An oscillation result of a third order linear differential equation with entire periodic coefficients

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Dedicated to the memory of Steven Bank

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We prove that the periodic equation $f''' - K f' + e^2 f = 0$ admits a solution with finite exponent of convergence if and only if $K = (n + 1)^2/9$ where $n$ is a non-negative integer satisfying a certain $(n + 1) \times (n + 1)$-determinant condition. Moreover, we obtain explicit representations for such solutions. Our result is somewhat similar to a result due to Bank, Laine and Langley [5] for a second order equation.

Keywords: differential equation; periodic coefficients oscillation result

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1. INTRODUCTION

We are concerned with the number of zeros of a third order linear differential equation with entire periodic coefficients. Our domain will be the entire complex plane and we shall employ Nevanlinna value

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distribution theory. For details of this theory we refer the reader to the

Let \( f(z) \) be a solution of an arbitrary linear differential equation with
its zeros \( a_1, a_2, a_3, \ldots \) ordered by increasing moduli. We define the exponent of convergence \( \lambda(f) \) of \( f \) to be \( \inf \{ \lambda : \sum_{i=1}^{\infty} 1/|a_i|^\lambda < \infty \} \). The theory of complex oscillation of differential equations is to investigate
how the quantity \( \lambda(f) \) is affected by the coefficients of the equation
and what value it takes including infinity. See Bank and Laine [2], and
Laine [12].

Results concerning linear differential equations with entire periodic
coefficients are particularly interesting. In fact, Bank and Laine [3] were
able to find explicit representations for solutions \( f \) of such equations
in the second order case, provided \( \lambda(f) < \infty \). These representations
depend upon whether \( f(z) \) and \( f(z+\omega) \) are linearly independent, where \( \omega \) is the period of the coefficients. Their results have been generalized
to some higher order equations by Bank and Langley [8], see Lemma D
below.

In another paper [4], Theorem 2, Bank, Laine and Langley proved a
specific result concerning the complex oscillation of a periodic second
order differential equation.

\textbf{THEOREM A.} Let \( K \in \mathbb{C} \) and suppose that

\[
f'' + (e^z - K)f = 0 \tag{1.1}
\]

has a non-trivial solution \( f(z) \) such that \( \lambda(f) < \infty \). Then

\[
K = \frac{q^2}{16}, \tag{1.2}
\]

where \( q \) is an odd positive integer. Conversely, if \( K \) is of the form (1.2),
then the equation (1.1) admits two linearly independent solutions \( f_1 \) and
\( f_2 \) each with \( \lambda(f_i) \leq 1, i = 1, 2 \).

Of course, Theorem A is a special case of the following Theorem B,
where \( \lambda(f) \) is more restricted, see Bank, Laine and Langley [5],
Theorem 3.3.

\textbf{THEOREM B.} Let \( P \) be a polynomial of degree \( n \geq 1 \), and let \( Q \) be an
entire function of order \( \sigma(Q) < n \). Suppose that the equation

\[
f'' + (e^p + Q)f = 0 \tag{1.3}
\]
admits a non-trivial solution \( f(z) \) with \( \lambda(f) < n \). Then \( f \) has no zeros, \( Q \) is a polynomial and

\[
Q = -\frac{1}{16} (P')^2 + \frac{1}{4} P''.
\]

Clearly, the equation (1.3) reduces to (1.1), provided \( P(z) \equiv z \). We remark that (1.1) plays an important role in a number of recent papers, see Bank and Langley [6], [7], Chiang [9] and Wang [13], [14].

In [10], Theorem 3.2, Theorem B was extended to some third order equations. As a special case, we recall

**THEOREM C.** Let \( P \) be a polynomial of degree \( n \geq 1 \), and \( Q_1, Q \) be entire functions each of order \( < n \). Suppose that

\[
f''' + Q_1 f' + (e^p + Q)f = 0
\]  \hspace{1cm} (1.4)

admits a solution \( f \) such that \( \lambda(f) < n \). Then \( f \) has no zeros, \( Q_1 \) and \( Q \) are polynomials such that

\[
Q_1 = -\frac{1}{9} (P')^2 + \frac{2}{3} P''
\]

and

\[
Q = \frac{1}{3} P^{(3)} - \frac{1}{9} P' P''.
\]

