Hilbert’s Axioms

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1 Flaws in Euclid

The description of “a point between two points, line separating the plane into two sides, a segment is congruent to another segment, and an angle is congruent to another angle,” are only demonstrated in Euclid’s Elements.

2 Axioms of Betweenness

Points on line are not unrelated. We assume that there is a ternary relation among points, named as “point B is between point A and point C,” abbreviated as

\[ A \ast B \ast C \]

Given distinct collinear points \( A, B, C, D \). We use

\[ A \ast B \ast C \ast D \]

to denote the following simultaneous relations of betweenness

\[ A \ast B \ast C, \quad A \ast B \ast D, \quad A \ast C \ast D, \quad B \ast C \ast D. \quad (1) \]

**Betweenness Axiom 1 (BA1) (Collinearity and symmetrization).** If \( A \ast B \ast C \), then \( A, B, C \) are three distinct points all lying on the same line, and \( C \ast B \ast A \).

**Betweenness Axiom 2 (BA2) (Extension).** Given two distinct points \( B \) and \( D \) on a line \( l \). There exist points \( A, C, E \) lying on line \( l \) such that \( A \ast B \ast D, B \ast C \ast D, \) and \( B \ast D \ast E \); see Figure 1.

![Figure 1: Betweenness Axiom 2](image)

**Betweenness Axiom 3 (BA3) (Uniqueness).** Let \( A, B, C \) br three distinct points on a line. Then one and only one of the three points is between the other two.

**Definition 1 (Line, segment, and ray).** The line determined by two distinct points \( A \) and \( B \) is denote by

\[ AB. \]
We also use $\overline{AB}$ to denote the set of all points incident with the line determined by points $A$ and $B$. A segment with endpoints $A$ and $B$, denoted

$$AB,$$

is the set of points $A, B$, and all points between $A$ and $B$. A ray emanating from a point $A$ to another point $B$, denoted

$$r(A, B),$$

is the set of all points on $AB$ and all points $C$ such that $A * B * C$. An open ray emanating from a point $A$ to another point $B$ is the set

$$\hat{r}(A, B) := r(A, B) \setminus \{A\}.$$

**Proposition 2.1.** For any two distinct points $A$ and $B$,

$$AB = r(A, B) \cap r(B, A), \quad \overline{AB} = r(A, B) \cup r(B, A).$$

*Proof.* Note that $AB \subseteq r(A, B) \cap r(B, A)$ by definition of segment and ray. For each point $P \in r(A, B) \cap r(B, A)$, we have $P \in r(A, B)$ and $P \in r(B, A)$. Suppose $P \notin AB$. By definition of ray, we have $A * B * P$ by $P \in r(A, B)$ and $P * A * B$ by $P \in r(B, A)$. Then $A, B, P$ are three distinct collinear points by BA1. This is contradictory to BA3 that there is only one point of the three $A, B, P$ between the other two.

It is clear that $r(A, B) \cup r(B, A) \subseteq \overline{AB}$. For each $P \in \overline{AB}$, if $P \in AB$, it is clear that $P \in r(A, B) \cup r(B, A)$. Assume $P \notin AB$, then $A, B, P$ are three distinct points by BA1, and one of them is between the other two by BA3. Since $P$ is not between $A$ and $B$, we have either $A$ is between $B$ and $P$ or $B$ is between $A$ and $P$. In the formal case, we have $P \in r(B, A)$; in the latter case, we have $P \in r(A, B)$. Hence $P \in r(A, B) \cup r(B, A)$. Hence $P \in r(A, B) \cup r(B, A)$. \qed

**Definition 2** (Same side and opposite side). Two points $A, B$ not on a line $l$ are said to be on the same side of $l$ if $A = B$ or the segment $AB$ does not meet $l$. Two points $A, B$ not on a line $l$ are said to be on opposite sides of $l$ if $AB$ does not meet $l$.

**Betweenness Axiom 4** (BA4) (Plane separation). Let $A, B, C$ be three distinct points not on a line $l$.

(i) If $A, B$ are on the same side of $l$ and $B, C$ are on the same side of $l$, then $A, C$ are on the same side of $l$.

(ii) If $A, B$ are on opposite sides of $l$ and $B, C$ are on opposite sides of $l$, then $A, C$ are on the same side of $l$.

The relation of being on the same side of a fixed line $l$ is an equivalence relation on the set of points not on the line $l$, since it is reflexive, symmetric, and transitive by definition and Betweenness Axiom 4(i). Each equivalence class is called an open half-plane bounded by $l$. For each point $P$ not on $l$, we denoted by

$$\hat{H}(l, P)$$

the open half-plane that contains $P$. The set

$$H(l, P) := \hat{H}(l, P) \cup l$$

is called a half-plane (or closed half-plane) bounded by $l$.

**Corollary 2.2.** For each line $l$ there are exactly two half-planes bounded by $l$.

(iii) If $A, B$ are on opposite sides of $l$ and $B, C$ are on the same side of $l$, then $A, C$ are on opposite sides of $l$. 

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Proof. Let \( A, B \) be two points on opposite sides of a line \( l \). We have two distinct half-planes \( H(l, A) \) and \( H(l, B) \). Given an arbitrary point \( C \) not on \( l \). If \( A, C \) are on the same side of \( l \), then \( H(l, C) = H(l, A) \). If \( A, C \) are on opposite sides, then \( B, C \) are on the same side of \( l \) by Betweenness Axiom 4(ii). Thus \( H(l, C) = H(l, B) \). Therefore there are at most two half-planes bounded by \( l \).

Given a point \( B \) on \( l \) and a point \( D \) not on \( l \). By Betweenness Axiom 2 there exist points \( A, C, E \) such that \( A \star B \star D, B \star C \star D \) and \( B \star D \star E \). Then \( A, D \) are on opposite sides of \( l \). So there are at least two half-planes bounded by \( l \).

Proposition 2.3 (Linearity rules). Let \( A, B, C, D \) be distinct points on a line \( l \). Then
\[
\begin{align*}
(a) & \quad A \star B \star C, A \star C \star D \Rightarrow A \star B \star C \star D, \\
(b) & \quad B \star C \star D, A \star B \star D \Rightarrow A \star B \star C \star D, \\
(c) & \quad A \star B \star C, B \star C \star D \Rightarrow A \star B \star C \star D. 
\end{align*}
\]

Proof. (a) Pick a point \( E \) outside \( l \) and make line \( \overline{EC} \); see Figure 2. Then \( C \) is the unique intersection of \( l \) and \( \overline{EC} \). The points \( A, B \) must be on the same side of line \( \overline{EC} \). (Otherwise, \( AB \) meets \( \overline{EC} \) at \( C \); we then have \( A \star C \star B \), which contradicts \( A \star B \star C \).) Since \( A \star C \star D \), then \( A, D \) are on opposite sides of \( \overline{EC} \). Hence \( B, D \) are on opposite sides of \( \overline{EC} \) by Corollary 2.2, i.e., \( BD \) meets \( \overline{EC} \) at \( C \). We then obtain \( B \star C \star D \).

Draw line \( \overline{EB} \); the point \( B \) is the unique intersection of \( l \) and \( \overline{EB} \); see Figure 2. Since \( A \star B \star C \), then \( A, C \) are on opposite sides of \( \overline{EB} \). Since \( B \star C \star D \), we must have \( C, D \) on the same side of \( \overline{EB} \). (Otherwise \( B \) would be between \( C \) and \( D \), contradicting to \( B \star C \star D \).) Thus \( A, D \) are on opposite sides of \( \overline{EB} \) by Corollary 2.2, i.e., \( AD \) meets \( \overline{EB} \) at \( B \) between \( A \) and \( D \). We then obtain \( A \star B \star D \).

(b) is similar to (a) by reversing the order.

(c) Note that \( A, B, C \) are distinct and \( B, C, D \) are distinct. If \( A = D \), then \( B \star C \star D \) becomes \( B \star C \star A \), which is contradictory to \( A \star B \star C \). So \( A, B, C, D \) are distinct.

