Week 1-2: Graphs and Subgraphs

September 26, 2016

1 Graphs

Definition of Graphs:

- A **graph** $G$ is an ordered pair $(V, E)$ consisting of a set $V$ of **vertices** and a set $E$ (disjoint from $V$) of **edges**, together with an **incidence function** $\text{End} : E \rightarrow M_2(V)$, where $M_2(V)$ is set of all 2-element sub-multisets of $V$. We usually write $V = V(G)$, $E = E(G)$, and $\text{End} = \text{End}_G$.

- If $e$ is an edge and $u, v$ are vertices such that $\text{End}(e) = \{u, v\}$, we say that $e$ **joins** $u$ and $v$, or, $u$ and $v$ are **incident** with $e$, or, $u$ and $v$ are **adjacent** by $e$, and say that $u, v$ are **end-vertices** of $e$. We say that $e$ is a **link** if $u \neq v$ and a **loop** if $u = v$.

- Two edges are said to be **parallel** if their end-vertices are identical.

Simple Graphs, Multigraphs, Complete Graphs, Bipartite Graphs:

- A graph is said to be **simple** if it has no loops and parallel edges. A graph with possible loops and parallel edges is referred to a **multigraph**.

- A graph is said to be **finite** if its vertex set and edge set are finite. We assume that all graphs are finite in our course.

- The graph with empty vertex set (and hence empty edge set) is called a **null graph**.

- A graph is said to be **trivial** if it has only one vertex. All other graphs are said to be **nontrivial**.

- A graph is called an **empty graph** if it does not contain any edge.

- A **complete graph** is a simple graph that every pair of vertices are adjacent. A complete graph with $n$ vertices is denoted by $K_n$.

- A graph $G$ is said to be **bipartite** if its vertex set $V(G)$ can be partitioned into two disjoint nonempty parts $X, Y$ such that every edge has one end-vertex in $X$ and the other in $Y$; such a partition $\{X, Y\}$ is called a **bipartition** of $G$, and such a bipartite graph is denoted by $G[X, Y]$. 

A bipartite graph $G[X,Y]$ is called a **complete bipartite graph** if every vertex in $X$ is joined to every vertex in $Y$; we abbreviate $G[X,Y]$ to $K_{m,n}$ if $|X| = m$ and $|Y| = n$.

**Neighbors, Degree:**

- Two adjacent vertices called **neighbors**. The set of neighbors of a vertex $v$ in a graph $G$ is the set of all vertices adjacent with $v$, denoted $N_v(G)$ or $G[v]$.
- The **degree** of a vertex $v$ in a graph $G$, denoted by $d_G(v)$, is the number of edges incident with the vertex, where loops are counted twice. A vertex is said to be **isolated** if its degree is 0. For a simple graph $G$, $d_G(v) = |N_v(G)|$.
- A graph is said to be **regular** if every vertex has the same degree. A graph is said to be **$k$-regular** if every vertex has degree $k$.
- For any graph $G = (V,E)$, 
  
  $$2|E| = \sum_{v \in V} d_G(v).$$

- In any graph, the number of vertices of odd degree is even.
- A graph is said to be **even** if the degree of its every vertex is an even number.

**Proposition 1.1.** Let $G[X,Y]$ be a bipartite graph without isolated vertices. If $d(x) \geq d(y)$ for all edge $xy$ with $x \in X$ and $y \in Y$, then $|X| \leq |Y|$, and the equality holds iff $d(x) = d(y)$ for all edges $xy$ with with $x \in X$ and $y \in Y$.

**Proof.** since $d(x) \geq d(y)$ for all edges $xy$ with $x \in X$ and $y \in Y$, we have

$$|X| = \sum_{x \in X} \sum_{\substack{y \in Y \atop xy \in E}} \frac{1}{d(x)} = \sum_{x \in X, y \in Y} \frac{1}{d(x)} \leq \sum_{x \in X, y \in Y} \frac{1}{d(y)} = \sum_{y \in Y} \sum_{x \in X} \frac{1}{d(y)} = |Y|.$$ 

It is clear that if $d(x) = d(y)$ for all edges $xy$ with $x \in X$ and $y \in Y$ then $|X| = |Y|$. Conversely, if $|X| = |Y|$, the above middle inequality must be equality. It forces that $d(x) = d(y)$ for all edges $xy$ with $x \in X$. 

**Incidence Matrix, Adjacency Matrix:**

- The **incidence matrix** of a graph $G$ is a matrix $M = [m_{ve}]$, whose rows are indexed by vertices and whose columns are indexed by the edges of $G$, such that (i) the entry $m_{ve} = 0$ at $(v,e)$ if the vertex $v$ is not incident with the edge $e$, (ii) $m_{ve} = 1$ if $v$ is incident with $e$ once (i.e., $e$ is a link), and (iii) $m_{ve} = 2$ if $v$ is incident with $e$ twice (i.e., $e$ is a loop).
- The **adjacency matrix** of a graph $G$ is a square matrix $A = [a_{uv}]$, whose rows and columns are indexed by vertices of $G$, where $a_{uv}$ is the number of edges between the vertices $u$ and $v$, each loop is counted twice.

