MATH306 SUPPLEMENTARY MATERIAL

A BRIEF INTRODUCTION TO BESSEL and RELATED SPECIAL FUNCTIONS

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© Draft date December 1, 2008
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Chapter 1

Trigonometric and Gamma Functions

1.1 Trigonometric Functions

Pythagoras Theorem:
(1.1) \( \sin^2 x + \cos^2 x = 1 \)

holds for all real \( x \).

Addition Theorem:
(1.2) \( \sin(A + B) = \sin A \cos B + \sin B \cos A \)
(1.3) \( \cos(A + B) = \cos A \cos B - \sin A \sin B \)

hold for all angles \( A, B \).

1.2 Gamma Function

We recall that
(1.1) \( k! = k \times (k - 1) \times \cdots \times 3 \times 2 \times 1 \).

Euler was able to give a correct definition to \( k! \) when \( k \) is not a positive integer. He invented the Euler-Gamma function in the year 1729. That is,
(1.2) \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \).
So the integral will “converge” for all positive real $x$. Since
\begin{equation}
\Gamma(1) = \int_0^\infty e^{-t} \, dt = 1,
\end{equation}
so one can show
\begin{equation}
\Gamma(x + 1) = x\Gamma(x),
\end{equation}
and so for each positive integer $n$
\begin{equation}
\Gamma(n + 1) = n!.
\end{equation}
In fact, the infinite integral will not only “converge” for all positive real $x$ but can be “analytically continued” into the whole complex plane $\mathbb{C}$. We have
\begin{equation}
\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \quad \Re(z) > 0,
\end{equation}
which is analytic in $\mathbb{C}$ except at the simple poles at the negative integers including 0. So it is a meromorphic function.

Euler worked out that
\begin{equation}
\left(\frac{1}{2}\right)! = \frac{1}{2}\sqrt{\pi}.
\end{equation}
So
\begin{equation}
\left(\frac{5}{2}\right)! = \frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \left(\frac{1}{2}\right)!.
\end{equation}
One can even compute negative factorial:
\begin{equation}
\Gamma(1/2) = (-1/2)(-3/2)(-5/2)(-19/2)(-11/2)\Gamma(-11/2).
\end{equation}

1.3 Chebyshev Polynomials

**Theorem 1.3.1** (De Moivre’s Theorem). Let $z = r(\cos \theta + i \sin \theta)$ be a complex number. Then for every non-negative integer $n$,
\begin{equation}
z^n = r^n(\cos n\theta + i \sin n\theta).
\end{equation}

**Proof** It is clear that that the formula holds when $n = 1$. We assume it holds at the integer $n$. Then the addition formulae for the trigonometric theorem imply that
\begin{align*}
z^{n+1} &= r^{n+1}(\cos \theta + i \sin \theta)(\cos n\theta + i \sin n\theta) \\
&= r^{n+1} [\cos(n+1)\theta + i \sin(n+1)\theta].
\end{align*}
Thus, the result follows by applying the principle of induction. \qed
Exercise 1.3.2. Justify the formula for negative integers $n$.

Remark 1.3.1. De Moivre’s formula actually works for all real and complex $n$. See MATH304.

Example 1.3.3 (Multiple angle formulae). Let $n = 2$ in De Moivre’s formula yields

(1.2) \[ z^2 = \cos 2\theta + i\sin 2\theta. \]

But

(1.3) \[ z^2 = (\cos \theta + i\sin \theta)^2 = \cos^2 \theta - \sin^2 \theta + i(2\cos \theta \sin \theta). \]

Comparing the real and imaginary parts of (1.2) and (1.3) yields

\[
\begin{align*}
\cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\
&= 2\cos^2 \theta - 1 \\
&= 1 - 2\sin^2 \theta 
\end{align*}
\]

and

(1.4) \[ \sin 2\theta = 2\cos \theta \sin \theta. \]

Similarly, we have

(1.5) \[ z^3 = \cos 3\theta + i\sin 3\theta. \]

But

\[
\begin{align*}
z^3 &= (\cos \theta + i\sin \theta)^3 \\
&= \cos^3 \theta + 3\cos^2 \theta(i\sin \theta) + 3\cos \theta(i\sin \theta)^2 + (i\sin \theta)^3 \\
&= \cos^3 \theta - 3\cos \theta \sin^2 \theta + i(3\cos^2 \theta \sin \theta - \sin^3 \theta). 
\end{align*}
\]

Comparing the real and imaginary parts of (1.5) and (1.6) yields

\[
\begin{align*}
\cos 3\theta &= \cos^3 \theta - 3\cos \theta \sin^2 \theta \\
&= 4\cos^3 \theta - 3\cos \theta 
\end{align*}
\]

and

\[
\begin{align*}
\sin 3\theta &= 3\cos^2 \theta \sin \theta - \sin^3 \theta \\
&= 3\sin \theta - 4\sin^3 \theta. 
\end{align*}
\]
More generally, we have
\[
\cos n\theta + i \sin n\theta = z^n = (\cos \theta + i \sin \theta)^n
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k} \theta (i \sin \theta)^k
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} i^k \cos^{n-k} \theta \sin^k \theta.
\]
(1.7)

Equating the real parts of the last equation and noting that \(i^2l = (-1)^l\) give
\[
\cos n\theta = \cos^n \theta - \binom{n}{2} \cos^{n-2} \theta \sin^2 \theta + \binom{n}{4} \cos^{n-4} \theta \sin^4 \theta + \cdots
\]
\[
+ (-1)^{\lfloor n/2 \rfloor} \binom{n}{\lfloor n/2 \rfloor} \cos^{n-2\lfloor n/2 \rfloor} \theta \sin^{2\lfloor n/2 \rfloor} \theta.
\]
(1.8)

But each of the sine function above has even power. Thus the substitution of \(\sin^2 \theta = 1 - \cos^2 \theta\) in (1.8) yields
\[
\cos n\theta = \sum_{l=0}^{\lfloor n/2 \rfloor} \binom{n}{2l} \cos^{n-2l} \theta \sum_{k=0}^{l} (-1)^k \binom{l}{k} \cos^{2k} \theta.
\]
(1.9)