Pick a point \( E \) outside \( l \) and draw the line \( \overline{EC} \). Since \( B \star C \star D \), then \( B, D \) are on opposite sides of \( \overline{EC} \) by definition. Likewise, \( A \star B \star C \) implies that \( A, B \) are on the same side of \( \overline{EC} \). (Otherwise, \( A, B \) are on opposite sides of \( \overline{EC} \), i.e., \( AB \) meet \( \overline{EC} \) at \( C \); so \( A \star C \star B \), contradicting to \( A \star B \star C \).) It follows from Corollary 2.2 that \( A, D \) are on opposite sides of \( \overline{EC} \). Hence \( AD \) meets \( \overline{EC} \) at \( C \) between \( A \) and \( D \), i.e., \( A \star C \star D \).

Definition 3 (Strict total order). A binary relation \( \prec \) on a set \( X \) is called a strict total order if

\( \text{(TO1) Irreflexivity: } x \not\prec x \text{ for all } x \in X; \)
\( \text{(TO2) Transitivity: if } x \prec y \text{ and } y \prec z \text{ then } x \prec z; \)
\( \text{(TO3) Completeness: either } x \prec y \text{ or } y \prec x \text{ but not both for all } x, y \in X \text{ with } x \neq y. \)

For a strict total order on \( X \), the relation \( \leq \), defined on \( X \) by \( x \leq y \) if \( x = y \) or \( x \prec y \), is called a total order. For an order relation, we also write \( x \prec y \) and \( x \preceq y \) as \( y \succ x \) and \( y \succeq x \) respectively. The set \( X \) with a total order is said to be totally ordered.
Proposition 2.4 (Strict total order of line). For each line \( l \) with two distinct points \( A, B \) there exists a unique total order on \( l \) such that \( A \prec B \) and if \( C \neq D \neq E \) then either \( C \prec D \prec E \) or \( E \prec D \prec C \).

But not both.

Proof. Define \( A \prec B \). For each point \( P \) of \( l \) other than \( A, B \), we define

1. \( P \prec A \) and \( P \prec B \) if \( P \neq A \neq B \),
2. \( A \prec P \) and \( P \prec B \) if \( A \neq P \neq B \),
3. \( A \prec P \) and \( B \prec P \) if \( A \neq B \neq P \).

For any two distinct points \( P, Q \) other than \( A \) and other than \( B \), we define

\[ P \prec Q \text{ if one of the following holds:} \]

(I) \( P \neq Q \neq A \neq B \), (II) \( P \neq A \neq Q \neq B \), (III) \( P \neq A \neq B \neq Q \), (IV) \( A \neq P \neq Q \neq B \), (V) \( A \neq P \neq B \neq Q \), (VI) \( A \neq B \neq P \neq Q \). We claim that \( \prec \) is a strict total order on \( l \).

It is clear that \( \prec \) satisfies irreflexive and completeness. For transitivity, let \( P \prec Q \) and \( Q \prec R \), we claim \( P \prec R \). If \( \{P, Q, R\} \cap \{A, B\} \neq \emptyset \), we clearly have \( P \prec R \) by definition of \( \prec \). If \( \{P, Q, R\} \cap \{A, B\} = \emptyset \), we verify the six cases.

Case I. \( P \neq Q \neq A \neq B \).

(I.1) \( Q \neq R \neq A \neq B \): Since \( P \neq Q \neq A \neq R \), then \( P \neq Q \neq R \neq A \) by Proposition 2.3(b). Hence \( P \prec R \) by definition.

(I.2) \( Q \neq A \neq R \neq B \): Since \( P \neq A \neq R \neq Q \), then \( P \neq Q \neq A \neq R \) by Proposition 2.3(c). Hence \( P \prec R \) by definition.

(I.3) \( Q \neq A \neq B \neq R \): Since \( P \neq A \neq B \neq R \), then \( P \neq A \neq B \neq R \) by Proposition 2.3(c). By definition \( P \prec R \).

Case II. \( P \neq A \neq Q \neq B \).

(II.1) \( A \neq Q \neq R \neq B \): Since \( P \neq A \neq B \neq R \), then \( P \neq A \neq R \neq B \) by Proposition 2.3(b). By definition \( P \prec R \).

(II.2) \( A \neq Q \neq B \neq R \): Since \( P \neq A \neq B \neq R \), then \( P \neq A \neq B \neq R \) by Proposition 2.3(c). By definition \( P \prec R \).

Case III. \( P \neq A \neq B \neq Q \).

(III.1) \( A \neq B \neq Q \neq R \): Since \( P \neq A \neq B \neq R \), then \( P \neq A \neq B \neq R \) by Proposition 2.3(c). By definition \( P \prec R \).

Case IV. \( A \neq P \neq Q \neq B \).

(IV.1) \( A \neq Q \neq R \neq B \): Since \( A \neq P \neq Q \neq R \), then \( A \neq P \neq Q \neq R \) by Proposition 2.3(a). Since \( P \neq Q \neq B \neq R \), then \( P \neq Q \neq R \neq B \) by Proposition 2.3(b). We then have \( A \neq P \neq R \neq B \). Thus \( A \neq P \neq R \neq B \) by Proposition 2.3(c). By definition \( P \prec R \).

(IV.2) \( A \neq Q \neq B \neq R \): Since \( A \neq P \neq B \neq R \), then \( A \neq P \neq B \neq R \) by Proposition 2.3(a). By definition \( P \prec R \).

Case V. \( A \neq P \neq B \neq Q \).

(V.1) \( A \neq B \neq Q \neq R \): Since \( A \neq P \neq B \neq R \), then \( A \neq P \neq B \neq R \) by Proposition 2.3(a). By definition \( P \prec R \).

Case VI. \( A \neq B \neq P \neq Q \).

(VI.1) \( A \neq B \neq Q \neq R \): Since \( B \neq P \neq Q \neq R \), then \( B \neq P \neq Q \neq R \) by Proposition 2.3(a). Since \( B \neq P \neq R \neq A \neq B \neq R \), then \( A \neq B \neq P \neq R \) by Proposition 2.3. By definition \( P \prec R \).
Proposition 2.5 (Line separation). Let $A, B, O$ be three distinct points such that $A \neq O \neq B$. Then

$$r(O, A) \cap r(O, B) = \{O\}, \quad r(O, A) \cup r(O, B) = \overline{AB}.$$ 

If $P \in \overline{AB}$, then either $P \in r(O, A)$ or $P \in r(O, B)$. The rays $r(O, A)$ and $r(O, B)$ are said to be opposite each other.

Proof. Let $\prec$ be the strict total order on the line $l$ such that $A \prec B$. By definition of the total order $\preceq$, the rays $r(O, A)$, $r(O, B)$, and the segment $AB$, we have

$$r(O, A) = \{P \in l : P \preceq O\}, \quad r(O, B) = \{P \in l : O \preceq P\}, \quad AB = \{P \in l : A \preceq P \preceq B\}.$$ 

Then $r(O, A) \cap r(O, B) = \{O\}$ and $r(O, A) \cup r(O, B) = \overline{AB}$ by the total ordering property of $\prec$.

Corollary 2.6 (Line separation). Let $l, m$ be two distinct lines intersecting at a point $O$. Let $\prec$ be a strict total order on $l$. Then the two sets

$$\hat{r}(O, -) := \{P \in l : P \prec O\}, \quad \hat{r}(O, +) := \{P \in l : O \prec P\}$$

are on opposite sides of $m$. We also call them on opposite sides of $O$ on $l$.

Proof. Let $A, B$ be two distinct points on $l$. If $A, B \in \hat{r}(O, -)$, i.e., $A \prec O, B \prec O$, then for all $P$ between $A$ and $B$, we have either $A \prec P \prec B$ or $B \prec P \prec A$. In either case we have $P \prec O$ by transitivity. So $AB$ is contained in $\hat{r}(O, -)$. Clearly, $AB$ does not meet $m$ (since $O$ is the unique intersection of $l$ and $m$). Hence $A, B$ are on the same side of $m$ by definition. Likewise, if $A, B \in \hat{r}(O, +)$, i.e., $O \prec A, O \prec B$, then $A, B$ are on the same side of $m$. If $A \prec O \prec B$ or $B \prec O \prec A$, then in either case $AB$ meets $m$ at $O$ between $A$ and $B$; so $A, B$ are on opposite sides of $m$ by definition.