**Walk, Trail, Path, Cycle, Connectedness:**

Walk, Trail, Path, Cycle, Connectedness:
A walk from a vertex $u$ to a vertex $v$ in a graph $G$ is a sequence

$$W = v_0 e_1 v_1 \cdots e_{\ell-1} v_{\ell}$$

of vertices and edges with $v_0 = u$ and $v_{\ell} = v$, whose terms are alternate between vertices and edges of $G$, such that the edge $e_i$ is incident with the vertices $v_{i-1}$ and $v_i$, $1 \leq i \leq \ell$. The vertex $v_0$ called the initial vertex, $v_{\ell}$ the terminal vertex of $G$, and the number of $\ell$ the length of $W$. A walk is said to be closed if its initial and terminal vertices are identical.

- A walk is called a trail if its edge terms are distinct.
- A walk is called a path if its vertex terms are distinct (so are its edge terms), except possible identical initial and terminal vertices, for which it is referred to a closed path. If $P = v_0 e_1 v_1 \cdots e_{\ell-1} e_{\ell} v_{\ell}$ is a path, then $v_0, v_1, \ldots, v_{\ell}$ are distinct, or, $v_0 = v_{\ell}$, $v_1, v_2, \ldots, v_{\ell-1}$ are distinct, and $v_1, v_2, \ldots, v_{\ell-1}$ are called internal vertices of $P$. The underlying graph of a closed path is called a cycle.
- A subgraph $C$ of a graph $G$ is a cycle iff $C$ is connected and 2-regular.
- A graph is said to be connected if there is a path between any two vertices of the graph.
- A graph is a cycle iff it is connected and 2-regular.
- An Euler trail of a graph $G$ is a trail that uses every edge of $G$. A closed Euler trail is called an Euler tour.
- A Hamilton path of graph $G$ is a path that uses every vertex of $G$. A closed Hamilton path is called a Hamilton cycle.

Union, Intersection, Cartesian Product:

- Two graphs are said to be disjoint if they have no common vertices, and to be edge-disjoint if they have no common edges.
- The union of two graphs $G$ and $H$ is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $G$ and $H$ are disjoint, then we write their union as $G + H$.
- The intersection of two graphs $G$ and $H$ is the graph $G \cap H$ with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$. If $G$ and $H$ are disjoint, then $G + H$ is the null graph.
- The Cartesian product of two simple graphs $G, H$ is the graph $G \square H$, whose vertex set is $V(G) \times V(H)$ and whose edge set is

$$\{(u, x)(v, x) : uv \in E(G)\} \cup \{(u, x)(u, y) : xy \in E(H)\}.$$
A directed graph (or digraph) is an ordered pair \( D = (V, A) \) consisting of a set \( V \) of vertices and a set of arcs, together with an incidence function \( \text{End} : A \to V \times V \). If \( a \in A \) is an arc and \( \text{End}(a) = (u, v) \), we call the arc \( a \) a directed edge from \( u \) to \( v \), the vertex \( u \) a tail, and \( v \) a head of \( a \). We usually write \( V = V(D) \), \( A = A(D) \), and \( \text{End} = \text{End}_D \).

Let \( v \) be a vertex in a digraph \( D \). The out-degree of \( v \) is the number of arcs of which \( v \) is tail, denoted \( d^+_D(v) \). The in-degree of \( v \) is the number of arcs of which \( v \) is a head, denoted \( d^-_D(v) \).

Let \( a \) be an arc in a digraph \( D \) such that \( \text{End}(a) = (u, v) \). We call \( u \) an in-neighbor of \( v \), and \( v \) an out-neighbor of \( u \). We denote by \( N^+_D(v) \) the set of all out-neighbors of a vertex \( v \), and by \( N^-_D(v) \) the set of all in-neighbors of \( v \).

An orientation of an edge \( e \) incident with two vertices \( u, v \) in a graph \( G \) is an assignment of signs to the pairs \( (u, e) \) and \( (v, e) \) such that their product is negative. So a loop must be assigned two opposite signs at its end-vertex. A link edge has exactly two orientations, i.e., \( + - \) and \( - + \). A loop also has exactly two orientations, i.e., \( \pm \) and \( \mp \).

An orientation on a graph \( G \) is an assignment that each edge is given an orientation. An orientation of \( G \) can be viewed as a multi-valued function \( \varepsilon : V \times E \to \{ -1, 0, 1 \} \) such that (i) \( \varepsilon(v, e) = 0 \) if the vertex \( v \) is not incident with the edge \( e \), (ii) \( \varepsilon(u, e)\varepsilon(v, e) = -1 \) if \( e \) is a link joining the vertices \( u \) and \( v \), (iii) \( \varepsilon(u, e) = \pm \) or \( \varepsilon(u, e) = \mp \) if \( e \) is a loop at \( u \), where \( u = v \). A graph \( G \) together with an orientation \( \varepsilon \) is called an oriented graph, denoted \( (G, \varepsilon) \).