In view of the relation between Theorem A and Theorem B, it is natural to ask, whether a similar result related to Theorem C holds in the case of \( P(z) \equiv z \), i.e., provided the coefficients of (1.4) are periodic. We prove such a result below, including explicit representations of solutions. Observe that A. Baesch determines, in a forthcoming paper [1], all solutions \( f \) of

\[
f^{(k)} + \sum_{j=1}^{k-2} A_j f^{(j)} + A_0(z) f = 0, \quad k \geq 3,
\]  \hspace{1cm} (1.5)

where \( A_1, \ldots, A_{k-2} \) are constants and \( A_0(z) \) is a nonconstant periodic entire function rational in \( e^z \), such that \( \log^+ N(r, 1/f) = o(r) \). She proves that this situation appears if and only if at least one of certain \( k^2 \) linear differential equations with polynomial coefficients admits a non-trivial polynomial solution. Our result below deals with a special
case of (1.5) only. However, our characterization is of a more simple, constructive type. The open determinant problem described in Section 4 is perhaps of some independent interest.

2. THE MAIN RESULT

THEOREM 1. Let $K \in \mathbb{C}$, and suppose that

$$f''' - K f' + e^f = 0$$  \hspace{1cm} (2.1)

admits a non-trivial solution $f$ such that

$$\log^+ N(r, 1/f) = o(r)$$

as $r \to \infty$. Then there exist two integers $r$ and $s$, $r + s \geq 0$, such that

$$K = \frac{(r + s + 1)^2}{9}. \hspace{1cm} (2.2)$$

Moreover, if $n = r + s > 0$, then $n$ satisfies the following tridiagonal $(n + 1) \times (n + 1)$-determinant condition:

$$\det A = 0, \hspace{1cm} (2.3)$$

where the non-zero diagonals of $A$ are determined by

$$\left\{ \begin{array}{ll}
    a_{j,j-1} := (j - 1)j(j + 1) - 2jn - jn^2, & j = 1, \ldots, n, \\
    a_{j,j} := -3j(j + 1) + 2n + n^2, & j = 0, \ldots, n, \\
    a_{j,j+1} := 3(j + 1), & j = 0, \ldots, n - 1. 
\end{array} \right. \hspace{1cm} (2.4)$$

Furthermore, $f$ admits one of the following representations:

$$f_i(z) = e^{-\frac{z-1}{3}} \psi(e^{\xi^3}) \exp(c_i e^{\xi^3}), \hspace{1cm} (2.5)$$

where $c_i^3 + 27 = 0$, $i = 1, 2, 3$, and

$$\psi(\xi) = \sum_{j=-r}^{s} d_j \xi^j, \quad d_{-r}d_s \neq 0. \hspace{1cm} (2.6)$$

Conversely, suppose $K$ takes the form (2.2) and, if $n = r + s > 0$, then $n$ satisfies (2.3) and (2.4). Then there exists a rational function of
the form (2.6) such that the three functions defined by (2.5) are linearly independent solutions of (2.1) each with \( \lambda(f_i) \leq 1 \) for \( i = 1, 2, 3 \).

**Remark**  The hypothesis \( \log^+ N(r, 1/f) = o(r) \) as \( r \to \infty \) that we have made above is in fact weaker than \( \lambda(f) < \infty \), see Lemma D below.

The proof of Theorem A depends heavily on the explicit representation of solutions of periodic differential equations obtained by Bank and Laine [3], and a special non-linear second order differential equation in \( E = f_1 f_2 \), where \( f_1 \) and \( f_2 \) are two linearly independent solutions, see [3], p. 6. For higher order equations, such useful differential equation in \( E \) has been found. Our argument depends on the following representation lemma obtained by Bank and Langley for higher order equations, see [8], Theorem 2:

**Lemma D.** Suppose that \( k \geq 3, \) that \( A_0 \) is a non-constant periodic entire function, rational in \( e^z \), and that \( A_1, \ldots, A_{k-2} \) are constants. Suppose finally that \( f \) is a non-trivial solution of

\[
y^{(k)} + \sum_{j=0}^{k-2} A_j(z)y^{(j)} = 0
\]

such that

\[
\log^+ N(r, 1/f) = o(r)
\]

as \( r \to \infty \). Then there exists an integer \( q \) with \( 1 \leq q \leq k \), a constant \( d \), and rational functions \( \psi(\xi) \) and \( S(\xi) \), analytic on \( 0 < |\xi| < \infty \) such that

\[
f(z) = \psi(e^{z/q}) \exp \left( dz + S(e^{z/q}) \right).
\]