Theorem 2.7 (Pasch’s Theorem). Let $A, B, C$ be distinct points of not collinear. Let $l$ be a line meeting $AB$ at a point $D$ between $A$ and $B$. Then one and only one of the three holds: (i) $l$ meets $AC$ at a point between $A$ and $C$, (ii) $l$ meets $BC$ at a point between $B$ and $C$, (iii) $l$ meets both $AC$ and $BC$ at a point $C$.

Intuitively, this theorem says that if a line “goes into” a triangle through one side then it must “come out” through another side.

![Figure 3: A line passes through a triangle](image)

Proof. The points $A, B$ are on the opposite sides of the line $l$. If $C$ is on $l$, then $l$ does not meet $AC$ between $A$ and $C$, otherwise $l = \overline{AC}$; and $l$ does not meet $BC$ between $B$ and $C$. If $C$ is not on $l$, then either $A, C$ are on the same side of $l$, or $B, C$ are on the same side of $l$, but not both. In the formal case, then $B, C$ are the opposite sides of $l$. Thus $l$ meets $BC$ at a point between meets $B$ and $C$, and is disjoint from $AC$. In the latter case, $l$ meets $AC$ at a point between $A$ and $C$, and is disjoint from $BC$. 

\[\]
Definition 4 (Interior of angle). Given points \( A, O, B \) not collinear. The interior of an angle \( \angle AOB \), denoted \( \angle AOB \), is the set of points \( P \) such that \( P, A \) are on the same side of line \( \overline{OB} \), and \( P, B \) are on the same side of line \( \overline{OA} \), in other words,
\[
\angle AOB := \overline{H(\overline{OB}, A)} \cap \overline{H(\overline{OA}, B)};
\]
see Figure 4. We also define
\[
\angle AOB := \overline{H(\overline{OB}, A)} \cap \overline{H(\overline{OA}, B)}.
\]
It is convenient to consider a closed half-plane as a flat angle.

![Figure 4: Interior of an angle](image)

Proposition 2.8 (Between-Cross Lemma). Given an angle \( \angle AOB \) and a point \( P \) on \( AB \). Then \( P \) belongs to \( \angle AOB \) if and only if \( A \ast P \ast B \).

Proof. \( \Rightarrow \): The point \( P \) belongs to \( \angle AOB \). By definition \( P, B \) are on the same side of line \( \overline{OA} \). Suppose \( P \ast A \ast B \). Then \( P, B \) are opposite sides of \( \overline{OA} \), since \( PB \) meets \( \overline{OA} \) at \( A \) between \( P \) and \( B \). This is a contradiction. Likewise, \( A \ast B \ast P \) leads to a similar contradiction. Then we must have \( A \ast P \ast B \) by trichotomy of betweenness.

\( \Leftarrow \): We have \( A \ast P \ast B \). Note that line \( \overline{AB} \) meets line \( \overline{OB} \) at the unique point \( B \). Then \( AP \) does not meet \( \overline{OB} \). So \( A, P \) are on the same side of \( \overline{OB} \). Likewise, points \( B, P \) are on the same side of \( \overline{OB} \). Hence by definition \( P \) belongs to \( \angle AOB \). \qed

Proposition 2.9. Let \( P \) be a point in \( \angle AOB \). Then
\( (a) \) The open ray \( \overline{r(O, P)} \) is contained in \( \angle AOB \).
\( (b) \) The opposite ray to \( \overline{r(O, P)} \) is disjoint from \( \angle AOB \). See the left of Figure 5.
\( (c) \) If \( B \ast O \ast B' \), then \( A \) belongs to \( \angle POB' \).

![Figure 5: Property of interior of an angle and Crossbar Theorem](image)

Proof. \( (a) \) Let \( Q \) be a point on the open ray \( \overline{r(O, P)} \). It is clear that \( PQ \) is disjoint from \( \overline{OA} \) (since the intersection of the two lines \( \overline{PQ}, \overline{OA} \) are the unique point \( O \)). This means that \( P, Q \) are on the same side of \( \overline{OA} \) by definition. Since \( P \in \angle AOB \), i.e., \( P, B \) are on the same side of \( \overline{OA} \), then \( B, Q \) are on the same side of \( \overline{OA} \). Likewise, \( A, Q \) are on the same side of \( \overline{OB} \). Thus \( Q \) is an interior point of \( \angle AOB \).
(b) Let \( P \) be a point on the opposite ray of \( r(O, P) \); see the left of Figure 5. Then \( P, P' \) are on opposite sides of \( \overline{OB} \). Since \( A, P \) are the same side of \( \overline{OB} \), then \( A, P' \) are on the opposite sides of \( \overline{OB} \). Thus \( P' \) is not an interior point of \( \angle AOB \) by definition.

(c) Note that \( P, A \) are on the same side of \( \overline{OB} \) (since \( \overline{OB} = \overline{OB} \) and \( P \in \angle AOB \)). We claim that \( A, B' \) are on the same side of \( \overline{OP} \). If so, we have \( A \in \angle POB' \) by definition.

Suppose that \( A, B' \) are on opposite sides of \( \overline{OP} \), i.e., \( \overline{OP} \) intersects \( AB' \) at \( C \) between \( A \) and \( B' \). Then \( A \ast C \ast B' \) and \( C \in \angle AOB' \) by Proposition 2.8. Since \( C \in \overline{OP} \) and \( C \neq O \), we have either \( C \in \hat{r}(O, P) \) or \( C \in \hat{r}(O, P') \).

If \( C \in \hat{r}(O, P) \), then \( P \in \hat{r}(O, C) \), which is contained in \( \angle AOB' \) by part (a). Thus \( P, B' \) are on the same side of \( \overline{OA} \) (since \( P \in \angle AOB' \)). Since \( P, B \) are the same side of \( \overline{OA} \), we see that \( B, B' \) are on the same side of \( \overline{OA} \). This is a contradiction.

If \( C \in \hat{r}(O, P') \), then \( P' \in \hat{r}(O, C) \), which is contained in \( \angle AOB' \) by part (a). Thus \( A, P' \) are on the same side of \( \overline{OB} = \overline{OB} \) by definition. Since \( P', P \) are on opposite sides of \( \overline{OB} \), we see that \( A, P \) are on the opposite sides of \( \overline{OB} \). This is a contradiction. \( \square \)

**Definition 5 (Between rays).** A ray \( r(O, P) \) is **between** two non-opposite rays \( r(O, A) \) and \( r(O, B) \) if \( P \) is in the interior of \( \angle AOB \) (independent of the choice of \( P \) on the ray \( r(O, P) \)).

**Proposition 2.10 (Crossbar Theorem).** If a ray \( r(O, P) \) is between two rays \( r(O, A) \) and \( r(O, B) \), then \( r(O, P) \) intersects \( AB \) at \( C \) between \( A \) and \( B \). See the right of Figure 5. The interior of \( \angle AOB \) is a disjoint union of interiors \( \hat{\angle AOP}, \hat{\angle BOP}, \) and open ray \( \hat{r}(O, P) \).