Finally, it is possible to re-write the summation formula as
\[
\cos n\theta = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \sum_{j=k}^{\lfloor n/2 \rfloor} \binom{n}{2j} \binom{j}{k} \cos^{n-2k} \theta.
\]
(1.10)

Writing \(\theta = \arccos x\) for \(0 \leq \theta \leq \pi\), and \(T_n(x) = \cos(n \arccos x)\). We deduce from the above formula that \(T_n(x)\) is a polynomial in \(x\) of degree \(n\). The polynomial \(T_n\) is called the \textbf{Chebyshev polynomial of the first kind of degree \(n\)}. The first few polynomials are given by
\[
T_0(x) = 1;
T_1(x) = x;
T_2(x) = 2x^2 - 1;
T_3(x) = 4x^3 - 3x;
T_4(x) = 8x^4 - 8x^2 + 1;
T_5(x) = 16x^5 - 20x^3 + 5x;
\ldots \ldots
\]

\footnote{P. L. Chebyshev (1821–1894) Russian mathematician. The collection of the polynomials \(\{T_n(x)\}_{0}^{\infty}\) is a complete orthogonal polynomial set on \([-1, 1]\).}
1.3. **CHEBYSHEV POLYNOMIALS**

The set of Chebyshev polynomials \( \{T_n(x)\} \) can generate a \( L^2[-1, 1] \) space. That is, it is a Hilbert space.

CHAPTER 1. TRIGONOMETRIC AND GAMMA FUNCTIONS
Chapter 2

Bessel Functions

2.1 Power Series

We define the Bessel function of first kind of order $\nu$ to be the complex function represented by the power series

\[
J_\nu(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{\Gamma(\nu+k+1) k!} = z^\nu \sum_{k=0}^{+\infty} \frac{(-1)^k (1/2)^{\nu+2k}}{\Gamma(\nu+k+1) k!} z^{2k}.
\]

Here $\nu$ is an arbitrary complex constant and the notation $\Gamma(\nu)$ is the Euler Gamma function defined by

\[
\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt, \quad \Re(z) > 0.
\]

2.2 Bessel Equation

A common introduction to the series representation for Bessel functions of first kind of order $\nu$ is to consider Frobenius method: Suppose

\[
y(z) = \sum_{k=0}^{+\infty} c_k z^{\alpha+k},
\]

where $\alpha$ is a fixed parameter and $c_k$ are coefficients to be determined. Substituting the above series into the Bessel equation

\[
z \frac{d^2 y}{dz^2} + x \frac{dy}{dz} + (z^2 - \nu^2) y = 0.
\]
and separating the coefficients yields:

\[ 0 = z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - \nu^2) y \]

\[ = \sum_{k=0}^{\infty} c_k (\alpha + k)(\alpha + k - 1) z^{\alpha+k} + \sum_{k=0}^{\infty} c_k (\alpha + k) z^{\alpha+k} \]

\[ + (z^2 - \nu^2) \cdot \sum_{k=0}^{\infty} c_k z^{\alpha+k} \]

\[ = \sum_{k=0}^{\infty} c_k [(\alpha + k) - \nu^2] z^{\alpha+k} + \sum_{k=0}^{\infty} c_k z^{\alpha+k+2} \]

\[ = c_0 (\alpha^2 - \nu^2) z^\alpha + \sum_{k=0}^{\infty} \{ c_k [(\alpha + k)^2 - \nu^2] + c_{k-2} \} z^{\alpha+k} \]

The first term on the right side of the above expression is

\[ c_0 (\alpha^2 - \nu^2) z^\alpha \]

and the remaining are

\[ c_1 [(\alpha + 1)^2 - \nu^2] + 0 = 0 \]
\[ c_2 [(\alpha + 2)^2 - \nu^2] + c_0 = 0 \]
\[ c_3 [(\alpha + 3)^2 - \nu^2] + c_1 = 0 \]
\[ \ldots \]

(2.2)

\[ c_k [(\alpha + k)^2 - \nu^2] + c_{k-2} = 0 \]

(2.3)

Hence a series solution could exist only if \( \alpha = \pm \nu \). When \( k > 1 \), then we require

\[ c_k [(\alpha + k)^2 - \nu^2] + c_{k-2} = 0, \]

and this determines \( c_k \) in terms of \( c_{k-2} \) unless \( \alpha - \nu = -2\nu \) or \( \alpha + \nu = 2\nu \) is an integer. Suppose we discard these exceptional cases, then it follows from (2.2) that

\[ c_1 = c_3 = c_5 = \cdots = c_{2k+1} = \cdots = 0. \]

Thus we could express the coefficients \( c_{2k} \) in terms of

\[ c_{2k} = \frac{(-1)^k c_0}{(\alpha - \nu + 2)(\alpha - \nu + 4) \cdots (\alpha - \nu + 2k) \cdot (\alpha + \nu + 2)(\alpha + \nu + 4) \cdots (\alpha + \nu + 2k)}. \]
2.2. BESSEL EQUATION

If we now choose $\alpha = \nu$, then we obtain

\begin{equation}
(2.4) \quad c_0 z^{\nu} \left[ 1 + \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{1}{2} z \right)^{2k}}{k!(\nu + 1)(\nu + 2) \cdots (\nu + k)} \right].
\end{equation}

Alternatively, if we choose $\alpha = -\nu$, then we obtain

\begin{equation}
(2.5) \quad c'_0 z^{-\nu} \left[ 1 + \sum_{k=0}^{\infty} \frac{(-1)^k \left( \frac{1}{2} z \right)^{2k}}{k!(-\nu + 1)(-\nu + 2) \cdots (-\nu + k)} \right].
\end{equation}

Since $c_0$ and $c'_0$ are arbitrary, so we choose them to be

\begin{equation}
(2.6) \quad c_0 = \frac{1}{2^{\nu} \Gamma(\nu + 1)}, \quad c'_0 = \frac{1}{2^{-\nu} \Gamma(-\nu + 1)}
\end{equation}

so that the two series \((2.4)\) and \((2.5)\) can be written in the forms:

\begin{equation}
(2.7) \quad J_{\nu}(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k \left( \frac{1}{2} z \right)^{\nu + 2k}}{\Gamma(\nu + k + 1) k!}, \quad J_{-\nu}(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k \left( \frac{1}{2} z \right)^{-\nu + 2k}}{\Gamma(-\nu + k + 1) k!}
\end{equation}

and both are called the Bessel function of order $\nu$ if the first kind.

