_- \]

whenever $l(w' w'') = l(w') + l(w'')$.

It is also easy to see that this map is injective.
Example 9.6. Consider $G = SL_3(\mathbb{C})$. Consider $u = s_1s_2, v = e$. The image in $G^{s_1s_2,e} = B_{s_1s_2}B \cap B_-$ is

$$x_{(1,2)}(a; t_1, t_2) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t_2 & 1 \end{pmatrix}$$

where $a_1a_2a_3 = 1$. We see that $B_{s_1s_2}B \cap B_-$ can be described by

$$\Delta_{3,1} = 0, \quad \Delta_{2,1} \neq 0, \quad \Delta_{3,2} \neq 0.$$ 

Furthermore, we can recover our parameters from some minors (say, with consecutive columns) by monomial transforms, e.g.

$$a_1 = \Delta_{1,1}, a_2 = \Delta_{12,12} \Delta_{1,1}, t_1 = \frac{\Delta_{1,1} \Delta_{2,1}}{\Delta_{12,12}}, t_2 = \frac{\Delta_{12,12} \Delta_{23,12}}{\Delta_{2,1}}$$

Example 9.7. On the other hand, the image in $G^{s_2s_1,e} = B_{s_2s_1}B \cap B_-$ is

$$x_{(1,2)}(a; t_1, t_2) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t_2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & 0 & 0 \\ a_2 t_1 & a_2 & 0 \\ a_3 t_1 t_2 & a_3 t_1 & a_3 \end{pmatrix}$$

We see that in this case $B_{s_2s_1}B \cap B_-$ can be described by

$$\Delta_{23,12} = 0, \quad \Delta_{3,1} \neq 0, \quad \Delta_{3,2} \neq 0.$$ 

and similarly we can recover our parameters by monomial transforms:

$$a_1 = \Delta_{1,1}, a_2 = \Delta_{12,12} \Delta_{1,1}, t_1 = \frac{\Delta_{12,12} \Delta_{13,12}}{\Delta_{1,1}}, t_2 = \frac{\Delta_{1,1} \Delta_{2,1}}{\Delta_{12,12}}$$

In general, the Bruhat cells can be described by conditions of the form $\Delta(x) = 0, \Delta(x) \neq 0$. (see Proposition 9.19) and the parameters can be recovered by certain collection $F(i)$ of minors (see Definition 9.21). More precisely,

$$H \times \mathbb{C}_m^{\neq 0} \simeq_{monomial} G^{u^{-1},v^{-1}} \simeq_{twisting} G^{u,v}$$

for some twisting $G^{u,v} \rightarrow G^{u^{-1},v^{-1}}$ which is a biregular isomorphism, and the parameters of $x'$ can be expressed as Laurent monomials from the minors in $F(i)$. 

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9.2 Combinatorial data

From a reduced word $i = (i_1, ..., i_{l(u)+l(v)})$, we will construct a rectangular matrix $\tilde{B} = \tilde{B}(i)$ which defines our cluster algebra of geometric type.

Let 

$$M := [-1, r] \cup [1, l(u) + l(v)]$$

and let $m := |M| = r + l(u) + l(v)$.

- Let us add $i_{-r}, ..., i_{-1}$ at the beginning of $i$ by setting $i_{-j} = -j$ for $j \in [1, r]$.
- For $k \in M$, let $k^+$ be smallest index $l$ such that $k < l$ and $|i_l| = |i_k|$. If it does not exist we set $k^+ = l(u) + l(v) + 1$.
- $k$ is called $i$-exchangeable if both $k, k^+ \in [1, l(u), l(v)]$.
- Let $e(i)$ be the set of $i$-exchangeable indices. Let 
  $$n := |e(i)| = l(u) + l(v) - |\text{Supp}(u, v)|$$

- Let $\tilde{B}(i)$ be a $m \times n$ matrix. The rows are labeled by the set $M$ and columns are labeled by the set $e(i)$.

**Definition 9.8.** The quiver $\Gamma(i)$ has vertices set $M$. For vertices $k, l$ with $k < l$, it is connected by an edge iff either $k$ or $l$ (or both) are $i$-exchangeable, and

1. $l = k^+$
2. $l < k^+ < l^+, c_{|i_k|, |i_l|} \neq 0$ and $\epsilon(i_l) = \epsilon(i_{k^+})$
3. $l < l^+ < k^+, c_{|i_k|, |i_l|} \neq 0$ and $\epsilon(i_l) = -\epsilon(i_{l^+})$

- In case (1), the (horizontal) edge is $k \rightarrow l$ if $\epsilon(i_l) = +$, and vice versa.
- In case (2) and (3), the (inclined) edge is $k \rightarrow l$ if $\epsilon(i_l) = -$, and vice versa.