**Proof.** Note that \( B, B' \) are on opposite sides of \( \overline{OP} \), and \( B', A \) are on the same side of \( \overline{OP} \); see the left of Figure 5. Then \( A, B \) are on opposite sides of \( \overline{OP} \). Thus \( \overline{OP} \) intersects \( AB \). Since the ray \( r(O, P') \) (opposite to the ray \( r(O, P) \)) is disjoint from the interior of \( \angle AOB \), and since the open segment \( (A, B) \) is contained in the interior \( \angle AOB \), then the open ray \( \hat{r}(O, P) \) must intersect \( AB \) at \( C \) between \( A \) and \( B \); see the right of Figure 5. \( \square \)

**Definition 6 (Interior of triangle).** The **interior** of a triangle \( \triangle ABC \) is the intersection of interiors of its three angles, denoted \( \hat{\triangle ABC} \). The **boundary** of \( \triangle ABC \) is the union of the three sides, i.e.,

\[
\partial \hat{\triangle ABC} := \overline{AB} \cup \overline{AC} \cup \overline{BC}.
\]

We also use \( \hat{\triangle ABC} \) to denote the union of the interior and the boundary of \( \triangle ABC \).

**Proposition 2.11.** Given a triangle \( \triangle ABC \) and \( O \in \hat{\triangle ABC} \). Let \( l = \overline{AB}, m = \overline{AC}, n = \overline{BC} \). Then

(a) \( \hat{\triangle ABC} = \hat{H}(l, O) \cap \hat{H}(m, O) \cap \hat{H}(n, O) \).

(b) Any ray \( r(O, P) \) meets the boundary of \( \hat{\triangle ABC} \) at a unique point \( Q \).

**Proof.** (a) Trivial by \( \hat{\angle ABC} = \hat{H}(l, O) \cap \hat{H}(n, O) \) and other two interiors of angles.

(b) Let \( l = \overline{OP} \). The line \( \overline{OA} \) meet \( BC \) at \( D \) between \( B \) and \( C \). We then have \( A \ast O \ast D \), the open ray \( \hat{r}(D, O) \) is contained in \( \hat{H}(n, O) \), and its opposite half-line is contained in the opposite side of \( \hat{n} \). So \( (A, D) := AD \setminus \{A, D\} \) is contained in the interior of \( \hat{\triangle ABC} \).

Case 1. \( \overline{OP} = \overline{OA} \). Then \( Q = A \) if \( A \ast P \ast O; Q = D \) if \( A \ast O \ast P \). See the left of Figure 6.

Case 2. \( \overline{OP} \neq \overline{OA} \). The line \( \overline{OA} \) separates the triangle \( \triangle ABC \) into two triangles \( \hat{\triangle ABD} \) and \( \hat{\triangle ACD} \). Since \( \overline{OP} \) meets \( AD \) at \( O \) between \( A \) and \( D \), then \( \overline{OP} \) meets the boundary of \( \hat{\triangle ABD} \) at a unique point \( E \) and the boundary of \( \hat{\triangle ACD} \) at a unique point \( F \). Moreover, \( E \in AB \cup BD \) and \( F \in AC \cup CD \). If \( r(O, P) = r(O, E) \), then \( Q = E \). If \( r(O, P) = r(O, F) \), then \( Q = F \). \( \square \)
3 Axioms of Segment Congruence

Segments are not unrelated. We assume that there is a binary relation between segments, named as “segment $AB$ is congruent to segment $CD$,” abbreviated as $AB \cong CD$.

**Congruence Axiom 1 (CA1).** Given two distinct points $A, B$, and a ray $r$ emanating from a point $A'$. There exists exactly one point $B'$ on $r$ such that $B' \neq A'$ and $AB \cong A'B'$. Moreover, if $r = r(A, B)$, then $B' = B$; if $r = r(B, A)$, then $B' = A$.

**Proposition 3.1.** (1) $AB \cong AB$, $AB \cong BA$. (2) If $AB \cong CD$, then $CD \cong AB$.

**Proof.** (1) It follows from the latter part of CA1. (2) Let $CD \cong AB'$, where $B'$ is a point on the ray $r(A, B)$. Then $AB \cong AB'$ by transitivity. Hence $B' = B$ by CA1. We then have $CD \cong AB$.

**Congruence Axiom 2.** (CA2) If $AB \cong CD$ and $CD \cong EF$, then $AB \cong EF$.

**Proposition 3.2 (Segment subtraction).** Given $A \ast B \ast C$ and $A' \ast B' \ast C'$. If $AB \cong A'B'$ and $AC \cong A'C'$, then $BC \cong B'C'$.

**Proof.** Let $BC \cong B'P$, where $P$ is a point on the ray $r(A', B')$. Then $AC \cong A'P$ by CA2. Since $AC \cong A'C'$, then $A'P \cong A'C'$ by CA2. Thus $P = C'$ by CA1. So $BC \cong B'C'$.

**Proposition 3.3 (Betweenness preserving by congruence of segments).** Given $AC \cong A'C'$ and $A \ast B \ast C$. Then there exists a unique point $B'$ between $A'$ and $C'$ such that $AB \cong A'B'$ and $BC \cong B'C'$.

**Proof.** Let $AB \cong A'B'$, where $B'$ is the unique point on the ray $r(A', C')$. Let $BC \cong B'P$, where $P$ is the unique point such that $A' \ast B' \ast P$. Since $AB \cong A'B'$ and $BC \cong B'P$, then $AC \cong A'P$ by CA3. Since $AC \cong A'C'$, then $P = C'$ by CA2. So $A' \ast B' \ast C'$.
Proposition 3.4 (Congruence of lines). For any two lines \( l \) and \( l' \), there exists a one-to-one correspondence \( f : l \to l' \) such that \( AB \cong f(A)f(B) \) for distinct points \( A, B \in l \) and if \( A \ast B \ast C \) then 
\[
f(A) \ast f(B) \ast f(C).
\]

Proof. Pick two points \( O \in l \) and \( O' \in m \). We have open rays \( \tilde{r}(O, -), \tilde{r}(O, +) \) of \( l \) and open rays \( \tilde{r}'(O', -), \tilde{r}'(O', +) \) of \( l' \). Define \( f(O) = O' \). For each \( P \in \tilde{r}(O, -) \), there exists a unique point \( P' \in \tilde{r}'(O', -) \) such that \( OP \cong O'P' \); define \( f(P) = P' \). For each \( Q \in \tilde{r}(O, +) \), there exists a unique point \( Q' \in \tilde{r}'(O', +) \) such that \( OQ \cong O'Q' \); define \( f(Q) = Q' \). We then have a map \( f : l \to l' \). Likewise we have a map \( f' : l' \to l \) defined in similar fashion. Then \( f' \circ f : l \to l \) and \( f \circ f' : l' \to l' \) are identity maps. So \( f \) and \( f' \) are bijections.

Given distinct points \( A, B \in l \). If \( A \ast B \ast O \) or \( B \ast A \ast O \), then either \( A, B \in \tilde{r}(O, -) \) or \( A, B \in \tilde{r}(O, +) \); thus \( AB \cong f(A)f(B) \) by segment subtraction. If \( A \ast O \ast B \), then \( AB \cong f(A)f(B) \) by segment addition.

If \( A \ast B \ast C \) on \( l \), then there exists a unique point \( B'' \) be between \( f(A) \) and \( f(C) \) such that \( AB \cong f(A)B'' \) and \( B''f(C) \) by the congruence of preserving betweenness. Since \( AB \cong f(A)f(B) \), we must have \( f(B) = B'' \). Hence \( f(A) \ast f(B) \ast f(C) \). \( \square \)

Definition 7 (Linear order of segments). For segments \( AB, CD \), if there exists a point \( E \) between \( C \) and \( D \) such that \( AB \cong CE \), we write \( AB < CD \) or \( CD > AB \).

Theorem 3.5 (Strict total order of segments). For two segments \( AB \) and \( CD \), one and only one of the three holds: \( AB < CD \), \( AB \cong CD \), \( AB > CD \) (trichotomy). Moreover,

(a) If \( AB \cong CD \) and \( CD < EF \), then \( AB < EF \).
(b) If \( AB < CD \) and \( CD \cong EF \), then \( AB < EF \).
(c) If \( AB < CD \) and \( CD < EF \), then \( AB < EF \).