An oriented graph \( (G, \varepsilon) \) can be viewed as a digraph \( D \) with the vertex set \( V(G) \), where each edge \( e \) with end-vertices \( u, v \) is an arc (or directed edge) from \( u \) to \( v \) if \( \varepsilon(u, e) = 1 \) and \( \varepsilon(v, e) = -1 \). Conversely, a digraph \( D \) can be viewed as an oriented graph \( (G, \varepsilon) \) with the vertex set \( V(D) \), where each directed edge (or arc) \( e \) from a vertex \( u \) to a vertex \( v \) is oriented by \( \varepsilon(u, e) = 1 \) and \( \varepsilon(v, e) = -1 \).

A tournament is a directed complete graph.

**Theorem 1.2.** Every tournament has a directed Hamilton path.

*Proof.* Let \( D \) be a tournament with \( n \) vertices. We proceed by induction on \( n \). For \( n = 2, 3 \), it is trivial to check directly. Now remove one vertex \( v \) from \( D \) to obtain a digraph \( D' = D - v \) with \( n - 1 \) vertices. By induction hypothesis, \( D' \) has a directed Hamilton path \( P = v_1v_2\ldots v_{n-1} \) from \( v_1 \) to \( v_{n-1} \). The situation can be divided into the following cases.

**Case 1.** \( (v, v_1) \) is a directed edge in \( D \). Then \( P_1 = vv_1v_2\ldots v_{n-1} \) is a directed Hamilton path for \( D \).

**Case 2.** \( (v_1, v) \) and \( (v, v_2) \) are directed edges in \( D \). Then \( P_2 = v_1vv_2\ldots v_{n-1} \) is a directed Hamilton path for \( D \).

**Case 3.** \( (v_1, v), (v_2, v), \) and \( (v, v_3) \) are directed edges in \( D \). Then \( P_3 = v_1v_2vv_3\ldots v_{n-1} \) is a directed Hamilton path for \( D \).

**Case k.** \( (v_1, v), (v_2, v), \ldots, (v, v_k) \) are directed edges in \( D \). Then \( P_k = v_1\ldots v_{k-1}vv_k\ldots v_{n-1} \) is a directed Hamilton path for \( D \).
Case \( n \). \((v_1, v), (v_2, v), \ldots, (v_{n-1}, v)\) are directed edges in \( D \). Then \( P_n = v_1v_2 \ldots v_{n-1}v \) is a directed Hamilton path for \( D \).

Isomorphism, Automorphism, Homomorphism:

- Two graphs \( G \) and \( H \) are said to be identical if \( V(G) = V(H) \) and \( E(G) = E(H) \).
- A graph \( G \) is said to be isomorphic to a graph \( H \) if there exist bijective mappings \( f : V(G) \rightarrow V(H) \) and \( g : E(G) \rightarrow E(H) \) such that \( \text{End}_G(e) = \{u, v\} \) iff \( \text{End}_H(g(e)) = \{f(u), f(v)\} \); such a pair \((f, g)\) of mappings is called an isomorphism from \( G \) to \( H \).
- An isomorphism from a graph \( G \) to itself if called an automorphism of \( G \). The set of all automorphisms of \( G \) froms a group under the composition of mappings, called the automorphism group of \( G \), denoted \( \text{Aut}(G) \).
- A homomorphism from a graph \( G \) to a graph \( H \) if there exist maps \( f : V(G) \rightarrow V(H) \) and \( g : E(G) \rightarrow E(H) \) such that if vertices \( u, v \) are adjacent by an edge \( e \) then the vertices \( f(u), f(v) \) are adjacent by the edge \( g(e) \). [The concept of homomorphism of graphs is not yet standardized. We rarely use the concept in our course.]

Labeled Graphs:

- Given a finite set \( V \). A simple graph \( G = (V, E) \) on \( V \) can be considered as a subset of \( \binom{V}{2} \), the set of all 2-element subsets of \( V \). A simple graph whose vertices are labeled, but whose edges are not labeled, is referred to a labeled simple graph.
- Given a set \( V \) of \( n \) elements. There are \( 2^{\binom{n}{2}} \) labeled simple graphs with the vertex set \( V \). We denote by \( G(V) \) the set of all labeled simple graphs with vertex set \( V \).
- Let \( G \) be an unlabeled graph with \( n \) vertices. Then the number of labelings of \( G \) is \( \frac{n!}{\text{Aut}(G)} \), where \( \text{Aut}(G) \) is understood as the automorphism group of \( G \) with any labeling. Then
  \[
  \sum_{G \text{ unlabeled graph with } n \text{ vertices}} \frac{n!}{\text{Aut}(G)} = 2^{\binom{n}{2}}.
  \]
- The number of unlabeled graphs with \( n \) vertices is at least \( \lceil 2^{\binom{n}{2}}/n! \rceil \).