It can be shown that the Wronskian of $J_{\nu}$ and $J_{-\nu}$ is given by (G. N. Watson “A Treatise On The Theory Of Bessel Functions”, pp. 42–43):

\begin{equation}
(2.8) \quad W(J_{\nu}, J_{-\nu}) = -\frac{2 \sin \nu \pi}{\pi z}.
\end{equation}

This shows that the $J_{\nu}$ and $J_{-\nu}$ forms a fundamental set of solutions when $\nu$ is not equal to an integer. In fact, when $\nu = n$ is an integer, then we can easily check that

\begin{equation}
(2.9) \quad J_{-n}(z) = (-1)^n J_n(z).
\end{equation}

Thus it requires an extra effort to find another linearly independent solution. It turns out a second linearly independent solution is given by

\begin{equation}
(2.10) \quad Y_{\nu}(z) = \frac{J_{\nu}(z) \cos \nu \pi - J_{-\nu}(z)}{\sin \nu \pi}
\end{equation}

when $\nu$ is not an integer. The case when $\nu$ is an integer $n$ is defined by

\begin{equation}
(2.11) \quad Y_n(z) = \lim_{\nu \to n} \frac{J_{\nu}(z) \cos \nu \pi - J_{-\nu}(z)}{\sin \nu \pi}.
\end{equation}
The $Y_\nu$ so defined is linearly independent with $J_\nu$ for all values of $\nu$.

In particular, we obtain

$$Y_\nu(z) = \frac{-1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left( \frac{z}{2} \right)^{2k-n}$$

$$+ \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{n+2k}}{k! (n+k)!} \left[ 2 \log \frac{z}{2} - \psi(k+1) - \psi(k+n+1) \right]$$

for $|\arg z| < \pi$ and $n = 0, 1, 2, \cdots$ with the understanding that we set the sum to be 0 when $n = 0$. Here the $\psi(z) = \Gamma'(z)/\Gamma(z)$. We note that the function is unbounded when $z = 0$.

2.3 Basic Properties of Bessel Functions


2.3.1 Zeros of Bessel Functions


2.3.2 Recurrence Formulae for $J_\nu$

We consider arbitrary complex $\nu$.

$$\frac{d}{dz} z^\nu J_\nu(z) = \frac{d}{dz} \frac{(-1)^k z^{2\nu+2k}}{2\nu+2k k! \Gamma(\nu + k + 1)}$$

$$= \frac{d}{dz} \frac{(-1)^k z^{2\nu-1+2k}}{2\nu-1+2k k! \Gamma(\nu + k)}$$

$$= z^\nu J_{\nu-1}(z).$$

But the left side can be expanded and this yields

$$z J'_\nu(z) + \nu J_\nu(z) = z J_{\nu-1}(z).$$
2.3. BASIC PROPERTIES OF BESSEL FUNCTIONS

Similarly,

\begin{equation}
\frac{d}{dz} z^{-\nu} J_\nu(z) = -z^{-\nu} J_{\nu+1}(z).
\end{equation}

and this yields

\begin{equation}
z J'_\nu(z) - \nu J_\nu(z) = -z J_{\nu+1}(z).
\end{equation}

Subtracting and adding the above recurrence formulae yield

\begin{equation}
J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z)
\end{equation}

\begin{equation}
J_{\nu-1}(z) - J_{\nu+1}(z) = 2J'_\nu(z).
\end{equation}

2.3.3 Generating Function for \( J_n \)

Jacobi in 1836 gave

\begin{equation}
e^{\frac{1}{2}z(t - \frac{1}{t})} = \sum_{k=-\infty}^{+\infty} t^k J_k(z).
\end{equation}

Many of the formulae derived above can be obtained from this expression.

\begin{equation}
e^{\frac{1}{2}z(t - \frac{1}{t})} = \sum_{k=-\infty}^{+\infty} c_k(z) t^k
\end{equation}

for \( 0 < |t| < \infty \). We multiply the power series

\begin{equation}
e^{\frac{z}{2}t} = 1 + \frac{(z/2)}{1!} t + \frac{(z/2)^2}{1!} t^2 + \cdots
\end{equation}

and

\begin{equation}
e^{-\frac{z}{2}t} = 1 - \frac{(z/2)}{1!} t + \frac{(z/2)^2}{1!} t^2 - \cdots
\end{equation}

Multiplying the two series and comparing the coefficients of \( t^k \) yield

\begin{equation}
c_n(z) = J_n(z), \quad n = 0, 1, \cdots
\end{equation}

\begin{equation}
c_n(z) = (-1)^n J_{-n}(z), \quad n = -1, -2, \cdots.
\end{equation}

Thus

\begin{equation}
e^{\frac{1}{2}z(t - \frac{1}{t})} = J_0(z) + \sum_{k=1}^{+\infty} J_k[t^k + (-1)^k t^{-k}].
\end{equation}
2.3.4 Lommel’s Polynomials

Iterating the recurrence formula
\[ J_{\nu+1}(z) = \frac{2\nu}{z} J_{\nu}(z) - J_{\nu-1} \]
with respect to \( \nu \) a number of times give
\[ J_{\nu+k}(z) = P(1/z)J_{\nu}(z) - Q(1/z)J_{\nu-1}. \]
Lommel (1871) found that
\[ J_{\nu+k}(z) = R_{k,\nu}(z)J_{\nu}(z) - R_{k-1,\nu+1}J_{\nu-1}. \]

2.3.5 Bessel Functions of half-integer Orders

One can check that
\[ J_{-\frac{1}{2}}(z) = \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} \cos z, \quad J_{\frac{1}{2}}(z) = \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} \sin z. \]
Moreover,
\[ J_{n+\frac{1}{2}}(z) = (-1)^n \sqrt{\frac{2}{\pi}} z^{n+\frac{1}{2}} \left( \frac{d}{dz} \right)^n \frac{\sin z}{z}, \quad n = 0, 1, 2, \ldots. \]

Thus applying a recurrence formula and using the Lommel polynomials yield
\[ J_{n+\frac{1}{2}}(z) = R_{n,\nu}(z)J_{\frac{1}{2}}(z) - R_{n-1,\nu+1}J_{-\frac{1}{2}}(z) \]
That is, we have
\[ J_{n+\frac{1}{2}}(z) = R_{n,\nu}(z) \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} \sin z - R_{n-1,\nu+1} \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} \sin z. \]

2.3.6 Formulae for Lommel’s polynomials

For each fixed \( \nu \), the Lommel polynomials are given by
\[ R_{n,\nu}(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (n-k)! (\nu)_{n-k} (2^2)^{n-2k}}{k!(n-2k)! (\nu)_{k}} \left( \frac{2}{\pi z} \right)^{n-2k} \]
where the \( \lfloor x \rfloor \) means the largest integer not exceeding \( x \).