3. **Proof of Theorem 1**

Under the hypothesis of Theorem 1 and by Lemma D (2.7), we may write \( f \) as

\[
f(z) = e^{dz} G(e^{z/q}),
\]

where \( G(\xi) = \psi(\xi) \exp(S(\xi)) \), \( 1 \leq q \leq 3 \), \( d \) is a constant and both \( \psi \) and \( S \) are rational and analytic on \( 0 < |\xi| < \infty \).
By substituting \(f(z)\) of (3.1) into (2.1) and denoting \(\zeta = e^{z/q}\), we have
\[
\zeta^3 G^{(3)}(\zeta) + (3dz^q + 3)\zeta^2 G''(\zeta) + (3d^2 q^2 + 3dz^q + 1 - q^2K)\zeta G'(\zeta) + q^3(\zeta^q + d^3 - Kd)G(\zeta) = 0.
\] (3.2)

We denote now
\[
\psi(\zeta) = \sum_{j=-r}^{s} c_j \zeta^j
\] (3.3)
and
\[
S(\zeta) = \sum_{j=-n}^{m} d_j \zeta^j.
\] (3.4)

Since \(f\) must be of infinite order, we have \((m, n) \neq (0, 0)\). We may also assume that \(s \geq -r\) and \(m \geq -n\). Then we have, for \(m \geq 1\),
\[
\frac{G'(\zeta)}{G(\zeta)} = \alpha \zeta^{m-1} + O(\zeta^{m-2}), \quad \frac{G''(\zeta)}{G(\zeta)} = \alpha^2 \zeta^{m-2} + O(\zeta^{2m-3})
\]
and
\[
\frac{G^{(3)}(\zeta)}{G(\zeta)} = \alpha^3 \zeta^{m-3} + O(\zeta^{3m-4})
\]
as \(\zeta \to \infty\) and \(\alpha \neq 0\) is a constant. It follows from (3.2) that \((m - 1) + 3 = q\), and since \(1 \leq q \leq 3\), we deduce readily that \(q = 3\) and \(m = 1\) in (3.4). Therefore, we must have \(m \leq 1\). Moreover, by considering \(G_1(t) = G(1/t)\), we have, again from (3.2), the following equation:
\[
t^3G^{(3)}_1(t) + (1 - dq)_t^2G''_1(t) + (3d^2 q^2 - 3dq + 1 - Kq^2)_tG'_1(t) - q^3(t^{-q} + d^3 - Kd)G_1(t) = 0.
\] (3.5)

Likewise, we deduce, for \(n \geq 1\),
\[
G'_1(t)/G_1(t) \sim \beta t^{n-1}, \quad G''_1(t)/G_1(t) \sim \beta^2 t^{2n-2}
\]
and
\[
G^{(3)}_1(t)/G_1(t) \sim \beta^3 t^{3n-3}
\]
for some constant \(\beta \neq 0\) as \(t \to \infty\). It follows from (3.5) that \((3n - 3) + 3 = 0\) and hence \(n = 0\). Therefore we must have \(n \leq 0\). However,
recalling that \((m, n) \neq (0, 0)\) and \(m \geq -n\), we have \(m = 1, n = 0\) and so \(G\) may be written as

\[ G(\xi) = \psi(\xi) \exp(c\xi) \quad (3.6) \]

for some non-zero constant \(c\). From this expression, we have \(G^{(j)}(\xi)/G(\xi) \sim c^j\) as \(\xi \to \infty, j = 1, 2, 3\). Substituting these estimates into (3.2) once more, we deduce that \(c^3 + \hat{q}^3 = 0\), i.e., \(c^3 + 27 = 0\).

Substituting now (3.6) into (3.2), and making use of \(q = 3\), we get

\[
\xi^3 \psi''(\xi) + (3c^3 + (9d + 3)x^2) \psi''(\xi) + (3c^2x^3 + 2c(9d + 3)x^2 \\
+ (27d^2 + 9d + 1 - 9K)x) \psi'(\xi) + (c^2(9d + 3)x^2 \\
+ c(27d^2 + 9d + 1 - 9K)x + 27(d^3 - Kd)) \psi(\xi) = 0.
\quad (3.7)
\]