Intersection Graphs, Interval Graphs, Polyhedral Graphs, Cayley Graphs:

- Let \( F \) be a family of subsets of set \( V \). The intersection graph of \( F \) is a graph whose vertex set is \( F \), and two members of \( F \) are adjacent if their intersection is nonempty.
- Let \( V = \mathbb{R} \) and \( F \) be a set of some closed intervals of \( \mathbb{R} \). The intersection graph of \( F \) is called an interval graph.
- Given a polytope \( P \) of \( \mathbb{R}^3 \). The vertices and edges of \( P \) form a graph, called a polyhedral graph.
- Let \( \Gamma \) be a group. Given a subset \( S \) of \( \Gamma \) such that \( S \) does not contain the identity element and is closed under inverse operation. The Cayley graph of \( \Gamma \) with respect to \( S \) is a graph \( G(\Gamma, S) \) with vertex set \( \Gamma \) in which two vertices \( x, y \) are adjacent if \( xy^{-1} \in S \).
2 Subgraphs

Definition of Subgraphs:

- A graph $H$ is called a subgraph of a graph $G$ if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and $End_H : E(H) \to M_2(V(H))$ is the restriction of $End_G : E(G) \to M_2(V(G))$ to $E(H)$. We then say that $G$ contains $H$ or $H$ is contained in $G$.

- A copy of a graph $H$ in a graph $G$ is a subgraph of $G$ which is isomorphic to $H$. Such a subgraph is also referred to as an $H$-subgraph of $G$.

- An embedding of graph $H$ in a graph $G$ is an isomorphism from $H$ to a subgraph of $G$. For each copy of $H$ in $G$, there are $|Aut(H)|$ embeddings in $G$, whose image subgraph is fixed.

- A maximal connected subgraph of $G$ is called a connected component of $G$. The number of connected components of $G$ is denoted by $c(G)$.

Deletion, Contraction:

- Let $v$ be a vertex in a graph $G$. We denote by $G \setminus v$ the graph obtained from $G$ by deleting the vertex $v$ and all edges incident with $v$. Such an operation is referred to as a vertex deletion, and $G \setminus v$ as a vertex-deleted subgraph.

- Let $e$ be an edge of graph $G$. We denote by $G \setminus e$ the graph obtained from $G$ by deleting the edge $e$ but leaving the end-vertices of $e$. Such an operation is referred to as an edge deletion, and $G \setminus e$ as an edge-deleted subgraph. If $S \subseteq E(G)$, we denote by $G \setminus S$ the graph obtained from $G$ by deleting all edges of $S$.

- Let $e$ be an edge of a graph $G$. We denote by $G/e$ the graph obtained from $G$ by deleting the edge $e$ and gluing the end-vertices of $e$ to become one vertex. Such an operation is called a contraction, and $G/e$ an edge-contracted minor of $G$. Note that there are edges (other than $e$) joining the end-vertices of $e$, then those edges become loops in $G/e$. If $S \subseteq E(G)$, we denote by $G/S$ the graph obtained from $G$ by contracting all edges of $S$.

Theorem 2.1. A graph $G$ whose every vertex has degree at least 2 contains a cycle.

Proof. Let $P := v_0 e_1 v_1 \cdots e_\ell v_\ell$ be a longest path in $G$. Such a path does exist since $G$ is finite. If $v_0 = v_\ell$, then the underlying graph of $P$ is already a cycle. Otherwise, the degree of $v_0$ in $P$ is 1. Since the degree of $v_0$ in $G$ is at least 2, there is an edge $e_0$ (not in $P$) joining $v_0$ to another vertex $v$. If $v = v_i$ for some $i$ with $0 \leq i \leq \ell$, then the underlying graph of $P_i := v e_0 v_0 v_1 \cdots e_i v_i$ is a cycle. Otherwise, $Q := v e_0 P$ is a longer path, a contradiction. [QED]

Corollary 2.2. A graph without cycles has at least one vertex of degree 0 or degree 1.

Acyclic Graphs (Forests):

- A graph is said to be acyclic if it does not contain any cycle. An acyclic graph is also called a forest. A connected forest is called a tree.
• A vertex of degree 1 in a tree is called a leaf of the tree.
• A tree with at least one edge has at least two leaves.

Spanning Subgraphs, Induced Subgraphs:
• A spanning subgraph $H$ of a graph $G$ is subgraph such that $V(H) = V(G)$.
• The symmetric difference of spanning subgraph $G_1$ and $G_2$ of graph $G$ is a spanning subgraph of $G$ whose edge is $E(G_1) \Delta E(G_2)$.
• Let $X$ be a vertex subset of a graph. An induced subgraph by $X$ is a graph $G[X]$ whose vertex is $X$ and whose edge set consists of all edges of $G$ which have end-vertices in $X$.
• Let $S$ be an edge subset of a graph $G$. An induced subgraph by $S$ is a graph $G[S]$ whose edge set is $S$ and whose vertex set consists of all end-vertices of edges in $S$.

