Week 5-6: The Binomial Coefficients

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1 Pascal Formula

Theorem 1.1 (Pascal’s Formula). For integers $n$ and $k$ such that $n \geq k \geq 1$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}. $$

The numbers $\binom{n}{2} = \frac{n(n-1)}{2}$ ($n \geq 2$) are triangle numbers, that is,

\[ \bullet \bullet \bullet \bullet \bullet \bullet \]

The pentagon numbers are 1, 5, 12, 22, . . . , defined as the numbers of points of dilated pentagons. Then $a_n = a_{n-1} + 3n + 1$ for $n \geq 1$ with $a_0 = 1$. Then $a_n = \frac{3}{2}n^2 + \frac{5}{2}n + 1$, $n \geq 1$. The k-gon numbers are 1, $k$, $3k - 3$, $6k - 8$, . . .

The numbers $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$ ($n \geq 3$) are tetrahedral numbers, i.e., $\binom{n}{3}$ is the number of lattice points of the tetrahedron $\Delta^3(n)$ defined by

$$\Delta^3(n) = \{(x, y, z) : x \geq 0, y \geq 0, z \geq 0, x + y + z \leq n - 3\}.$$

Theorem 1.2. The number of nondecreasing coordinate paths from $(0, 0)$ to $(m, n)$ with $m, n \geq 0$ equals

$$\binom{m+n}{m}.$$

2 Binomial Theorem

Theorem 2.1 (Binomial Expansion). For integer $n \geq 1$ and variables $x$ and $y$,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k},$$

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$
3 Binomial Identities

Definition 3.1. For any real number $\alpha$ and integer $k$, define the binomial coefficients

$$\binom{\alpha}{k} = \begin{cases} 
0 & \text{if } k < 0 \\
1 & \text{if } k = 0 \\
\alpha(\alpha - 1) \cdots (\alpha - k + 1)/k! & \text{if } k > 0
\end{cases}$$

Proposition 3.2. (1) For real number $\alpha$ and integer $k$,

$$\binom{\alpha}{k} = \binom{\alpha - 1}{k} + \binom{\alpha - 1}{k - 1}.$$

(2) For real number $\alpha$ and integer $k$,

$$k\binom{\alpha}{k} = \alpha\binom{\alpha - 1}{k - 1}.$$

(3) For nonnegative integers $m$, $n$, and $k$ such that $m + n \geq k$,

$$\binom{m + n}{k} = \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k - i}.$$

Proposition 3.3. For integers $n, k \geq 0$,

$$\binom{n + 1}{k + 1} = \sum_{m=0}^{n} \binom{m}{k}$$

Proof. Applying the Pascal formula again and again, we have

$$\binom{n + 1}{k + 1} = \binom{n}{k + 1} + \binom{n}{k}$$

$$= \binom{n - 1}{k + 1} + \binom{n - 1}{k} + \binom{n}{k}$$

$$= \cdots$$

$$= \binom{0}{k + 1} + \binom{0}{k} + \binom{1}{k} + \cdots + \binom{n}{k}.$$ 

Note that $\binom{0}{k + 1} = 0$. 

4 Multinomial Theorem

Theorem 4.1 (Multinomial Expansion). For any positive integer $n$,

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_1+n_2+\cdots+n_k=n}^{n_{1, n_2, \ldots, n_k \geq 0}} \binom{n}{n_{1, n_2, \ldots, n_k}} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k},$$
where the coefficients 
\[ \binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!} \]
are called multinomial coefficients.

Proof.

\[
(x_1 + x_2 + \cdots + x_k)^n = \sum u_1 u_2 \cdots u_n \quad (u_i = x_1, x_2, \ldots, x_k, 1 \leq i \leq n)
\]

\[
= \sum_{n_1 + n_2 + \cdots + n_k = n} \left\{ \text{number of permutations of the multiset } \{n_1 x_1, n_2 x_2, \ldots, n_k x_k\} \right\}
\]

\[
= \sum_{n_1 + n_2 + \cdots + n_k = n} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.
\]

5 Newton Binomial Theorem

Theorem 5.1 (Newton’s Binomial Expansion). Let \( \alpha \) be a real number. If \( 0 \leq |x| < |y| \), then

\[
(x + y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha - k},
\]

where

\[
\binom{\alpha}{k} = \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{k!}.
\]

Proof. Apply the Taylor expansion formula for the function \((x + y)^\alpha\) of two variables.