Lommel is a German mathematician who made a major contribution to Bessel functions around 1870s.
2.3.7 "Pythagoras’ Theorem" for Bessel Function

These Lommel polynomials have remarkable properties. Since

\[ J_{-\frac{1}{2}}(z) = \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} \cos z \quad J_{\frac{1}{2}}(z) = \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} \sin z \]

and \( \sin^2 x + \cos^2 x = 1 \); we now have

\[ J_{n+\frac{1}{2}}(z) + J_{-n-\frac{1}{2}}(z) = 2(-1)^n \frac{R_{2n, \frac{1}{2}-n}(z)}{\pi z} \]

That is, we have

\[ J_{n+\frac{1}{2}}(z) + J_{-n-\frac{1}{2}}(z) = \frac{2}{\pi z} \sum_{k=0}^{n} \frac{(2z)^{2n-2k}(2n-k)!(2n-2k)!}{(n-k)!^2 k!} \]

A few special cases are

1. \( J_{\frac{1}{2}}^{2}(z) + J_{-\frac{1}{2}}^{2}(z) = \frac{2}{\pi z} \);
2. \( J_{\frac{3}{2}}^{2}(z) + J_{-\frac{3}{2}}^{2}(z) = \frac{2}{\pi z} \left( 1 + \frac{1}{z^2} \right) \);
3. \( J_{\frac{5}{2}}^{2}(z) + J_{-\frac{5}{2}}^{2}(z) = \frac{2}{\pi z} \left( 1 + \frac{3}{z^2} + \frac{9}{z^4} \right) \);
4. \( J_{\frac{7}{2}}^{2}(z) + J_{-\frac{7}{2}}^{2}(z) = \frac{2}{\pi z} \left( 1 + \frac{6}{z^2} + \frac{45}{z^4} + \frac{225}{z^6} \right) \)

2.3.8 Orthogonality of the Lommel Polynomials

Let us set

\[ h_{n, \nu}(z) = R_{n, \nu}(1/z) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k(n-k)!(\nu)_{n-k}}{k!(n-2k)!} \left( \frac{z}{2} \right)^{n-2k} \]

then the set \( \{ h_{n, \nu}(z) \} \) is called the modified Lommel polynomials. Since the Bessel functions \( J_{\nu}(z) \) satisfies a three-term recurrence relation, so the Lommel polynomials inherit this property:

\[ 2z(n + \nu)h_{n, \nu}(z) = h_{n+1, \nu}(z) + h_{n-1, \nu}(z) \]
with initial conditions

\begin{equation}
(2.26) \quad h_{0, \nu}(z) = 1, \quad h_{1, \nu}(z) = 2\nu z.
\end{equation}

If one start with a different set of initial conditions

\begin{equation}
(2.27) \quad h^*_{0, \nu}(z) = 0, \quad h^*_{1, \nu}(z) = 2\nu,
\end{equation}

then the sequence \( \{h^*_{n, \nu}(z)\} \) generated by the (2.25) is called the associated Lommel polynomials. It is known that a three-term recurrence relation for polynomials with the coefficients of as in (2.25) will generate a set of orthogonal polynomials on \((−∞, +∞)\). That is, there is a probability measure \( \alpha \) on \((−∞, +∞)\) with \( \int_{−∞}^{+∞} d\alpha = 1 \)

**Theorem 2.3.1** (A. A. Markov, 1895). Suppose the set of \( \{p_n(z)\} \) of orthogonal polynomials with its measure \( \alpha \) supported on a bounded interval \([a, b]\), then

\begin{equation}
(2.28) \quad \lim_{n \to \infty} \frac{p^*_{n, \nu}(z)}{p_{n, \nu}(z)} = \int_{a}^{b} \frac{d\alpha(t)}{z - t}
\end{equation}

holds uniformly for \( z \notin [a, b] \).

Since we know

**Theorem 2.3.2** (Hurwitz). The limit

\begin{equation}
(2.29) \quad \lim_{n \to \infty} \frac{(z/2)^{\nu+n}R_{n, \nu+1}(z)}{\Gamma(n + \nu + 1)} = J_\nu(z),
\end{equation}

holds uniformly on compact subsets of \( \mathbb{C} \).

So we have

**Theorem 2.3.3.** For \( \nu > 0 \), the polynomials \( \{h_n, \nu(z)\} \) are orthogonal with respect to a discrete measure \( \alpha_\nu \) normalized to have \( \int_{−∞}^{+∞} d\alpha(t) = 1 \), and

\begin{equation}
(2.30) \quad \int_{\mathbb{R}} \frac{d\alpha(t)}{z - t} = 2\nu \frac{J_\nu(1/z)}{J_{\nu-1}(1/z)}.
\end{equation}

One can use this formula to recover the measure \( \alpha(t) \). Thus the Lommel “polynomials” was found to have very distinct properties to be complete in \( L^2(-1, +1) \).
2.4 Integral formulae

\[(2.1)\]
\[J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\frac{1}{2})\Gamma(\nu + \frac{1}{2})} \int_{-1}^{1} (1 - t^2)^{-\nu/2} \cos zt \, dt, \quad \Re(\nu) > -1/2, \quad |\arg z| < \pi.\]

Or equivalently,

\[(2.2)\]
\[J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\frac{1}{2})\Gamma(\nu + \frac{1}{2})} \int_{0}^{\pi} \cos(z \cos \theta) \sin^{2\nu} \theta \, d\theta, \quad \Re(\nu) > -1/2, \quad |\arg z| < \pi,\]

where \(t = \cos \theta\).