Decomposition, Coverings:
• A decomposition of a graph $G$ is a family of edge-disjoint subgraphs of $G$ such that
  $$E(G) = \bigcup_{H \in \mathcal{F}} E(H).$$
• A covering or cover of graph $G$ is a family $\mathcal{F}$ of not necessarily edge-disjoint subgraphs of $G$ such that
  $$E(G) = \bigcup_{H \in \mathcal{F}} E(H).$$
  A covering $\mathcal{F}$ is referred to a path (cycle) covering if the family $\mathcal{F}$ consists entirely of path (cycles) of $G$.
• A covering of a graph $G$ is said to be a uniform if every edge of $G$ is covered the same number of times by $\mathcal{F}$. When this number is $k$, the covering is called a $k$-cover. A 2-cover is usually called a double cover.

Theorem 2.3. A graph admits a cycle decomposition iff it is even.

Proof. The necessity is trivial, for every cycle is 2-regular and the degree of a vertex in the graph is a sum of 2’s. The sufficiency is as follows.

Let $G$ be an even graph. If $G$ contains some edges, then $G$ contains a cycle $C_1$ by Theorem 2.1. Remove the edge of $C_1$ from $G$ to obtain a graph $G_1$, which is still even. Then by Theorem 2.1 again there is a cycle $C_2$ in $G_1$. Remove the edges of $C_2$ from $G_1$ to obtain a graph $G_2$, which is even. Continue this procedure, we obtain a cycle decomposition of $G$.

Theorem 2.4. Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a family of bipartite graphs. If $\mathcal{F}$ is a decomposition of $K_n$, then $k \geq n - 1$. 

Cuts, Bonds, Even Graphs:
• let $X$ and $Y$ be vertex subsets of a graph $G$ or digraph $D$. We introduce the edge subset and the arc subset

$[X, Y] = \text{set of edges with one vertex in } X \text{ and the other vertex in } Y,$

$(X, Y) = \text{set of arcs having the tail in } X \text{ and the head in } Y.$

• An edge cut or just a cut of graph $G$ is a nonempty edge subset of the form $[X, X^c]$, where $X$ is a vertex subset and $X^c$ is its complement in $V(G)$. We also write

$\partial(X) = \partial_G(X) := [X, X^c].$

If $\partial X$ is a cut, then $X$ must be a proper subset of $V(G)$.

• For each vertex subset $X$ of a graph $G$,

$|X, X^c| + 2|X, X| = \sum_{v \in X} d_G(v).$

• A bond of a graph $G$ is a minimal cut, i.e., an edge cut none of whose proper subset is an edge cut.

• Deleting the edges of a cut increases the number of connected components. Deleting the edges of a bond increases exactly by one the number of connected components.

• An even graph is a graph whose every vertex has even degree. A connected even graph is called an Eulerian graph.

**Theorem 2.5.** A graph $G$ is even iff every cut of $G$ has even number of edges.

*Proof.* If $G$ is even, then for each subset $X \subseteq V(G)$, the cardinality

$|X, X^c| = -2||X, X|| + \sum_{v \in X} d_G(v)$

is clearly even. Conversely, for each vertex $v \in V(G)$, we have the cut $\{|v\}, \{v\}^c\}$, and clearly, $d_G(v) = ||\{v\}, \{v\}^c|| + 2||\{v\}, \{v\}||$ is even.

**Proposition 2.6.** The difference of two cuts is a cut or empty set. For vertex subsets $X, Y$ of a graph $G$,

$[X, X^c] \Delta [Y, Y^c] = [X \Delta Y, (X \Delta Y)^c].$

*Proof.* Note that $\{X \cap Y, X \cap Y^c, X^c \cap Y, X^c \cap Y^c\}$ is a partition of $V(G)$.

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Since \([X, X^c] \cap [Y, Y^c] = [X \cap Y, X^c \cap Y^c]\), we have
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[X, X^c] \Delta [Y, Y^c] = [X, X^c] \cup [Y, Y^c] - [X \cap Y, X^c \cap Y^c]
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\[
= [X \cap Y, X \cap Y^c] \cup [X \cap Y^c, X \cap Y] \cup
[X \cap Y^c, X^c \cap Y^c] \cup [X^c \cap Y, X^c \cap Y^c].
\]
Since \(X \Delta Y = (X \cap Y^c) \cup (X^c \cap Y)\) and \((X \Delta Y)^c = (X \cap Y) \cup (X^c \cap Y^c)\), we have
\[
[X \Delta Y, (X \Delta Y)^c] = [(X \cap Y^c) \cup (X^c \cap Y), (X \cap Y) \cup (X^c \cap Y^c)]
\]
\[
= [X \cap Y^c, X \cap Y] \cup [X^c \cap Y, X \cap Y] \cup
[X \cap Y^c, X^c \cap Y^c] \cup [X^c \cap Y, X^c \cap Y^c].
\]
Note that \([X \cap Y, X \cap Y^c] = [X \cap Y^c, X \cap Y]\) and \([X \cap Y, X^c \cap Y] = [X^c \cap Y, X \cap Y]\).

\[\Box\]

**Proposition 2.7.** Let \(B\) be an edge subset of a connected graph \(G\). Then \(B\) is a bond iff there is a vertex subset \(X\) such that both \(G[X]\) and \(G[X^c]\) are connected and \(B = [X, X^c]\).