Proof. Given segments \( AB \) and \( CD \). Let \( AB \cong CE \), where \( E \) is the unique point on the ray \( r(C, D) \). We have one and only one of the three: \( C \ast E \ast D \), \( E = D \), \( C \ast D \ast E \). These are exactly the three cases: \( AB < CD \), \( AB \cong CD \), \( AB > CD \).

(a) Let \( P \) be a point such that \( E \ast P \ast F \) and \( CD \cong EP \). Then \( AB \cong EP \) by CA2. Thus \( AB < EF \) by definition.

(b) Let \( P \) be a point such that \( C \ast P \ast D \) and \( AB \cong CP \) by definition. Then there exists a point \( Q \) such that \( E \ast Q \ast F \) and \( CP \cong EQ \) by Proposition 3.3 (congruence of preserving betweenness). Then \( AB \cong EQ \) by CA2. Thus \( AB < EF \) by definition.

(c) Let \( P \) be such that \( AB \cong CP \) and \( C \ast P \ast D \). Let \( R \) be such that \( E \ast R \ast F \) and \( CD \cong ER \). Then there exists a point \( Q \) such that \( E \ast Q \ast R \) and \( CP \cong EQ \). Thus \( AB \cong EQ \). \( E \ast Q \ast R \ast F \), and of course \( E \ast Q \ast F \). Therefore \( AB < EF \). \( \square \)

4 Axioms of Angle and Triangle Congruence

Angles are not unrelated. We assume that there is a binary relation between angles, named as “\( \angle ABC \) is congruent to \( \angle DEF \),” abbreviated as

\[
\angle ABC \cong \angle DEF.
\]

Congruence Axiom 4 (CA4). Given an angle \( \angle AOB \) and a ray \( r(O', A') \), where the rays \( r(O, A), r(O, B) \) are not opposite. There exists a unique ray \( r(O', B') \) on each side of the line \( \overline{OA} \) such that \( \angle A'O'B' \cong \angle AOB \). Moreover, if \( r(O', A') = r(O, A) \) and the side of \( \overline{OA} \) is the half-plane \( H(\overline{OA}, B) \), then \( r(O, B') = r(O, B) \). If \( r(O', A') = r(O, B) \) and the side of \( \overline{OB} \) is the half-plane \( H(\overline{OB}, A) \), then \( r(O, B') = r(O, A) \).
**Congruence Axiom 5 (CA5).** If $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$.

It is easy to see that angle congruence is reflexive, symmetric, and transitive. So angle congruence is an equivalence relation on angles.

**Definition 8 (Congruence of triangles).** A triangle is a collection of three non-collinear points $A, B, C$ together with three segments $AB, AC, BC$ (called sides), and three angles $\angle ABC, \angle ACB, \angle BAC$, denoted $\triangle ABC$. The point set of $\triangle ABC$, denoted by the same notation, is

$$\triangle ABC := \angle ABC \cap \angle ACB \cap \angle BAC.$$ The interior of $\triangle ABC$ is the point set

$$\triangle ABC := \operatorname{Int}(\triangle ABC).$$

Two triangles are said to be congruent if there is a one-to-one correspondence between their vertices such that the corresponding sides are congruent and the corresponding angles are congruent. More specifically, if a triangle with vertices $A, B, C$ is congruent to a triangle with vertices $A', B', C'$ by the one-to-one correspondence $A \leftrightarrow A'$, $B \leftrightarrow B'$, $C \leftrightarrow C'$, written

$$\triangle ABC \cong \triangle A'B'C'$$

then $AB \cong A'B'$, $AC \cong A'C'$, $BC \cong B'C'$, and $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, $\angle C \cong \angle C'$.

**Congruence Axiom 6 (Side-angle-side) (SAS).** If two sides and the included angle of a triangle are congruent respectively to two sides and the included angle of another triangle, then we say that the two triangles are congruent. More precisely, given two triangles with vertices $A, B, C$ and vertices $A', B', C'$. If $AB \cong A'B'$, $AC \cong A'C'$, and $\angle A \cong \angle A'$, then $\triangle ABC \cong \triangle A'B'C'$.

**Corollary 4.1.** Given a triangle $\triangle ABC$ and a segment $A'B' \cong AB$. Then there exists a unique point $C'$ on each side of the line $\overline{AB}$ such that $\triangle ABC \cong \triangle A'B'C'$.

*Proof.* Choose a side of line $\overline{AB}$. There exists one and only one ray $r(A', P)$ such that $\angle A'B'P \cong \angle BAC$ by CA4. Then there exists a unique point $C'$ on $r(A', P)$ such that $A'C' \cong AC$ by CA1. Thus $\triangle B'C' \cong \triangle BAC$ by SAS. \square

**Proposition 4.2.** Given a triangle $\triangle ABC$. If $AB \cong AC$, then $\angle B \cong \angle C$.

*Proof.* Consider the one-to-one correspondence $A \leftrightarrow A$, $B \rightarrow C$, $C \rightarrow B$. We have $AB \cong AC$, $\angle BAC \cong \angle CAB$, $AC \cong AB$. Then $\triangle ABC \cong \triangle ACB$ by SAS. Thus $\angle B \cong \angle C$ by definition of congruence of triangles. \square

**Definition 9 (Supplementary angle, opposite angle, right angle).** Supplementary angles and opposite angles are defined as before. A right angle is an angle which is congruent to its supplement. A closed half-plane is not an angle by our definition of angles; it is convenient to call it a flat angle.

**Proposition 4.3 (Supplementary, opposite, right angle congruence rules).** (a) Supplements of congruent angles are congruent.

(b) Opposite angles are congruent each other.

(c) Any angle congruent to a right angle is a right angle.
Proof. Given two congruent angles $\angle AOB \cong \angle A'O'B'$. Pick a point $C$ on the opposite ray of $r(O, A)$ with $C \neq O$. Pick a point $C'$ on the opposite ray of $r(O', A')$ with $C' \neq O$. We may assume $OA \cong O'A'$, $OB \cong O'B'$, $OC \cong O'C'$. See Figure 8.

(a) We need to show $\angle BOC \cong \angle B'O'C'$. Since $OA \cong O'A'$, $\angle AOB \cong \angle A'O'B'$, $OB \cong O'B'$, then $\triangle AOB \cong \triangle A'O'B'$ by SAS. Then $AC \cong A'C'$ by CA3; $AB \cong A'B'$ and $\angle BAC \cong \angle B'A'C'$ by definition of congruence triangles. Thus $\triangle BAC \cong \triangle B'A'C'$ by SAS. Since $OC \cong O'C'$, $\triangle OCB \cong \triangle O'C'B'$ and $CB \cong C'B'$, then $\triangle OCB \cong \triangle O'C'B'$ by SAS. We see $\angle BOC \cong \angle B'O'C'$.

(b) Consider opposite angles $\angle AOB$ and $\angle COD$ in the left of Figure 8. Both are supplementary to $\angle BOC$. So $\angle AOB \cong \angle COD$ by (a).

(c) Let $\angle AOB$ be a right angle. Need to show that $\angle A'O'B'$ is a right angle. Notice that $\angle AOB \cong \angle BOC$ by definition of right angles, $\angle B'O'C' \cong \angle BOC$ by (a), and $\angle AOB \cong \angle A'O'B'$ by given condition. Then $\angle A'O'B' \cong \angle B'O'C'$ by transitivity. This means that $\angle A'O'B'$ is a right angle. \hfill \square

**Proposition 4.4 (Existence of perpendicular line).** For each line $l$ and each point $P$ not on $l$, there exists a unique line $m$ through $P$ perpendicular to $l$.

![Figure 9: Construction of perpendicular lines](image)

Proof. Pick two distinct points $A, B$ on $l$. Draw segment $AP$. Then there exists a unique ray $r(A, C)$ on the opposite side of line $l$ such that $\angle BAP \cong \angle BAC$. Mark a point $P'$ on the ray $r(A, C)$ such that $AP \cong AP'$. Draw line $\overline{PP'} = m$. We claim that $m \perp l$. See Figure 9.