![Diagram](image-url)
Definition 9.9. The matrix $\tilde{B}$ is defined by

1. $b_{kl} = 0$ iff there are no edges connecting $k$ and $l$.
2. $b_{kl} > 0$ if $k \rightarrow l$, $b_{kl} < 0$ if $k \leftarrow l$
3. $|b_{kl}| = \begin{cases} 1 & |i_k| = |i_l| \text{ (horizontal edge)} \\ -c_{|i_k|,|i_l|} & |i_k| \neq |i_l| \text{ (inclined edge)} \end{cases}$

Example 9.10. For $G = SL_3$, $r = 2$. Take $u = v = w_0$. We have $l(u) = l(v) = 3$, $m = 8, n = 4$. Take $i = (1, 2, 1, -1, -2, -1)$. Then $e(i) = \{1, 2, 3, 4\}$. The graph has vertices $\{-2, -1, 1, 2, 3, 4, 5, 6\}$ but we label them by $i$. The vertices $\in e(i)$ is highlighted in red. Then $\tilde{B}$ is given by

$$
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
-2 & -1 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 \\
1 & 0 & -1 & 1 & 0 \\
2 & 1 & 0 & -1 & 1 \\
3 & -1 & 1 & 0 & -1 \\
4 & 0 & -1 & 1 & 0 \\
5 & 0 & 1 & 0 & -1 \\
6 & 0 & 0 & 0 & 1
\end{array}
$$

with the (skew-symmetric) principal part $B$ highlighted in red.

Proposition 9.11. The matrix $\tilde{B}(i)$ has full rank $n$. Its principal part $B(i)$ is skew-symmetrizable.

Proof. Enough to show that the determinant of the $n \times n$ submatrix $\Delta$ of $\tilde{B}$ labeled by the row set

$$
e(i)^- := \{k \in M : k^+ \in e(i)\}$$

is nonzero. Note that if $k \in e(i)^-$ and $l \in e(i)$, then

$$|b_{kl}| = \begin{cases} 1 & k^+ = l \\
0 & k^+ < l \end{cases}$$
hence $\Delta$ is triangular (after reindexing) with diagonal entries $\neq 0$.

\[
\begin{array}{c|cccc}
  & 1 & 2 & 3 & 4 \\
\hline
-1 & 1 & 0 & 0 & 0 \\
-2 & -1 & 1 & 0 & 0 \\
1 & 0 & -1 & 1 & 0 \\
3 & -1 & 1 & 0 & -1
\end{array}
\]

The matrix $\Delta$ ($-1$ and $-2$ interchanged):

Cartan matrix is symmetrizible $\implies B(i)$ is skew-symmetrizable by our definition.

Hence by previous results, the matrix $\vec{B}(i)$ give rise to a well-defined upper cluster algebra $\mathcal{A}(i)$ of geometric type, which coincides with the upper bound $U(S)$ for the seed $S(i) = (x, \vec{B}(i))$. The ambient field $\mathcal{F}$ of $\mathcal{A}(i)$ is the field of rational functions over $\mathbb{Q}$ in $m$ independent variables $\vec{x} = \{x_k : k \in M\}$. The cluster variables in $x$ are labeled by the set $e(i)$, and the coefficient group $P$ is generated by the remaining indices.

**Example 9.12.** In our previous example for $i = (-2, -1, 1, 2, 1, -1, -2, -1)$, we have $x = \{x_1, x_2, x_3, x_4\}$, $P = \langle x_{-2}^\pm, x_{-1}^\pm, x_{2}^\pm, x_{6}^\pm \rangle$, and the exchange relation

\[
\begin{align*}
x_1x_1' &= x_{-1}x_2 + x_{-2}x_3 \\
x_2x_2' &= x_{-2}x_3x_5 + x_1x_4 \\
x_3x_3' &= x_1x_4 + x_2 \\
x_4x_4' &= x_2x_6 + x_3x_5
\end{align*}
\]

The algebra $\mathcal{A}(i)$ consists of all rational functions in $\mathcal{F} = \mathbb{Q}(x_{-2}, x_{-1}, x_1, \ldots, x_6)$ that can be written as Laurent polynomials in each of the 5 clusters:

$x = (x_1, x_2, x_3, x_4), x_1 = (x_1', x_2, x_3, x_4), \ldots, x_4 = (x_1, x_2, x_3, x_4')$

References