**Proof.** “⇒” Since \(B\) is a cut, there is a vertex subset \(X\) such that \(B = [X, X^c]\). We claim that both \(G[X]\) and \(G[X^c]\) are connected. Suppose \(G[X]\) is not connected, say, \(X = X_1 \sqcup X_2, X_i \neq \emptyset, i = 1, 2,\) and \([X_1, X_2] = \emptyset\). Then both \([X_1, X^c]\) and \([X_2, X^c]\) are nonempty cuts, and are contained in \([X, X^c]\); this is contradict to that \(B\) is a minimal cut. So \(G[X]\) is connected. Likewise, \(G[X^c]\) is connected.

“⇐” Clearly, \(B = [X, X^c]\) is a cut. Suppose \(B\) is not minimal, i.e., there exists a proper subset \(B_1 \subsetneq B\) such that \(B_1\) is also a cut. Then \(G - B_1\) is disconnected. However, the edges of \(B - B_1\) are between \(G[X]\) and \(G[X^c]\), and both \(G[X]\), \(G[X^c]\) are connected. So \(G - B_1\) is connected, contradictory to that \(G - B_1\) is disconnected.

**Proposition 2.8.** An edge subset of \(G\) is a cut iff it is a disjoint union of bonds.

**Proof.** The sufficiency is trivial. For necessity, consider an edge cut \([X, X^c]\) of \(G\). Let \(G[X]\) be decomposed into connected components, and let \(G_1, \ldots, G_m\) be those components having a vertex adjacent to a vertex in \(X^c\). Set \(X_i = V(G_i), i = 1, \ldots, m\). Then \([X, X^c]\) is a disjoint union of the cuts \([X_i, X_i^c]\), \(1 \leq i \leq m\). For each \(i\), let \(G[X_i]\) be decomposed into connected components, and let \(G_i, \ldots, G_{im}\) be those components having a vertex adjacent to a vertex in \(X_i\). Set \(X_{ij} = V(G_{ij})\). Then \([X_{ij}, X_{ij}^c]\) is a disjoint union of cuts \([X_{ij}, X_{ij}^c]\). It suffices to show that each cut \([X_{ij}, X_{ij}^c]\) is a bond. In fact, \([X_{ij}, X_{ij}^c]\) consists of the edges between the connected subgraphs \(G_{ij}\) and \(G_i\). Then \([X_{ij}, X_{ij}^c]\) is a bond in \(G[X_i \cup X_{ij}]\) by Proposition 2.7. So is \([X_{ij}, X_{ij}^c]\) in \(G\). (In fact, suppose \([X_{ij}, X_{ij}^c]\) is not a bond, i.e., there is a proper nonempty subset \(S\) of \([X_{ij}, X_{ij}^c]\) such that \(S = [Y, Z]\) is a cut of \(G\), where \(Y \subset X_{ij}\) and \(Z \subset X_i\). There is no path from \(G[Y]\) to \(G[Z]\). However, there is an edge \(e \notin S\) such that \(e = uv\) and \(u \in X_{ij}, v \in X_i\). Note that there exist a path from \(G[Y]\) to \(u\) and a path from \(G[Z]\) to \(v\). So there is a path from \(G[Y]\) to \(G[Z]\), a contradiction.)

**Proposition 2.9.** For spanning subgraphs \(G_1, G_2\) of a graph \(G\),
\[
\partial_{G_1 \Delta G_2}(X) = \partial_{G_1}(X) \Delta \partial_{G_2}(X), \quad i.e.,
\]
\[
[X, X^c]_{G_1 \Delta G_2} = [X, X^c]_{G_1} \Delta [X, X^c]_{G_2}.
\]
Proof. For each edge $e \in \partial G(X)$, we see that $e \in \partial_{G_1 \Delta G_2}(X)$ iff either $e \in E(G_1) - E(G_2)$ or $e \in E(G_2) - E(G_1)$ iff either $e \in \partial G_1(X)$ or $e \in \partial G_2(X)$, i.e., $e \in \partial G_1(X) \Delta \partial G_2(X)$. \qed

Theorem 2.10. The difference of two even graphs is an even graph.