Corollary 5.2. If \(|z| < 1\), then

\[
(1 + z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k,
\]

\[
\frac{1}{(1 - z)^\alpha} = \sum_{k=0}^{\infty} \binom{-\alpha}{k} (-z)^k = \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k} z^k.
\]

The identity

\[
\binom{-\alpha}{k} = (-1)^k \binom{\alpha + k - 1}{k}.
\]

is called the reciprocity law of binomial coefficients.
Proof. Apply the Taylor expansion formula.

In particular, since \( \frac{1}{1-z} = \sum_{i=0}^{\infty} z^i \), we have
\[
\frac{1}{(1-z)^n} = \left( \sum_{i_1=0}^{\infty} z^{i_1} \right) \cdots \left( \sum_{i_n=0}^{\infty} z^{i_n} \right)
= \sum_{k=0}^{\infty} z^k \sum_{i_1+\cdots+i_n=k} 1
= \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k.
\]
This shows again that the number of nonnegative integer solutions of the equation
\[ x_1 + x_2 + \cdots + x_n = k \]
equals the binomial coefficient
\[
\left\langle \frac{n}{k} \right\rangle = \binom{n+k-1}{k}.
\]

6 Unimodality of Binomial Coefficients

Definition 6.1. A sequence \( s_0, s_1, s_2, \ldots, s_n \) is said to be unimodal if there is an integer \( k \) \((0 \leq k \leq n)\) such that
\[ s_0 \leq s_1 \leq \cdots \leq s_k \geq s_{k+1} \geq \cdots \geq s_n. \]

Theorem 6.2. Let \( n \) be a positive integer. The sequence of binomial coefficients
\[
\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}
\]
is an unimodal sequence. More precisely, if \( n \) is even,
\[
\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{n/2} > \cdots > \binom{n}{n-1} > \binom{n}{n};
\]
and if \( n \) is odd,
\[
\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{(n-1)/2} = \binom{n}{(n+1)/2} > \cdots > \binom{n}{n-1} > \binom{n}{n}.
\]

Proof. Note that the quotient
\[
\binom{n}{k} / \binom{n}{k-1} = \frac{n-k+1}{k} = \begin{cases} \geq 1 & \text{if } k \leq (n+1)/2 \\ \leq 1 & \text{if } k \geq (n+1)/2 \end{cases}
\]
The unimodality follows immediately. \( \Box \)
A sequence \(s_0, s_1, \ldots, s_n\) of positive numbers is said to be log-concave if
\[
s_i^2 \geq s_{i-1}s_{i+1}, \quad i = 1, \ldots, n - 1.
\]
The condition implies that the sequence \(\log s_1, \log s_2, \ldots, \log s_n\) are concave, i.e.,
\[
\log s_i \geq \frac{(\log s_{i-1} + \log s_{i+1})}{2}.
\]

**Proposition 6.3.** If a sequence \((s_i)\) is log-concave, then it is unimodal.

**Proof.** Assume the sequence is nonzero. The condition \(s_i^2 \geq s_{i-1}s_{i+1}\) is equivalent to
\[
\frac{s_{i-1}}{s_i} \leq \frac{s_i}{s_{i+1}}.
\]
If there exists an \(i_0\) such that \(s_{i_0} \leq s_{i_0+1}\), i.e., \(\frac{s_{i_0}}{s_{i_0+1}} \leq 1\), then \(\frac{s_{i-1}}{s_i} \leq 1\) for all \(i \leq i_0\), i.e.,
\[
s_0 \leq s_1 \leq \cdots \leq s_{i_0} \leq s_{i_0+1}.
\]
If there exists an \(i_0\) such that \(s_{i_0-1} \geq s_{i_0}\), i.e., \(\frac{s_{i_0-1}}{s_{i_0}} \geq 1\), then \(\frac{s_{i-1}}{s_i} \geq 1\) for all \(i \geq i_0\), i.e.,
\[
s_{i_0-1} \geq s_{i_0} \geq \cdots \geq s_{n-1} \geq s_n.
\]
Now for the nondecreasing numbers \(\frac{s_i}{s_{i+1}}\), there exists an index \(i_0\) such that
\[
\frac{s_{i_0-1}}{s_{i_0}} \leq 1 \leq \frac{s_{i_0}}{s_{i_0+1}}.
\]
It follows that
\[
s_0 \leq s_1 \leq \cdots \leq s_{i_0} \geq s_{i_0+1} \geq \cdots \geq s_n.
\]