Writing \(H^{(1)}_\nu(z) = J_\nu(z) + iY_\nu(z)\). Then

**Theorem 2.4.1** (Hankel 1869).

\[(2.3)\]
\[H^{(1)}_\nu(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i[z - \nu \pi/2 - \pi/4]} \frac{\exp(i\beta)}{\Gamma(\nu + 1/2)} \int_{0}^{\infty} e^{-u} u^{\nu - \frac{1}{2}} \left(1 + \frac{iu}{2z}\right)^{\nu - \frac{1}{2}} du,\]

where \(|\beta| < \pi/2\).

2.5 Asymptotic Behaviours of Bessel Functions

Expanding the integrand of Henkel’s contour integral by binomial expansion and after some careful analysis, we have

**Theorem 2.5.1.** For \(-\pi < \arg z < 2\pi\),

\[(2.1)\]
\[H^{(1)}_\nu(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{i[x - \frac{1}{2} \nu \pi - \frac{1}{4} \pi]} \left[\sum_{m=0}^{p-1} \frac{(1/2 - \nu)m(1/2 + \nu)m}{(2ix)^m m!} + R^{(1)}_p(x)\right],\]

and for \(-2\pi < \arg x < \pi\),

\[(2.2)\]
\[H^{(2)}_\nu(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{-i[x - \frac{1}{2} \nu \pi - \frac{1}{4} \pi]} \left[\sum_{m=0}^{p-1} \frac{(1/2 - \nu)m(1/2 + \nu)m}{(2ix)^m m!} + R^{(2)}_p(x)\right],\]

where

\[(2.3)\]
\[R^{(1)}_p(x) = O(x^{-p}) \quad \text{and} \quad R^{(2)}_p(x) = O(x^{-p}),\]

as \(x \to +\infty\), uniformly in \(-\pi + \delta < \arg x < 2\pi - \delta\) and \(-2\pi + \delta < \arg x < \pi - \delta\) respectively.
The two expansions are valid simultaneously in $-\pi < \arg x < \pi$.

We thus have $J_{\nu}(z) = \frac{1}{2}(H^{(1)}_{\nu}(z) + H^{(2)}_{\nu}(z))$. So

**Theorem 2.5.2.** For $|\arg z| < \pi$, we have

$$J_{\nu}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left[\cos \left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) \sum_{k=0}^{\infty} \frac{(-1)^k(\nu, 2k)}{(2z)^{2k}}
- \sin \left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right) \sum_{k=0}^{\infty} \frac{(-1)^k(\nu, 2k + 1)}{(2z)^{2k+1}}\right]$$

### 2.5.1 Addition formulae

Schläfli (1871) derived

$$J_{\nu}(z) = \sum_{k=-\infty}^{\infty} J_{\nu-k}(t)J_k(z).$$  \hspace{1cm} (2.4)

**Theorem 2.5.3** (Neumann (1867)). Let $z, Z, R$ form a triangle and let $\phi$ be the angle opposite to the side $R$, then

$$J_0\{\sqrt{(Z^2 + z^2 - 2Zz \cos \phi)}\} = \sum_{k=0}^{\infty} \epsilon_k J_k(Z)J_k(z) \cos k\phi,$$

where $\epsilon_0 = 1, \epsilon_k = 2$ for $k \geq 1$.

We note that $Z, z$ can assume complex values. There are generalizations to $\nu \neq 0$.

### 2.6 Fourier-Bessel Series

**Theorem 2.6.1.** Suppose the real function $f(r)$ is piecewise continuous in $(0, a)$ and of bounded variation in every $[r_1, r_2] \subset (0, a)$. Then if

$$\int_0^a \sqrt{r}|f(r)| dr < \infty,$$  \hspace{1cm} (2.1)
2.6. FOURIER-BESSEL SERIES

then the Fourier-Bessel series

\begin{equation}
(2.2) \quad f(r) = \sum_{k=1}^{\infty} c_k J_\nu\left(x_{\nu k} \frac{r}{a}\right)
\end{equation}

converges to $f(r)$ at every continuity point of $f(r)$ and to

\begin{equation}
(2.3) \quad \frac{1}{2} \left[ f(r + 0) + f(r - 0) \right]
\end{equation}

at every discontinuity point of $f(r)$. Here

\begin{equation}
(2.4) \quad c_k = \frac{2}{J^2_\nu(\lambda_k)} \int_0^1 x f(x) J_\nu(\lambda_k x) \, dx
\end{equation}

is the Fourier-Bessel coefficients of $f$.

The $x_{\nu k}$ are the zero of $J_\nu(x)$ $k = 0, 1, \cdots$. The discussion of the orthogonality of the Bessel functions is omitted. You may consult G. P. Tolstov’s “Fourier Series”, Dover 1962.

The situation here is very much like the classical Fourier series in terms of sine and cosine functions.
CHAPTER 2. BESSEL FUNCTIONS
Chapter 3

An New Perspective

3.1 Hypergeometric Equations

We shall base our consideration of functions on the finite complex plane \( \mathbb{C} \) in this chapter. Let \( p(z) \) and \( q(z) \) be meromorphic functions defined in \( \mathbb{C} \). A point \( z_0 \) belongs to \( \mathbb{C} \) is called a regular singular point of the second order differential equation

\[
(3.1) \quad \frac{d^2 f(z)}{dz^2} + p(z) \frac{df(z)}{dz} + q(z)f(z) = 0
\]

if

\[
(3.2) \quad \lim_{z \to z_0} (z - z_0)p(z) = A, \quad \lim_{z \to z_0} (z - z_0)^2q(z) = B
\]

both exist and being finite. We start by recalling that any second order linear equation with three regular singular points in the \( \mathbb{C} \) can always be transformed ([2, pp. 73–75]) to the hypergeometric differential equation (due to Euler, Gauss and Riemann)

\[
(3.3) \quad z(1-z)\frac{d^2 y}{dz^2} + [c - (a + b + 1)z]\frac{dy}{dz} - ab y = 0
\]

which has regular singular points at 0, 1, \( \infty \).