Proof. For even graphs $G_1, G_2$ and a vertex subset $X$, let $a = |[X, X^c]_{G_1} \cap [X, X^c]_{G_2}|$. By Proposition 2.9,

$$|[X, X^c]_{G_1} \Delta [X, X^c]_{G_2}| = |[X, X^c]_{G_1} - [X, X^c]_{G_2}| = |[X, X^c]_{G_1}| + |[X, X^c]_{G_2}| - 2a$$

which is an even number. Then by Theorem 2.5, $G_1 \Delta G_2$ is an even graph. \qed

Vector Spaces Associated to Graphs:

- Let $S$ be a set and $\mathbb{F}$ a field. Let $\mathbb{F}^S$ denote the set of all functions from $S$ to $\mathbb{F}$. Then $\mathbb{F}^S$ becomes a vector space over $\mathbb{F}$ under the addition and the scalar multiplication of functions, i.e., for functions $f, g \in \mathbb{F}^S$ and a scalar $a \in \mathbb{F}$,

$$(f + g)(s) = f(s) + g(s), \quad (af)(s) = af(s), \quad s \in S.$$

- Let $S$ be a set and $\mathbb{F}_2 = \{0, 1\}$ the field of two elements. There is a one-to-one correspondence between the power set $\mathcal{P}(S)$ and the vector space $\mathbb{F}_2^S$. In fact, each subset $A \subseteq S$ corresponds to the characteristic function $1_A : S \to \mathbb{F}_2$, where $1_A(s) = 1$ for $s \in A$ and $1_A(s) = 0$ for $s \in A^c$. Moreover, for subsets $A, B \subseteq S$,

$$A \triangle B = A \cup B - A \cap B \leftrightarrow 1_A + 1_B.$$

So $\mathcal{P}(S)$ can be viewed as a vector space of dimension $|S|$, whose addition is the symmetric difference, where the zero vector is the empty set, and the negative of a subset is itself.

- For a graph $G = (V, E)$, the vector space $\mathbb{F}_2^V$ is called the **vertex space** of $G$, and $\mathbb{F}_2^E$ the **edge space** of $G$.

- The set of even subgraphs of a graph $G$ is a vector subspace of its edge space, called the **cycle space** of $G$.

- The set of cuts of a graph $G$ is a vector subspace of its edge space, called the **bond space** of $G$.

Exercises

Ch1: 1.1.21; 1.1.22; 1.2.8; 1.3.9; 1.4.2; 1.5.6; 1.5.7; 1.5.12.

Ch2: 2.1.2(b); 2.1,11; 2.2.9; 2.2.12; 2.2.13; 2.4.1; 2.4.2; 2.4.4; 2.4.9; 2.5.2; 2.5.4; 2.5.7; 2.6.2; 2.6.4.
3 Chain Groups of Graphs

Definition of Graph and Orientation (revisited):

- A graph $G$ is a system $(V, E)$, where $V$ is a set whose elements are called vertices and $E$ a set of disjoint non-closed simple paths, called edges, such that the initial (and also terminal) point of each edge is glued to a vertex in $V$. Since each edge $e \in E$ is a non-closed simple path, we view $e$ as an embedding $e : [0, 1] \to \mathbb{R}^d$, and view the initial point $e(0)$ and the terminal point $e(1)$ as vertices of $G$. The edge $e$ is called a link if $e(0) \neq e(1)$ and a loop if $e(0) = e(1)$. The pairs $(0, e)$ and $(1, e)$ are called two ends of $e$, and may be viewed as the sub-paths $e[0, 1/2]$ and $e[1/2, 1]$. The collection of all ends is denoted by $\text{End}(G)$, i.e., $\text{End}(G) = \{(i, e) : e \in E, i = 0, 1\}$.

- An orientation of an edge $e$ is an assignment $\varepsilon$ of signs on the two ends of $e$ such that $\varepsilon(0, e) \varepsilon(1, e) = -1$. Each edge has exactly two (opposite) orientations. An edge with an orientation is called an oriented edge. If $\vec{e}$ denotes an oriented edge, the the same edge with opposite orientation is denoted by $-\vec{e}$. Let $\vec{E}(G)$ denote the set of all oriented edges of $G$. An orientation of a graph $G$ is an assignment that each edge is given one of its two orientations, and it can be viewed as a subset $\omega \subset \vec{E}(G)$ such that $\omega \cap (-\omega) = \emptyset$ and $\omega \cup (-\omega) = \vec{E}(G)$.

Chain Groups:

- Each vertex of a graph $G$ is referred to a 0-cell, and each edge of $G$ is referred to a 1-cell. So a graph $G$ can be viewed as a 1-dimensional cell complex.

- Let $A$ be an abelian group and $G = (V, E)$ a graph. Set $\vec{V} = \{\pm v : v \in V\}$. A 0-chain of $G$ is a function $p : \vec{V} \to A$ such that $p(-v) = -p(v)$. A 1-chain of is a function $f : \vec{E}(G) \to A$ such that $f(-e) = f(e)$. Let $C_i(G, A)$ denote the group of all $i$-chains of $G$, called the $i$-th chain group of $G$, $i = 0, 1$.