\(\square\)

The sequence \(s_i = \binom{n}{i}\) of binomial coefficients is log-concave. In fact,
\[
\frac{s_i^2}{s_{i-1}s_{i+1}} = \frac{(n - i + 1)(i + 1)}{i(n - i)} > 1, \quad i = 1, \ldots, n - 1.
\]

Given a graph \(G\) with \(n\) vertices. A coloring of \(G\) with \(t\) colors is said to be proper if no two adjacent vertices receive the same color. The number of proper colorings turns out to be a polynomial function of \(t\), called the chromatic polynomial of \(G\), denoted \(\chi(G, t)\), and it can be written as the form
\[
\chi(G, t) = \sum_{k=0}^{n} (-1)^{n-k} a_k t^k.
\]

**Conjecture 6.4** (Log-Concavity Conjecture). The coefficients of the above chromatic polynomial satisfies the log-concave equality:
\[
a_k^2 \geq a_{k-1}a_{k+1}.
\]

When the inequalities are strict inequalities, it is called the Strict Log-Concavity Conjecture.
A **cluster** of a set $S$ is a collection $\mathcal{A}$ of subsets of $S$ such that no one is contained in another. A **chain** is a collection $\mathcal{C}$ of subsets of $S$ such that for any two subsets, one subset is always contained in another subset. For example, for $S = \{a, b, c, d\}$, the collection
\[
\mathcal{A} = \{\{a, b\}, \{b, c, d\}, \{a, c\}, \{a, d\}\}
\]
is a cluster; while the collection
\[
\mathcal{C} = \{\emptyset, \{b, d\}, \{a, b, d\}, \{a, b, c, d\}\}
\]
is a chain. In more general context, a cluster is an antichain of a partially ordered set.

**Theorem 6.5** (Sperner). Every cluster of an $n$-set $S$ contains at most $\binom{n}{\lfloor n/2 \rfloor}$ subsets of $S$.

**Proof.** Let $S = \{1, 2, \ldots, n\}$. We actually prove the following stronger result by induction on $n$:

The power set $P(S)$ can be partitioned into disjoint chains $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ with
\[
m = \binom{n}{\lfloor n/2 \rfloor}.
\]
If so, then for each cluster $\mathcal{A}$ of $S$,
\[
|\mathcal{A} \cap \mathcal{C}_i| \leq 1 \quad \text{for all} \quad 1 \leq i \leq m.
\]

Consequently,
\[
|\mathcal{A}| = |\mathcal{A} \cap \bigcup_{i=1}^{m} \mathcal{C}_i| = \sum_{i=1}^{m} |\mathcal{A} \cap \mathcal{C}_i| \leq m = \binom{n}{\lfloor n/2 \rfloor}.
\]

For $n = 1$, $\binom{n}{\lfloor n/2 \rfloor} = \binom{1}{0} = 1$,
\[
\emptyset \subset \{1\}.
\]

For $n = 2$, $\binom{n}{\lfloor n/2 \rfloor} = \binom{2}{1} = 2$,
\[
\emptyset \subset \{1\} \subset \{1, 2\}, \quad \{2\}.
\]