It is known that the hypergeometric equation has a power series solution

\[
(3.4) \quad _2F_1 \left( \begin{array}{c} \frac{a}{c}, \frac{b}{c} \\ \frac{c}{c} \end{array} \right) |z = _2F_1 [a, b; c; z] = \sum_{k=0}^{+\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k
\]
which converges in $|z| < 1$. It is called the Gauss hypermetric series which depends on three parameters $a, b, c$. Here the notation is the standard Pochhammer notation (1891)

$$ (a)_k = a \cdot (a + 1) \cdot (z + 2) \cdots (a + k - 1), \quad (\nu)_0 = 1 $$

for each integer $k$.

**Theorem 3.1.1** (Euler 1769). Suppose $\Re(c) > \Re(b) > 0$, then we have

$$ 2F_1 \left( \begin{array}{c} a, b \\ c \end{array} \bigg| z \right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a} \, dt, $$

defined in the cut-plane $\mathbb{C}\setminus[1, +\infty)$, $\arg t = \arg(1-t) = 0$, and that $(1-xt)^{-a}$ takes values in the principal branch $(-\pi, \pi)$.

Thus Euler’s integral representation of $2F_1$ thus continues the function analytically into the cut-plane.

**Special cases of (3.4) include:** ...

**In 1812 Gauss gave a complete set of 15 contiguous relations ...**

We note that one can write familiar functions in terms of $pF_q$:

1. $e^x = 0F_0 \left( \begin{array}{c} - \\ x \end{array} \right)$;

2. $(1-x)^{-a} = 1F_0 \left( \begin{array}{c} a \\ x \end{array} \right)$;

3. $\log(1+z) = 2F_1 \left( \begin{array}{c} 1, 1 \\ 2 \end{array} \bigg| -x \right)$;

4. $\cos x = 0F_1 \left( \begin{array}{c} - \\ 1/2 \end{array} \bigg| -\frac{x^2}{4} \right)$;

5. $\sin^{-1} x = 2F_1 \left( \begin{array}{c} 1/2, 1/2 \\ 3/2 \end{array} \bigg| x^2 \right)$.
3.2 Confluent Hypergeometric Equations

Writing the (3.1) in the variable \( x = z/b \) results in the equation

\[
(3.1) \quad x(1 - x/b) \frac{d^2 y}{dx^2} + [c - (a + b + 1)x/b] \frac{dy}{dx} - ay = 0
\]

which has regular singular points at 0, b, \( \infty \). Letting \( b \) tends to \( \infty \) results in

\[
(3.2) \quad x \frac{d^2 y}{dx^2} + (c - x) \frac{dy}{dx} - ay = 0
\]

which is called the confluent hypergeometric equation. But the equation has a regular singular point at 0 and an irregular singular point at \( \infty \).

Substituting \( z/b \) in (3.4) and letting \( b \) to infinity results in a series

\[
(3.3) \quad \Phi(a, c; z) := _1F_1 \left[ a; c; z \right] = \sum_{k=0}^{+\infty} \frac{(a)_k}{(c)_k k!} z^k.
\]

This series is called a Kummer series. We note that the Gauss hypergeometric series has radius of convergence 1, and after the above substitution of \( z/b \) results in a series that has radius of convergence equal to \( |b| \). So the Kummer series has radius of convergence \( \infty \) after taking the limit \( b \to \infty \). We call all solutions to the equation (3.2) confluent hypergeometric functions because they are obtained from the confluence of singularities of the hypergeometric equations. Further details can be found in [4], better known as the Bateman project volume one.

3.3 A Definition of Bessel Functions

We define the Bessel function of first kind of order \( \nu \) to be the complex function represented by the power series

\[
(3.1) \quad J_\nu(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k \left(\frac{1}{2}z\right)^{\nu+2k}}{\Gamma(\nu + k + 1) k!} = z^\nu \sum_{k=0}^{+\infty} \frac{(-1)^k \left(\frac{1}{2}\right)^{\nu+2k}}{\Gamma(\nu + k + 1) k!} z^{2k}.
\]

Here \( \nu \) is an arbitrary complex constant and the notation \( \Gamma(\nu) \) is the Euler Gamma function defined by

\[
(3.2) \quad \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \quad \Re(z) > 0.
\]
CHAPTER 3. AN NEW PERSPECTIVE

It can be analytically continued onto the whole complex plane \( \mathbb{C} \), and has simple poles at the negative integers including 0. So it is a meromorphic function. In particular, \( \Gamma(k + 1) = k! = k(k - 1) \cdots 2 \cdot 1 \). So the Euler-Gamma function is a generalization of the factorial notation \( k! \).

Let us set \( a = \nu + \frac{1}{2} \), \( c = 2\nu + 1 \) and replace \( z \) by \( 2iz \) in the Kummer series.

That is,

\[
\Phi(\nu + 1/2, 2\nu + 1; 2iz) = \sum_{k=0}^{\infty} \frac{(\nu + \frac{1}{2})_k}{(2\nu + 1)_k k!} (2iz)^k
\]

\[
= \, _{1}F_{2} \left( \begin{array}{c} \nu + \frac{1}{2} \\ 2\nu + 1 \end{array} \right | 2iz \right)
\]

\[
= e^{iz} \, _{0}F_{1} \left( \begin{array}{c} - \\ \nu + 1 \end{array} \right | \frac{z^2}{2} \right)
\]

\[
= e^{iz} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(\nu + 1)_k k! 2^{2k}}
\]

\[
= e^{iz} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\nu + 1)}{\Gamma(\nu + k + 1) k!} \left( \frac{z}{2} \right)^{2k}
\]

\[
= e^{iz} \Gamma(\nu + 1) \left( \frac{z}{2} \right)^{-\nu} \cdot J_{\nu}(z)
\]

where we have applied Kummer’s second transformation

\[
_{1}F_{1} \left( \begin{array}{c} a \\ 2a \end{array} \right | 4x \right) = e^{2x} \, _{0}F_{1} \left( \begin{array}{c} - \\ a + \frac{1}{2} \end{array} \right | x^2 \right)
\]

in the third step above, and the identity \( (\nu)_k = \Gamma(\nu + k)/\Gamma(\nu) \) in the fourth step.