- For each chain $f$, the support of $f$ is the set $\text{supp}f = \{e \in E(G) : f(\vec{e}) \neq 0\}$. We usually write
  $$f = \sum_{e \in \text{supp}f} f(\vec{e})\vec{e}.$$  
  Here we do not care which oriented edge $\vec{e}$ is selected, since $\sum_{e \in \text{supp}f} f(-\vec{e})(-\vec{e}) = \sum_{e \in \text{supp}f} f(\vec{e})\vec{e}$.

- We introduce a pairing $\langle, \rangle : C_1(G, \Gamma) \times C_1(G, \Gamma) \to \Gamma$ by
  $$\langle f, g \rangle = \sum_{e \in \vec{E}(G)} f(\vec{e})g(\vec{e}).$$  
  Again, here it does not matter which oriented edge $\vec{e}$ is selected, since
  $$\sum_{e \in \vec{E}(G)} f(-\vec{e})g(-\vec{e}) = \sum_{e \in \vec{E}(G)} f(\vec{e})g(\vec{e}).$$
There is a **boundary map**

\[ \partial : C_1(G, \Gamma) \to C_0(G, \Gamma), \quad \partial \vec{e} = u - v, \text{ where } \vec{e} = \overrightarrow{uv}, \]

extended by group homomorphic property. The **flow group** \( F(G, A) \) of \( G \) with coefficients in \( A \) is the kernel \( \ker \partial \). Each member of \( F(G, A) \) is called a **flow** of \( G \) with values in \( A \).

There is a **co-boundary map**

\[ \delta : C_0(G, \Gamma) \to C_1(G, \Gamma), \quad (\delta f)(\vec{e}) = f(u) - f(v), \text{ where } \vec{e} = \overrightarrow{uv}, \]

which is a group homomorphism.

**Circuit Vectors, Cut vectors, Bond Vectors:**

- A **circuit** of a graph \( G \) is a nonempty edge subset \( C \) such that \( G[C] \) is a minimal even graph. A circuit is just a cycle. A **direction** of a circuit \( C \) is an orientation \( \omega_C \) on \( G[C] \) such that there is neither a source nor a sink. A circuit \( C \) with a direction \( \omega_C \) is referred to as a **directed circuit**, denoted \( (C, \omega_C) \).

- It is easy to see that there are exactly two directions \( \pm \omega_C \) (opposite each other) on \( C \). A **circuit vector** of \( G \), associated with a directed circuit \( (C, \omega_C) \), is a chain \( I_{(C, \omega_C)} : \vec{E} \to \mathbb{Z} \) defined by

\[
I_{(C, \omega_C)}(e) = \begin{cases} 
1 & \text{if } e \in \omega_C, \\
0 & \text{if } e \notin \omega_C \cup (-\omega_C).
\end{cases}
\]

Sometimes we simply use \( \omega_C \) to denote this chain \( I_{(C, \omega_C)} \).

- A **direction** of a cut \( U = [X, X^c] \) of a graph \( G \) is an orientation \( \omega_U \) on \( G[U] \) such that the each oriented edge has its tail in \( X \) and its head in \( X^c \). A cut \( U \) with a direction \( \omega_U \) is referred to as a **directed cut**, denoted \( (U, \omega_U) \).

- A **cut vector** of \( G \), associated with an oriented cut \( (U, \omega_U) \), is a chain \( I_{(U, \omega_U)} : \vec{E} \to \mathbb{Z} \) defined by

\[
I_{(U, \omega_U)}(e) = \begin{cases} 
1 & \text{if } e \in \omega_U, \\
0 & \text{if } e \notin \omega_U \cup (-\omega_U).
\end{cases}
\]

Sometimes we simply use \( \omega_U \) to denote this chain \( I_{(U, \omega_U)} \).

- A cut vector is referred to a **bond vector** if the cut is a bond.

**Proposition 3.1** (Berge). Let \( G \) be a graph.

(a) A chain \( f \in C_1(G, A) \) is a flow iff for each directed cut \( (U, \omega_U) \),

\[
\sum_{e \in \omega_U} f(e) = 0.
\]
(b) In particular, a digraph \((G, \omega)\) is a directed Eulerian iff for each directed cut \((U, \omega_U)\),
\[
\sum_{e \in \omega_U} [\omega, \omega_U](e) = 0.
\]

Tensions:

A **tension** of a graph \(G\) with values in an abelian group \(A\) is a chain \(g \in C_1(G, A)\) such that for each directed circuit \((C, \omega_C)\),
\[
\sum_{e \in \omega_C} g(e) = 0, \quad \text{i.e.,} \quad \langle g, \omega_C \rangle = 0.
\]

The **tension group** \(T(G, A)\) of \(G\) with coefficients in \(A\) is the group of all tensions of \(G\) with values in \(A\).

**Proposition 3.2.** For each potential \(p \in C_0(G, A)\), \(\delta p\) is a tension of \(G\).

**Theorem 3.3.** A 1-chain \(g\) of a graph \(G\) with values in an abelian group \(A\) is a tension iff for each flow \(f\) of \(G\) with values in \(A\),
\[
\langle f, g \rangle = 0.
\]