For $n = 3$, $\binom{n}{\lfloor n/2 \rfloor} = \binom{3}{1} = 3$,
\[
\emptyset \subset \{1\} \subset \{1, 2\} \subset \{1, 2, 3\}, \quad \{2\} \subset \{2, 3\}, \quad \{3\} \subset \{1, 3\}.
\]
For \( n = 4 \), \( \binom{n}{\lfloor n/2 \rfloor} = \binom{4}{2} = 6 \). The 6 chains can be obtained in two ways: (i) Attach a new subset at the end to each chain of the chain partition for \( n = 3 \) (this new subset is obtained by appending 4 to the last subset of the chain); (ii) delete the last subsets in all chains of the partition for \( n = 3 \) and append 4 to all the remaining subsets.

\[
\emptyset \subset \{1\} \subset \{1, 2\} \subset \{1, 2, 3\} \subset \{1, 2, 3, 4\},
\{2\} \subset \{2, 3\} \subset \{2, 3, 4\},
\{3\} \subset \{1, 3\} \subset \{1, 3, 4\},
\{4\} \subset \{1, 4\} \subset \{1, 2, 4\},
\{2, 4\},
\{3, 4\}.
\]

Note that the chain partition satisfies the properties: (i) Each chain is saturated in the sense that no subset can be added in between any two consecutive subsets; (ii) in each chain the size of the beginning subset plus the size of the ending subset equals \( n \). A chain partition satisfying the two properties is called a symmetric chain partition. The above chain partitions for \( n = 1, 2, 3, 4 \) are symmetric chain partitions.

Given a symmetric chain partition for the case \( n - 1 \); we construct a symmetric chain partition for the case \( n \): For each chain \( A_1 \subset A_2 \subset \cdots \subset A_k \) in the chain partition for the case \( n - 1 \),

- if \( k \geq 2 \), do \( A_1 \subset A_2 \subset \cdots \subset A_k \subset A_k \cup \{n\} \), and \( A_1 \cup \{n\} \subset A_2 \cup \{n\} \subset \cdots \subset A_{k-1} \cup \{n\} \);
- if \( k = 1 \), do \( A_1 \subset A_1 \cup \{n\} \).

It is clear that the chains constructed form a symmetric chain partition. In fact, the chains constructed are obviously saturated. Since \(|A_1| + |A_k| = n - 1\), then \(|A_1| + |A_k \cup \{n\}| = |A_1| + |A_k| + 1 = n\), and when \( k \geq 2 \),

\[ |A_1 \cup \{n\}| + |A_{k-1} \cup \{n\}| = |A_1| + |A_{k-1}| + 2 = |A_1| + |A_k| + 1 = n. \]

Now for each chain \( B_1 \subset B_2 \subset \cdots \subset B_l \) of the symmetric chain partition for the case \( n \), since \(|B_1| \leq |B_l|\), we must have \(|B_1| \leq n/2 \leq |B_l|\) (otherwise, if \(|B_l| < n/2\) then \(|B_1| + |B_2| < n\), or if \(|B_1| > n/2\) then \(|B_1| + |B_l| > n\). By definition of \([n/2]\) and \([n/2]\), we have

\[ |B_1| \leq [n/2] \leq [n/2] \leq |B_l|. \]

This means that \( B_1 \subset B_2 \subset \cdots \subset B_l \) contains exactly one \([n/2]\)-subset and exactly one \([n/2]\)-subset. Note that the number of \([n/2]\)-subsets of \( S \) is \( \binom{n}{[n/2]} \) and the number of \([n/2]\)-subsets of \( S \) is \( \binom{n}{[n/2]} \). It follows that the number of chains in the constructed
symmetric chain partition is

\[ \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}. \]

Thus every cluster of the power set \( P(S) \) has size less than or equal to \( \binom{n}{\lfloor n/2 \rfloor} \). The cluster \( P_{n/2}(S) \) is of size \( \binom{n}{\lfloor n/2 \rfloor} \). □

The proof of the Spencer theorem actually gives the construction of clusters of maximal size. When \( n = \text{even} \), there is only one such cluster,

\[ P_{n/2}(S) : \text{ the collection of all } \frac{n}{2}-\text{subsets of } S; \]

and when \( n = \text{odd} \), there are exactly two such clusters,

\[ P_{n-1/2}(S) : \text{ the collection of all } \frac{n-1}{2}-\text{subsets of } S, \text{ and} \]

\[ P_{n+1/2}(S) : \text{ the collection of all } \frac{n+1}{2}-\text{subsets of } S. \]

**Example 6.1.** (a) Let \( S = \{1\} \). Then \( n = 1 \) and \( \binom{1}{0} = \binom{1}{1} = 1 \). There are two clusters: \( \emptyset \) and \( \{1\} \).