Since the confluent hypergeometric function \( \Phi(\nu + 1/2, 2\nu + 1; 2iz) \) satisfies the hypergeometric equation

\[
z \frac{d^2 y}{dz^2} + (c - z) \frac{dy}{dz} - a y = 0
\]

with \( a = \nu + 1/2 \) and \( c = 2\nu + 1 \). Substituting \( e^{iz} \Gamma(\nu + 1) \left( \frac{z}{2} \right)^{-\nu} \cdot J_{\nu}(z) \) into the \( \text{(3.4)} \) and simplifying lead to the equation

\[
z^2 \frac{d^2 y}{dz^2} + x \frac{dy}{dz} + (z^2 - \nu^2)y = 0.
\]
3.3. A DEFINITION OF BESSEL FUNCTIONS

The above derivation shows that the Bessel function of the first kind with order \( \nu \) is a special case of the confluent hypergeometric functions with specifically chosen parameters.
Chapter 4

Physical Applications of Bessel Functions

We consider radial vibration of circular membrane. We assume that an elastic circular membrane (radius $\ell$) can vibrate and that the material has a uniform density. Let $u(x, y, t)$ denote the displacement of the membrane at time $t$ from its equilibrium position. We use polar coordinate in the $xy$–plane by the change of variables:

$$x = r \cos \theta, \quad y = r \sin \theta.$$  \hspace{1cm} (4.1)

Then the corresponding equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$  \hspace{1cm} (4.2)

can be transformed to the form:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right).$$  \hspace{1cm} (4.3)

Since the membrane has uniform density, so $u = u(r, t)$ that is, it is independent of the $\theta$. Thus we have

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right).$$  \hspace{1cm} (4.4)

The boundary condition is $u(\ell, t) = 0$, and the initial conditions take the form

$$u(r, t) = f(r), \quad \frac{\partial u(r, 0)}{\partial t} = g(r).$$  \hspace{1cm} (4.5)

Separation of variables method yields

$$u(r, t) = R(r)T(t),$$  \hspace{1cm} (4.6)
which satisfies the boundary condition \( u(\ell, t) = 0 \). Thus

\[
(4.7) \quad RT'' = c^2 \left( R''T + \frac{1}{r}R'T \right).
\]

Hence

\[
(4.8) \quad \frac{R''}{R} + \frac{(1/r)R'}{R} + \frac{T''}{c^2T} = -\lambda^2,
\]

where the \( \lambda \) is a constant. Thus

\[
(4.9) \quad R'' + \frac{1}{r}R' + \lambda^2 R = 0
\]

\[
(4.10) \quad T'' + c^2\lambda^2 T = 0.
\]

We notice that the first equation is Bessel equation with \( \nu = 0 \). Thus its general solution is given by

\[
(4.11) \quad R(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r).
\]

Since \( Y_0 \) is unbounded when \( r = 0 \), so \( C_2 = 0 \). Thus the boundary condition implies that

\[
(4.12) \quad J_0(\lambda \ell) = 0,
\]

implying that \( \mu = \lambda \ell \) is a zero of \( J_0(\mu) \). Setting \( C_1 = 1 \), we obtain for each integer \( n = 1, 2, 3, \ldots \),

\[
(4.13) \quad R_n(r) = J_0(\lambda_n r) = J_0(\frac{\mu_n}{\ell} r),
\]

where \( \mu_n = \lambda_n \ell \) is the \( n \)-th zero of \( J_0(\mu) \). Thus we have

\[
(4.14) \quad u_n(r, t) = (A_n \cos c\lambda_n t + B_n \sin c\lambda_n t) \cdot J_0(\lambda_n r),
\]

for \( n = 1, 2, 3, \ldots \). Thus the general solution is given by

\[
(4.15) \quad \sum_{n=1}^{\infty} u_n(r, t) = \sum_{n=1}^{\infty} (A_n \cos c\lambda_n t + B_n \sin c\lambda_n t) \cdot J_0(\lambda_n r),
\]

and by

\[
(4.16) \quad f(r) = u(r, 0) = \sum_{n=1}^{\infty} A_n \cdot J_0(\lambda_n r),
\]
and

\begin{equation}
(4.17) \quad g(r) = \frac{\partial u(r, 0)}{\partial t} \bigg|_{t=0} = \sum_{n=1}^{\infty} B_n c \lambda_n \cdot J_0(\lambda_n r).
\end{equation}

Fourier-Bessel Series theory implies that

\begin{equation}
(4.18) \quad A_n = \frac{2}{\ell^2 J_1^2(\mu_n)} \int_0^\ell r f(r) J_0(\lambda_n r) \, dr,
\end{equation}

\begin{equation}
(4.19) \quad B_n = \frac{2}{c \lambda_n \ell^2 J_1^2(\mu_n)} \int_0^\ell r g(r) J_0(\lambda_n r) \, dr.
\end{equation}
CHAPTER 4. PHYSICAL APPLICATIONS OF BESSEL FUNCTIONS
Chapter 5

Separation of Variables of Helmholtz Equations

The Helmholtz equation named after the German physicist Hermann von Helmholtz refers to second order (elliptic) partial differential equations of the form:

\[(\Delta^2 + k^2)\Phi = 0,\]

where \(k\) is a constant. If \(k = 0\), then it reduces to the Laplace equations.

In this discussion, we shall restrict ourselves in the Euclidean space \(\mathbb{R}^3\). One of the most powerful theories developed in solving linear PDEs is the method of separation of variables. For example, the wave equation

\[
\left(\Delta^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi(r, t) = 0,
\]

can be solved by assuming \(\Psi(r, t) = \Phi(r) \cdot T(t)\) where \(T(t) = e^{i\omega t}\). This yields

\[
\left(\Delta^2 - \frac{\omega^2}{c^2}\right)\Phi(r) = 0,
\]

which is a Helmholtz equation. The questions now is under what 3—dimensional coordinate system \((u_1, u_2, u_3)\) do we have a solution that is in the separation of variables form

\[
\Phi(r) = \Phi_1(u_1) \cdot \Phi_2(u_2) \cdot \Phi_3(u_3)
\]

Theorem 5.0.1 (Eisenhart 1934). There are a total of eleven curvilinear coordinate systems in which the Helmholtz equation separates.