If $A, P, P'$ are collinear, then $A$ is the intersection of lines $\overline{AB}$ and $\overline{PP'}$. Clearly, $\angle BAP$ and $BAP'$ are congruent supplementary angles. So they are right angles and $m \perp l$.

Assume that $A, P, P'$ are not collinear. Since $P, P'$ on opposite sides of $l$, then $r(P, P')$ intersects $l$ at a unique point $Q$. We have triangles $\triangle APQ$ and $\triangle AP'Q$. Since $AP \cong AP'$, $\angle PAQ \cong \angle P'AQ$, $AQ \cong AQ$, then $\triangle PAQ \cong \triangle P'AQ$ by SAS. Thus $\angle AQP \cong \angle AQP'$ by definition of congruence triangles, i.e., $m \perp l$. \hfill \square
Proposition 4.5 (Angle-side-angle criterion) (ASA). If two angles and the included side of a triangle are congruent to two angles and the included side of another triangle, then the two triangles are congruent.

\[ \triangle ABC, \triangle A'B'C', \text{ and } \angle BAC \cong \angle B'A'C', AB \cong A'B', \angle ABC \cong \angle A'B'C'. \]

Draw the unique point \( C'' \) on the ray \( r(A', C') \) such that \( AC \cong A'C'' \). Then \( \triangle ABC \cong \triangle A'B'C'' \) by SAS. Then \( \angle ABC \cong \angle A'B'C'' \) by definition of congruence of triangles. Since \( \angle ABC \cong \angle A'B'C'' \) by given condition, then \( \angle A'B'C'' \cong \angle A'B'C'' \) by transitivity. This means that \( B', C', C'' \) are collinear, i.e., \( C', C'' \) are on both lines \( B'C'' \) and \( A'C'' \). Since intersection point of two lines is unique, we have \( C' = C'' \). Hence \( \triangle ABC \cong \triangle A'B'C'' \). See Figure 10. Hence \( \triangle ABC \cong \triangle A'B'C'' \).

\[ \triangle AOC \sim \triangle A'O'C'. \]

Proposition 4.6 (Angle addition). Given two angles \( \angle AOC \) and \( \angle A'O'C' \). Let \( r(O, B) \) be a ray between rays \( r(O, A) \) and \( r(O, C) \). Let \( r(O', B') \) be a ray between rays \( r(O', A') \) and \( r(O', C') \). If \( \angle AOB \cong \angle A'O'B' \) and \( \angle BOC \cong \angle B'O'C' \), then \( \angle AOC \cong \angle A'O'C' \).

\[ \triangle AOC \sim \triangle A'O'C'. \]

Proof. We may assume that \( OA \cong O'A', OB \cong O'B', OC \cong O'C' \), and that \( B \) is a point on \( AC \) between \( A, C \). But we did not assume that \( B' \) is a point on \( A'C' \). See Figure 11. Then \( \triangle AOB \cong \triangle A'O'B' \) and \( \triangle BOC \cong \triangle B'O'C' \) by SAS. We see that the supplementary angles \( \angle OBA, \angle OBC \) are congruent to the angles \( \angle O'B'A', \angle O'B'C' \) respectively. Then the supplementary angle \( \angle O'B'C'' \) of \( \angle O'B'A' \) is congruent to \( \angle OBC \) by the Supplementary Angle Congruence Rule. Thus \( \angle O'B'C'' \cong \angle O'B'C' \) by transitivity. Since \( \angle O'B'C'' \) and \( \angle O'B'C' \) are on the same side of line \( O'B' \), it follows that \( \angle O'B'C'' = \angle O'B'C' \) by CA3. So \( A', B', C', C'' \) are collinear. Since \( AB \cong A'B', BC \cong B'C' \), then \( AC \cong A'C' \). Since \( \angle OAC \cong \angle O'A'C', \angle OCA \cong \angle O'C'A' \), we have \( \triangle AOC \cong \triangle A'O'C' \) by ASA. Therefore \( \angle AOC \cong \angle A'O'C' \).

Proposition 4.7 (Angle subtraction). Let \( r(O, B) \) be a ray between rays \( r(O, A) \) and \( r(O, C) \). Let \( r(O', B') \) be a ray between rays \( r(O', A') \) and \( r(O', C') \). If \( \angle AOB \cong \angle A'O'B' \), \( \angle AOC \cong \angle A'O'C' \), then \( \angle BOC \cong \angle B'O'C' \).
Proposition 4.8. Given a triangle \( \Delta ABC \). If \( \angle B \cong \angle C \), then \( AB \cong AC \).

Proof. Let \( A \mapsto A', B \mapsto C, C \mapsto B \). Since \( \angle ABC \cong \angle ACB, BC \cong CB, \angle ACB \cong \angle ABC \), then \( \Delta ABC \cong \Delta ACB \) by ASA. Thus \( AB \cong AC \) by definition of congruence of triangles. \( \square \)

Definition 10. An angle \( \angle AOB \) is less than an angle \( \angle A'O'C' \), written \( \angle AOB < \angle A'O'C' \), if there exists a ray \( r(O', B') \) between the rays \( r(O', A') \) and \( r(O', C') \), such that \( \angle AOB \cong \angle A'O'B' \).

Proposition 4.9 (Strict total order of angles). For any two angles \( \angle A \) and \( \angle B \), one and only one of the three holds: \( \angle A < \angle B, \angle A \cong \angle B, \angle B < \angle A \) (trichotomy). Moreover,

(a) If \( \angle A \cong \angle B, \angle B < \angle C \), then \( \angle A < \angle C \).
(b) If \( \angle A < \angle B, \angle B \cong \angle C \), then \( \angle A < \angle C \).
(c) If \( \angle A < \angle B, \angle B < \angle C \), then \( \angle A < \angle C \).

Proof. Given two angles \( \angle AOB \) and \( \angle A'O'B' \). There exists a unique open ray \( \hat{r}(O', C') \) in the open half-plane \( H(\overline{O'A}, B') \) such that \( \angle AOB \cong \angle A'O'C' \). If \( C' \) is on the ray \( r(O', B') \), then \( r(O', C') = r(O', B') \) and \( \angle AOB \cong \angle A'O'B' \). If \( C' \) is not on the ray \( r(O', B') \), there are two cases.

Case 1. Points \( C', A' \) are on the same side of \( \overline{OB} \).
Since \( C', B' \) are on the same side of \( \overline{OA} \), then \( C' \) is contained in the interior of \( \angle A'O'B' \); so is the open ray \( \hat{r}(O', C') \). Thus \( \angle AOB < \angle A'O'B' \).

Case 2. Points \( C', A' \) are on opposite sides of \( \overline{OB} \). Then \( r(O', B') \) meets \( A'C' \) at \( P' \) between \( A' \) and \( C' \) by Crossbar Theorem. Thus \( \hat{r}(O', B') \) is contained in \( \angle A'O'C' \). By definition \( \angle AOB > \angle A'O'B' \).

(a) Let \( \angle AOB \cong \angle A'O'B' < \angle A''O''C'' \). There exists a ray \( r(O'', B'') \) between rays \( r(O'', A'') \) and \( r(O'', C'') \) such that \( \angle A'O'B' \cong \angle A''O''B'' \) by definition. Then \( \angle AOB \cong \angle A''O''B'' \) by transitivity. Thus \( \angle AOB < \angle A''O''C'' \).

(b) Let \( \angle AOB < \angle A'O'C' \cong \angle A''O''C'' \). There exists a ray \( r(O', B') \) between the rays \( r(O', A') \) and \( r(O', C') \) such that \( \angle AOB \cong \angle A'O'B' \). Let \( r(O'', B'') \) be a ray between the rays \( r(O'', A'') \) and \( r(O'', C'') \) such that \( \angle AOB \cong \angle A''O''B'' \). Then \( \angle AOB \cong \angle A''O''B'' \) by transitivity. Thus \( \angle AOB < \angle A''O''C'' \).