(b) Let \( S = \{1, 2\} \). Then \( n = 2 \) and \( \binom{2}{1} = 2 \). There is only one cluster of maximal size: \( \{\{1\}, \{2\}\} \).

(c) Let \( S = \{1, 2, 3\} \). Then \( n = 3 \) and \( \binom{3}{1} = \binom{3}{2} = 3 \). There are two clusters of maximal size:

\[ \{\{1\}, \{2\}, \{3\}\} \text{ and } \{\{1, 2\}, \{1, 3\}, \{2, 3\}\} \]

(d) Let \( S = \{1, 2, 3, 4\} \). Then \( n = 4 \) and \( \binom{4}{2} = 6 \). There is only one cluster of maximal size:

\[ \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\} \]

(e) Let \( S = \{1, 2, 3, 4, 5\} \). Then \( n = 5 \) and \( \binom{5}{2} = \binom{5}{3} = 10 \). There are two clusters of maximal size:

\[ \left\{ \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\} \right\}, \]

\[ \left\{ \{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\} \right\}. \]

### 7 Dilworth Theorem

Let \( (X, \leq) \) be a finite partially ordered set. A subset \( A \) of \( X \) is called an **antichain** if any two elements of \( A \) are incomparable. In contrast, a **chain** is a subset \( C \) of \( X \) whose any two elements are comparable. Thus a chain is a linearly ordered subset of \( X \). It is
clear that any subset of a chain is also a chain, and any subset of an antichain is also an antichain. The important connection between chains and antichains is:

\[ |A \cap C| \leq 1 \text{ for any antichain } A \text{ and chain } C. \]

**Example 7.1.** Let \( X = \{1, 2, \ldots, 10\} \). The divisibility \( | \) makes \( X \) into a partially ordered set. The subsets

\[
\{2, 3, 5, 7\}, \{2, 5, 7, 9\}, \{3, 4, 5, 7\}, \{3, 4, 7, 10\}, \{3, 5, 7, 8\}, \{3, 7, 8, 10\},
\{4, 5, 6, 7, 9\}, \{4, 6, 7, 9, 10\}, \{5, 6, 7, 8, 9\}, \{6, 7, 8, 9, 10\}
\]

are antichains, they are actually maximal antichains; while the subsets

\[
\{1, 2, 4, 8\}, \{1, 3, 6\}, \{1, 3, 9\}, \{1, 5, 10\}, \{1, 7\}
\]

are chains and they are actually maximal chains.

Let \((X, \leq)\) be a finite poset. We are interested in partitioning \( X \) into disjoint union of antichains and partitioning \( X \) into disjoint union of chains. Let \( \mathcal{A} \) be an antichain partition of \( X \) and let \( C \) be a chain of \( X \). Since no two elements of \( C \) can be contained in any antichain in \( \mathcal{A} \), then

\[ |\mathcal{A}| \geq |C|. \]

Similarly, for any chain partition \( \mathcal{C} \) and an antichain \( A \) of \( X \), there are no two elements of \( A \) belonging to a chain of \( \mathcal{C} \), we then have

\[ |\mathcal{C}| \geq |A|. \]

**Theorem 7.1.** Let \((X, \leq)\) be a finite poset, and let \( r \) be the largest size of a chain. Then \( X \) can be partitioned into \( r \) but no fewer antichains. In other words,

\[ \min \{ |\mathcal{A}| : \mathcal{A} \text{ is an antichain partition} \} = \max \{ |\mathcal{C}| : \mathcal{C} \text{ is a chain} \}. \]