Each of the curvilinear coordinate is characterized by quadrics. That is, surfaces defined by

\[ Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + lz + J = 0. \]  

One can visit http://en.wikipedia.org/wiki/Quadric for some of the quadric surfaces. Curvilinear coordinate systems are formed by putting relevant orthogonal quadric surfaces. Wikipedia contains quite a few of these pictures. We list the eleven coordinate systems here:
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<td>(1) Cartesian</td>
<td>$x = x, \quad y = y, \quad z = z$</td>
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| (2) Cylindrical | $x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad z = z$  
|               | $\rho \geq 0, \quad -\pi < \phi \leq \pi$ |                      |
| (3) Spherical polar | $x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta$  
|                  | $r \geq 0, \quad 0 \leq \theta \leq \pi, \quad -\pi \leq \phi \leq \pi$ |                      |
| (4) Parabolic cylinder | $x = u^2 - v^2, \quad y = 2uv, \quad z = z$  
|                  | $u \geq 0, \quad -\infty < v < +\infty$ | Half-plane |
| (5) Elliptic cylinder | $x = f \cosh \xi \cos \eta, \quad y = f \sinh \xi \sin \eta, \quad z = z$  
|                  | $\xi \geq 0, \quad -\infty < \eta < +\infty$ | Infinite strip; Plane with straight aperture |
| (6) Rotation paraboloidal | $x = 2uv \cos \phi, \quad 2uv \sin \phi, \quad z = u^2 - v^2$  
|                  | $u, v \geq 0, \quad -\pi < \phi < \pi$ | Half-line |
| (7) Prolate spheroidal | $x = \ell \sinh u \sin v \cos \phi, \quad y = \ell \cosh u \sin v \sin \phi, \quad z = \ell \cosh u \cos v$  
|                  | $u \geq 0, \quad 0 \leq v \leq \pi, \quad -\pi < \phi \leq \pi$ | Finite line; segment Two half-lines |
| (8) Oblate spheroidal | $x = \ell \cosh u \sin v \cos \phi, \quad y = \ell \cosh u \sin v \sin \phi, \quad z = \ell \sinh u \cos v$  
|                  | $u \geq 0, \quad 0 \leq v \leq \pi, \quad -\pi < \phi \leq \pi$ | Circular plate (disc); Plane with circular aperture |
| (9) Paraboloidal | $x = \frac{1}{2} \ell (\cosh 2\alpha + 2 \cos 2\beta - \cosh 2\gamma), \quad y = 2\ell \cosh \alpha \cos \beta \sinh \gamma, \quad z = 2\ell \sinh \alpha \sin \beta \cosh \gamma$  
|                  | $\alpha, \gamma \geq 0, \quad -\pi < \beta \leq \pi$ | Parabolic plate; Plane with parabolic aperture |
| (10) Elliptic conal | $x = k \rho \sin \alpha \sin \beta; \quad y = (ik/k') \rho \cos \alpha \cos \beta; \quad z = (1/k') \rho \sin \alpha \sin \beta$,  
|                  | $r \geq 0, \quad -2K < \alpha \leq 2K, \quad -2K < \beta \leq 2K'$ | Plane sector; Including quarter plane |
### Chapter 5. Separation of Variables of Helmholtz Equations

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<td>Associated Legender</td>
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<td>(7) Prolate spheroidal</td>
<td>Associated Legender</td>
<td>Spheroidal wave</td>
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<tr>
<td>(8) Prolate spheroidal</td>
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<tr>
<td>(9) Paraboloidal</td>
<td>Mathieu</td>
<td>Whittaker-Hill</td>
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<tr>
<td>(10) Elliptic conal</td>
<td>Lame</td>
<td>Spherical Bessel, Lame</td>
</tr>
<tr>
<td>(11) Ellipsoidal</td>
<td>Lame</td>
<td>Ellipsoidal</td>
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</table>
1. Associated Legendre:
   \[(5.6) \quad (1 - x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + \left\{ n(n + 1) - \frac{m^2}{(1 - x^2)} \right\} y = 0 \]

2. Bessel:
   \[(5.7) \quad x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)y = 0 \]

3. Spherical Bessel:
   \[(5.8) \quad x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} + (x^2 - n(n + 1))y = 0 \]

4. Weber:
   \[(5.9) \quad \frac{d^2y}{dx^2} + (\lambda - \frac{1}{4}x^2)y = 0 \]

5. Confluent hypergeometric:
   \[(5.10) \quad x \frac{d^2y}{dx^2} + (\gamma - x) \frac{dy}{dx} - \alpha y = 0 \]

6. Mathieu:
   \[(5.11) \quad \frac{d^2y}{dx^2} + (\lambda - 2q \cos 2x)y = 0 \]

7. Spheroidal wave:
   \[(5.12) \quad (1 - x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + \left\{ \lambda - \frac{\mu^2}{(1 - x^2)} + \gamma^2(1 - x^2) \right\} y = 0 \]

8. Lame:
   \[(5.13) \quad \frac{d^2y}{dx^2} + (h - n(n + 1)k^2 \text{sn}^2 x)y = 0 \]

9. Whittaker-Hill:
   \[(5.14) \quad \frac{d^2y}{dx^2} + (a + b \cos 2x + c \cos 4x)y = 0 \]

10. Ellipsoidal wave:
    \[(5.15) \quad \frac{d^2y}{dx^2} + (a + bk^2 \text{sn}^2 x + qk^4 \text{sn}^4 x)y = 0 \]

**Remark 5.0.1.** The spheroidal wave and the Whittaker-Hill do not belong to the hypergeometric equation regime, but to the **Heun equation** regime (which has four regular singular points).
5.1 What is not known?

It is generally regarded that the Bessel functions, Weber functions, Legendre functions are better understood, but the remaining equations/functions are not so well understood.

1. Bessel functions. OK! Still some unknowns.

2. Confluent hypergeometric equations/functions. NOT OK.

3. Spheroidal wave, Mathieu, Lame, Whittaker-Hill, Ellipsoidal wave are poorly understood. Some of them are related to the Heun equation which has four regular singular points. Its research has barely started despite the fact that it has been around since 1910.

4. Mathematicians are separating variables of Laplace/Helmholtz equations, but in more complicated setting (such as in Riemannian spaces, etc)
Bibliography