(c) Let \( \angle AOB < \angle A'O'C' < \angle A''O''C'' \). There exists a ray \( r(O'', C'') \) between the rays \( r(O'', A'') \) and \( r(O'', D'') \) such that \( \angle A'O'C' \cong \angle A''O''C'' \). Then \( \angle AOB < \angle A''O''C'' \) by (b). Thus there exists a ray \( r(O'', B'') \) between the rays \( r(O'', A'') \) and \( r(O'', C'') \) such that \( \angle AOB \cong \angle A''O''B'' \). Therefore \( \angle AOB < \angle A''O''B'' \). \( \square \)

Proposition 4.10 (Side-side-side criterion) (SSS). Given triangles \( \Delta ABC \) and \( \Delta A'B'C' \). If \( AB \cong A'B', AC \cong A'C', BC \cong B'C', \) then \( \Delta ABC \cong \Delta A'B'C' \).

Proof. Let \( C'' \) be the unique point on the opposite side of \( H(\overline{A'B'}, C') \) bounded by \( \overline{A'B} \) such that \( \Delta ABC \cong \Delta A'B'C'' \). Draw the segment \( C'C'' \). The line \( \overline{A'B} \) meets \( C'C'' \) at \( D' \) between \( C' \) and \( C'' \), See Figure 12. Then \( A'C'' \cong AC \cong A'C'' \) and \( B'C' \cong BC \cong B'C'' \), i.e., \( \Delta A'C'C'' \) and \( \Delta B'C'C'' \) are isosceles triangles. Hence \( \angle A'C'C'' \cong \angle A'B'C' \) and \( \angle B'C'C'' \cong \angle B'C'C'' \).
If \( A' \ast D' \ast B' \), then the open ray \( \hat{r}(C', C'') \) is contained in \( \hat{\triangle} A'C'B' \), and the open ray \( \hat{r}(C'', C') \) is contained in \( \hat{\triangle} A'C''B' \) by Crossbar Theorem; thus \( \angle A'C'B' \cong \angle A'C''B' \) by angle addition. If \( A' \ast B' \ast D' \), then \( \hat{r}(C', B') \) is contained in \( \hat{\triangle} A'C'D' \), and \( \hat{r}(C'', B') \) is contained in \( \hat{\triangle} A'C''D' \) by Crossbar Theorem; thus \( \angle A'C'B' \cong \angle A'C''B' \) by angle subtraction. Since \( A'C'' \cong AC \cong A'C' \), \( \angle A'C'B' \cong \angle A'C''B' \), \( B'C' \cong BC \cong B'C'' \), we see that \( \triangle A'B'C' \cong \triangle A'B'C'' \) by SAS. Hence \( \triangle ABC \cong \triangle A'B'C'' \) by transitivity.

**Theorem 4.11 (Euclid’s Fourth Postulate).** All right angles are congruent to each other.

![Figure 12: Side-side-side criterion](image)

**Proof.** Given angles \( \angle AOC \cong BOC \) and \( \angle A'O'C' \cong B'O'C' \); see Figure 13. We need to show \( \angle AOC \cong \angle A'O'C' \). Let \( \hat{r}(O', P') \) be the unique open ray in \( \overline{AB}, C' \) such that \( \angle AOC \cong \angle A'O'P' \). It suffices to show that \( \hat{r}(O, P) = \hat{r}(O', C') \). Suppose \( \hat{r}(O', P') \neq \hat{r}(O, C') \). Then either \( \hat{r}(O', P') = \hat{r}(O', D') \), which is contained in \( \hat{\triangle} A'O'C' \), or \( \hat{r}(O', P') = \hat{r}(O', E') \), which is contained in \( \hat{\triangle} B'O'C' \).

In the former case, we have \( \angle A'O'D' < \angle A'O'C' \) and \( \angle B'O'C' < \angle B'O'D' \) by definition of order of angles. Since \( \angle A'O'C' \cong \angle B'O'C' \) by right angle property, then \( \angle A'O'D' < \angle B'O'D' \) by Proposition 4.9. Note that \( \angle BOC \cong \angle B'O'D' \) by Proposition 4.9(a), and \( \angle AOC \cong \angle A'O'D' \). Then \( \angle AOC < \angle BOC \). However, \( \angle AOC \cong \angle BOC \) by right angle property. In summary we have

\[
\angle AOC \cong \angle A'O'D' < \angle A'O'C' \cong \angle B'O'C' < \angle B'O'D' \cong \angle BOC \cong \angle AOC.
\]

So \( \angle AOC < \angle AOC \); this is a contradiction. In the latter case, we have

\[
\angle AOC \cong \angle A'O'E' > \angle A'O'C' \cong \angle B'O'C' > \angle B'O'E' \cong \angle BOC \cong \angle AOC.
\]

So \( \angle AOC > \angle AOC \); this is a contradiction.
5 Axioms of Continuity

Dedekind’s Axiom (Continuity Axiom). If a line \( l \) is partitioned into two nonempty subsets \( \Sigma_1, \Sigma_2 \), i.e., \( l = \Sigma_1 \cup \Sigma_2 \) and \( \Sigma_1 \cap \Sigma_2 = \emptyset \), such that no point of either subset is between two points of the other (equivalently both are convex), then there exists a unique point \( O \) on \( l \) such that one of \( \Sigma_1, \Sigma_2 \) is a ray with vertex \( O \) and the other is an open ray with vertex \( O \) opposite to the other ray. The pair \( \{ \Sigma_1, \Sigma_2 \} \) is called a Dedekind cut of \( l \).

Given two distinct points \( A, B \in l \). If \( A, B \in \Sigma_i \), we have \( AB \subset \Sigma_i \), i.e., \( \Sigma_i \) has no “hole.” Suppose we do not require \( \Sigma_1 \cap \Sigma_2 = \emptyset \) in Dedekind’s axiom. If \( A, B \in \Sigma_1 \cap \Sigma_2 \), then we must have \( A = B \). So the intersection \( \Sigma_1 \cap \Sigma_2 \) contains exactly one point \( O \). Thus \( \Sigma_1, \Sigma_2 \) are two rays with the vertex \( O \). So, when \( \Sigma_1 \cap \Sigma_2 = \emptyset \) is imposed, we say that the partition \( \{ \Sigma_1, \Sigma_2 \} \) determines one point on \( l \).

Definition 11. A subset \( \Omega \) of points is said to be convex provided that whenever two points \( P, Q \) are contained in \( \Omega \) then the segment \( PQ \) is contained in \( \Omega \).

For a line \( l \) with a total order \( \preceq \), the following subsets of \( l \) are convex sets, known as intervals: line \( l \); rays

\[
r(O, -) = \{ P \in l : P \preceq O \}, \quad r(O, +) = \{ P \in l : O \preceq P \};
\]

open rays

\[
\hat{r}(O, -) = \{ P \in l : P < O \}, \quad \hat{r}(O, +) = \{ P \in l : O < P \};
\]

closed interval (segment)

\[
[A, B] = AB = \{ P \in l : A \preceq P \preceq B \};
\]

open interval (segment)

\[
(A, B) = \{ P \in l : A < P < B \};
\]

half-closed and half-open intervals (segments)

\[
[A, B) = \{ P \in l : A \preceq P < B \}, \quad (A, B] = \{ P \in l : A < P \preceq B \}.
\]

The points \( O, A, B \) are called endpoints of \( I \). For a line not satisfying Dedekind’s axiom, a convex subset of the line is not necessarily an interval.

Proposition 5.1 (Extended Dedekind’s Axiom). Dedekind’s axiom is valid for any nonempty interval \( I \) of any line \( l \). More precisely, if a nonempty interval \( I \) is partitioned into two nonempty convex sets \( \Sigma_1, \Sigma_2 \), then both \( \Sigma_1, \Sigma_2 \) are intervals with an endpoint \( O \), one is closed and the other is open at \( O \).

Proof. Let \( l \) be totally ordered so that left and right sides of \( I \) are meaningful. If \( I \) is empty or contains exactly one point, nothing is to be proved for the statement is irrelevant. We assume that \( I \) contains at least two points.