**Proof.** It is enough to show that \( X \) can be partitioned into \( r \) antichains. Let \( X_1 = X \) and let \( A_1 \) be the set of all minimal elements of \( X_1 \). Let \( X_2 = X_1 - A_1 \) and let \( A_2 \) be the set of all minimal elements of \( X_2 \). Let \( X_3 = X_2 - A_2 \) and let \( A_3 \) be the set of all minimal elements of \( X_3 \). Continuing this procedure we obtain a decomposition of \( X \) into antichains \( A_1, A_2, \ldots, A_p \). By the previous argument we always have \( p \geq r \). On the other hand, for any \( a_p \in A_p \), there is an element \( a_{p-1} \in A_{p-1} \) such that \( a_{p-1} < a_p \). Similarly, there is an element \( a_{p-2} \in A_{p-2} \) such that \( a_{p-2} < a_{p-1} \). Continuing this process we obtain a chain \( a_1 < a_2 < \cdots < a_p \). Since \( r \) is the largest size of a chain, we then have \( r \geq p \). Thus \( p = r \).

\[ \square \]

The following dual version of the theorem is known as the **Dilworth Theorem**.
Theorem 7.2 (Dilworth). Let \((X, \leq)\) be a finite poset. Let \(s\) be the largest size of an antichain. Then \(X\) can be partitioned into \(s\), but not less than \(s\), chains. In other words,
\[
\min \{ |C| : C \text{ is a chain partition} \} = \max \{ |A| : A \text{ is an antichain} \}.
\]

Proof. It suffices to show that \(X\) can be partitioned into \(s\) chains. We proceed by induction on \(|X|\). Let \(|X| = n\). For \(n = 1\), it is trivially true. Assume that \(n \geq 2\). Let \(A_{\min}\) be the set of all minimal elements of \(X\), and \(A_{\max}\) the set of all maximal elements of \(X\). Both \(A_{\min}\) and \(A_{\max}\) are maximal antichains. We divide the situation into two cases.

CASE 1. \(A_{\min}\) and \(A_{\max}\) are the only maximal antichains of \(X\). Take an element \(x \in A_{\min}\) and an element \(y \in A_{\max}\) such that \(x \leq y\) (possibly \(x = y\)). Let \(X' = X - \{x, y\}\). If \(X' = \emptyset\), then \(X = \{x, y\}\) and \(x < y\), thus \(s = 1\) and \(x < y\) is the required chain partition. Assume \(X' \neq \emptyset\), then \(X'\) has only the maximal antichains \(A_{\min} - \{x\}\) and \(A_{\max} - \{y\}\). The largest size of antichains of \(X'\) is \(s - 1\). Since \(|X'| \leq n - 1\), by induction the set \(X'\) can be partitioned into \(s - 1\) chains \(C_1, \ldots, C_{s-1}\). Set \(C_s = \{x \leq y\}\). The collection \(\{C_1, \ldots, C_s\}\) is a chain partition of \(X\).

CASE 2. The set \(X\) has a maximal antichain \(A = \{a_1, a_2, \ldots, a_s\}\) of size \(s\) such that \(A \neq A_{\min}\) and \(A \neq A_{\max}\). Let

\[
A^- = \{x \in X : x \leq a_i \text{ for some } a_i \in A\},
\]

\[
A^+ = \{x \in X : x \geq a_i \text{ for some } a_i \in A\}.
\]

The sets \(A^+\) and \(A^-\) satisfy the following properties:

1. \(A^+ \subseteq X\). (Since \(A_{\min} \not\subseteq A\), i.e., there is a minimal element not in \(A\); this minimal element cannot be in \(A^+\), otherwise, it is larger than one element of \(A\) by definition.)

2. \(A^- \subseteq X\). (Since \(A_{\max} \not\subseteq A\), i.e., there is a maximal element not in \(A\); this maximal element cannot be in \(A^-\), otherwise, it is smaller than one element of \(A\) by definition.)

3. \(A^- \cap A^+ = A\). (It is always true that \(A \subseteq A^+ \cap A^-\). For each \(x \in A^+ \cap A^-\), there exist \(a_i, a_j \in A\) such that \(a_i \leq x \leq a_j\) by definition, then \(a_i \leq a_j\), which implies \(a_i = a_j\) so that \(i = j\), thus \(x = a_i = a_j \in A\).)

4. \(A^- \cup A^+ = X\). (Suppose there is an element \(x \not\in A^- \cup A^+\), then \(x\) is neither ahead nor behind any member of \(A\), thus \(A \cup \{x\}\) is an antichain of larger size than \(A\).)

Since \(A^-\) and \(A^+\) are smaller posets having the maximal antichain \(A\) of size \(s\), then by induction, by induction \(A^-\) can be partitioned into \(s\) chains \(C_1^-, C_2^- , \ldots, C_s^-\) with the minimal elements \(a_1, a_2, \ldots, a_s\) respectively, and \(A^+\) can be partitioned into \(s\) chains \(C_1^+, C_2^+ , \ldots, C_s^+\) with the minimal elements \(a_1, a_2, \ldots, a_s\) respectively. Thus we obtain a partition of \(X\) into \(s\) chains

\[
C_1^- \cup C_1^+, \quad C_2^- \cup C_2^+, \quad \ldots, \quad C_s^- \cup C_s^+.
\]

\(\square\)
Example 7.2. Let \( X = \{1, 2, \ldots, 20\} \) be the poset with the partial order of divisibility. Then the subset \( \{1, 2, 4, 8, 16\} \) is a chain of maximal size. The set \( X \) can be partitioned into five antichains

\[
\{1\}, \quad \{2, 3, 5, 7, 11, 13, 17, 19\}, \quad \{4, 6, 9, 10, 14, 15\}, \quad \{8, 12, 18, 20\}, \quad \{16\}.
\]

However, the size of the antichain \( \{2, 3, 5, 7, 11, 13, 17, 19\} \) of size 8 is not maximal. In fact,

\[
\{4, 6, 7, 9, 10, 11, 13, 15, 17, 19\}
\]

is an antichain of size 10. The set \( X \) can be partitioned into ten chains

\[
\{1, 2, 4, 8, 16\}, \quad \{3, 6, 12\}, \quad \{5, 10, 20\}, \quad \{7, 14\}, \quad \{9, 18\},
\]

\[
\{11\}, \quad \{13\}, \quad \{15\} \quad \{17\}, \quad \{19\}.
\]

This means that \( \{4, 6, 7, 9, 10, 11, 13, 15, 17, 19\} \) is an antichain of maximal size.

Example 7.3. Let \( X = \{(i, j) \in \mathbb{Z}^2 : 0 \leq i, j \leq 3, \} \) be a poset whose partial order \( \leq \) is defined by \((i, j) \leq (k, l)\) if and only if \(i \leq k\) and \(j \leq l\). The size of the longest chain is 7. For instance,

\[
(0, 0) < (1, 0) < (1, 1) < (1, 2) < (2, 2) < (2, 3) < (3, 3)
\]

is a chain of length 7. The following collection of subsets

\[
\{(0, 0)\}, \quad \{(1, 0), (0, 1)\}, \quad \{(2, 0), (1, 1), (0, 2)\}, \quad \{(3, 0), (2, 1), (1, 2), (0, 3)\},
\]

\[
\{(3, 1), (2, 2), (1, 3)\}, \quad \{(3, 2), (2, 3)\}, \quad \{(3, 3)\}
\]

is an antichain partition of \( X \). The maximal size of antichain is 4 and the poset \( X \) can be partitioned into 4 disjoint chains:

\[
(0, 0) < (0, 1) < (0, 2) < (0, 3) < (1, 3) < (2, 3) < (3, 3),
\]

\[
(1, 0) < (1, 1) < (1, 2) < (2, 2) < (3, 2),
\]

\[
(2, 0) < (2, 1) < (3, 1),
\]

\[
(3, 0).
\]

\[
\{(0, 0), (0, 1), (0, 2), (0, 3), (1, 3), (2, 3), (3, 3)\},
\]

\[
\{(1, 0), (1, 1), (1, 2), (2, 2), (3, 2)\},
\]

\[
\{(2, 0), (2, 1), (3, 1)\},
\]

\[
\{(3, 0)\}.
\]
Finding an antichain of maximal size for a poset is a difficult problem. So far there is no canonical way to do this job.