Let \( \Sigma_1 \) be on the left side of \( \Sigma_2 \). Since \( I \) is nonempty, the complement \( l \setminus I \) has one of the forms: (a) empty set \( \emptyset \), (b) a left (open) ray \( \Gamma_1 \), (c) a right (open) ray \( \Gamma_2 \), (d) disjoint union of a left (open) ray \( \Gamma_1 \) and a right (open) ray \( \Gamma_2 \).

Set \( \Sigma'_1 := \Sigma_1 \cup \Gamma_1 \) and \( \Sigma'_2 := \Sigma_2 \cup \Gamma_2 \). Then \( \Sigma'_1, \Sigma'_2 \) form a Dedekind cut of \( l \). By Dedekind’s axiom, there exists a unique point \( O \) on \( l \) such that \( \Sigma'_1 \) is the right closed (open) ray \( r(O, -) \) (\( \hat{r}(O, -) \)) with vertex \( O \), and \( \Sigma'_2 \) is the left open (closed) ray \( \hat{r}(O, +) \) (\( r(O, +) \)) with vertex \( O \). Hence \( \Sigma_1 \) is a right-closed (right-open) interval with right endpoint \( O \in \Sigma_1 \), and \( \Sigma_2 \) is left-open (left-closed) interval with left endpoint \( O \). \( \square \)
Example. Let $\mathbb{Q}$ be the field of rational numbers. Then $\mathbb{Q}^2$ forms an affine plane, called rational affine plane under its points and lines defined by one linear equation. Dedekind’s axiom is not satisfied by the rational plane. Consider the $x$-axis $l = \{(a,0) : a \in \mathbb{Q}\}$. Let $\Sigma_1 = \{(a,0) : a \in \mathbb{Q}, a^2 > 3, a > 0\}$ and $\Sigma_2 = l \cup \Sigma_1$. Then $\Sigma_1, \Sigma_2$ form a Dedekind cut, that is, they satisfy the conditions of Dedekind’s axiom. However, neither $\Sigma_1$ nor $\Sigma_2$ is an (open) ray of $l$. In fact, $\Sigma_1 = \{(a,0) : a \in \mathbb{Q}, a > \sqrt{3}\}$ is not an interval in $\mathbb{Q}^2$ (since it has no left endpoint).

Dedekind’s axiom implies all of the following axioms.

Euclid’s Proposition. For each segment there exists an equilateral triangle having one of its sides to be the given segment.

Definition 12. A point $P$ is said to be inside a circle of radius $OR$ with center $O$ if $OP < OR$.

Circular Continuity Principle. If each of two circles has one point inside but outside the other, then the two circles intersect at two points.

Elementary Continuity Principle. If one endpoint of a segment is inside a circle and the other endpoint is outside, then the segment intersects the circle.

Archimedes’[aˈkiːmiːdiːz] Axiom. Given a segment $AB$ and a ray $r$ with vertex $O$. For each point $P \neq O$ on $r$, there exist an integer $n$ and a point $Q$ on $r$, where $OQ \cong n \cdot AB$, such that either $Q = P$ or $O \ast P \ast Q$.

Aristotle’s[aˈriːstɔtli] Axiom. Given an acute angle $\angle AOB$ and a segment $CD$. There exists a point $Y$ on the ray $r(O,B)$ such that $XY > CD$, where $X$ is the foot of $Y$ on the ray $r(O,A)$.

Proposition 5.2 (Dedekind’s implies Archimedes’). Dedekind’s axiom implies Archimedes’ axiom.

Proof. Given a segment $AB$ and a ray $r$ with vertex $O$. A point $P \in r$ is said to be reachable by $AB$ if $P = O$ or there exist an positive integer $n$ and a point $Q$, such that $OQ \cong n \cdot AB$ and $O \ast P \ast Q$. Let $\Sigma_1$ be the set of points on $r$ reachable by $AB$, and points on the opposite ray of $r$; so $O \in \Sigma_1$. Let $\Sigma_2$ be the complement of $\Sigma_1$ in the line $l$ that contains $r$; $\Sigma_2$ is also the complement of $\Sigma_1$ in $r$. We claim that $\Sigma_2 = \emptyset$. (If so, all points on the ray $r$ are reachable by $AB$, which is Archimedes’ axiom.)

Suppose $\Sigma_2 \neq \emptyset$. We claim that $\{\Sigma_1, \Sigma_2\}$ is a Dedekind cut of $l$. One the one hand, let $P, Q \in \Sigma_1$ be distinct points. If both $P, Q$ are on $r$ or on the opposite ray of $r$, it is clear that $PQ \subset \Sigma_1$. If $P$ is on the opposite ray of $r$ and $Q \in r$, then $PO \subset \Sigma_1$ and $OQ \subset \Sigma_1$; so $PQ = PO \cup OQ \subset \Sigma_1$. On the other hand, let $P, Q \in \Sigma_2$ be distinct points. Suppose $PQ \not\subset \Sigma_2$; there exists a point $R \in \Sigma_1$ such that $R \ast P \ast Q$; since $R$ can be reached, so is $P$; thus $P \in \Sigma_1$, which is a contradiction. Therefore $\{\Sigma_1, \Sigma_2\}$ is a Dedekind cut of $l$, and determines a unique point $O'$ on $l$.

Case 1. $O' \in \Sigma_1$. Then $\Sigma_1$ is a ray with vertex $O'$. Since the opposite ray of $r$ is contained in $\Sigma_1$, then $O' \in r$ and $\Sigma_2$ is an open ray on $r$ with vertex $O'$. Let $O'$ be reached by laying off $n$ copies of $AB$ starting from $O$. Then by laying off one more copy of $AB$ on $r$ starting from $O'$, we get points of $\Sigma_2$ being reached by $AB$. This is impossible.

Case 2. $O' \in \Sigma_2$. Then $\Sigma_2$ is a ray on $r$ with vertex $O' \neq O$, and $\Sigma_1$ is the opposite open ray with vertex $O'$. Laying off one copy of $AB$ starting from $O'$ on the open ray $\Sigma_1$, we obtain a point $P'$ in $\Sigma_1$. Then any point $Q'$ such that $P' \ast Q' \ast O'$ is reachable by $AB$. Thus by laying one more copy of $AB$ starting from $Q'$, the point $O'$ is reachable. So $O' \in \Sigma_1$. This is a contradiction.
Proposition 5.3 (Dedekind’s implies Elementary Continuity). Dedekind’s axiom implies Elementary Continuity Principle.

Proof. Let γ be a circle with center O and radius OR. Let AB be a segment with A inside and B outside γ, i.e., OA < OR and OB > OR. Let Σ₁ denote the set of points on AB inside γ, and Σ₂ the subset of points on AB outside or on γ. Then Σ₁, Σ₂ form a Dedekind cut for the segment AB by trichotomy of segments. Dedekind’s axiom implies that there exists a unique point P on AB such that Σ₁, Σ₂ are intervals with endpoint P, one contains P and the other does not contain P. We claim that P is on γ, i.e., OP ∼ OR.

Case 1. OP < OR. Then P ∈ Σ₁. Take a point Q ∈ Σ₂ such that |PQ| = (|OR| − |OP|)/2. By triangle inequality we have

|OR| < |OQ| < |OP| + |PQ| = |OP|/2 + |OR|,

which is a contradiction.

Case 2. OP > OR. Then P ∈ Σ₂. Take a point Q ∈ Σ₁ such that A * Q * P and |PQ| ≤ (|OP| − |OR|)/2. Since |OQ| < |OR| and |QP| = |PQ|, then by triangle inequality

|OP| ≤ |OQ| + |QP| < |OR| + (|OP| − |OR|)/2 = |OR|/2 + |OP|,

which is a contradiction.

So we must have OP ∼ OR.

Relationship between the Axioms of Continuity.

Dedekind’s axiom ⇒ Archimedes’ axiom, Circular Continuity Principle

Archimedes’ axiom ⇒ Aristotle’s axiom

Circular Continuity Principle ⇒ Elementary Continuity Principle