Substituting (3.3) into (3.7), making use of \(c^3 + 27 = 0\), and collecting the coefficient of the highest term \(\xi^{t+2}\) in (3.7), we get

\[(3d + 1 + s)3c^2c_s = 0,\]

hence \(d = (-s - 1)/3\). Likewise, the coefficient of the lowest term \(\xi^{-r}\) is

\[
((-r)(-r - 1)(-r - 2) + 3(3d + 1)(-r)(-r - 1) \\
+ (27d^2 + 9d + 1 - 9K)(-r) + 27(d^3 - Kd)) c_{-r} = 0,
\]

and so we must have

\[ r^3 - 9dr^2 + 27d^2r - 27d^3 = 9(r - 3d)K. \]

Therefore,

\[ K = \frac{r^3 - 9dr^2 + 27d^2r - 27d^3}{9(r - 3d)} = \frac{1}{9}(r - 3d)^2 = \frac{1}{9}(r + s + 1)^2. \quad (3.8) \]

Gathering the results from above, we deduce that \(f\) takes the form

\[ f(z) = e^{-z^{1+3}} \psi(e^{z/3}) \exp(c e^{z/3}), \quad \quad (3.9) \]

where \(c^3 + 27 = 0\) and

\[ \psi(\xi) = \sum_{j=-r}^{s} d_j \xi^j, \quad d_{-r}d_s \neq 0. \]
It remains to verify the determinant condition (2.3). Setting \( n = r + s \) and assuming that \( n > 0 \), we rewrite \( f \) as
\[
f(z) = \Psi(e^{-z^3}) \exp(c e^{z^3} - z^3),
\]
where
\[
\Psi(\xi) = \sum_{j=0}^{n} e_j \xi^j, \quad e_j = d_{s-j} \text{ and } e_0 e_n = d_{s-d-r} \neq 0.
\]
Substituting (3.10) into (2.1), and making use of (3.8), we obtain
\[
\xi^3 \Psi^{(3)}(\xi) + 3(2\xi^2 - c\xi)\Psi''(\xi) + ((6-2n-n^2)\xi - 6c + 3c^2/\xi) \Psi'(\xi) - (n^2 + 2n)(1-c/\xi)\Psi(\xi) = 0.
\]
Then we substitute (3.11) into (3.12), and this gives
\[
\sum_{j=-1}^{n-1} B_j \xi^j = 0,
\]
where
\[
B_{-1} = (n^2 + 2n)ce_0 + 3c^2e_1,
\]
\[
B_j = (j-n)(j+n+2)(j+1)e_j - \left\{ 3(j+1)(j+2) - 2n - n^2 \right\} ce_{j+1} + 3c^2(j+2)e_{j+2}
\]
for \( 0 \leq j \leq n - 2 \), and
\[
B_{n-1} = - (2n^2 + n)(e_{n-1} + ce_n).
\]
Therefore, we must have \( B_j = 0 \) for all \( j = -1, \ldots, n - 1 \). Let now \( \mathbf{B} \) denote the tridiagonal determinant whose non-zero diagonals are determined by
\[
\begin{align*}
b_{j,j-1} &:= a_{j,j-1}, & j = 1, \ldots, n, \\
b_{j,j} &:= ca_{j,j}, & j = 0, \ldots, n, \\
b_{j,j+1} &:= c^2 a_{j,j+1}, & j = 0, \ldots, n - 1,
\end{align*}
\]
see (2.4). Then the above result can be rewritten as a matrix equation
\[
\mathbf{B} \times \begin{pmatrix} e_0 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.
\]
As \( e_0e_n \neq 0 \), the determinant \( \det(B) \) must be zero for (3.14) to admit a non-trivial solution. Since \( \det(B) = c^{n+1} \det(A) \), this proves the necessary part of Theorem 1.

To prove the converse, it is immediately seen that \( f_i(z) = \exp(c_i e^{z/3} - z/3) \), where \( c_i^2 + 27 = 0 \), \( i = 1, 2, 3 \), are linearly independent zero-free solutions of (2.1) for \( K = 1/9 \). Hence, we may assume that \( K = (n + 1)^2/9 \) where \( n = r + s > 0 \) and \( r, s \) are two given integers. We define \( \Psi(\xi) \) by (3.11) where the coefficients \( c_j \), \( 0 \leq j \leq n \) are as given after (3.13), satisfying (3.14). Therefore, by reversing the argument above, the function defined by (3.11) solves the equation (3.12), provided \( c^3 + 27 = 0 \). In particular, the function (3.10) then solves (2.1) and it can be written as \( f(z) = e^{-\xi^{n-1} / 3} \psi(e^{z/3}) \exp(c e^{z/3}) \), where \( \psi(\xi) = \sum_{j=-r}^{r} d_j / \xi^j \), which is precisely (2.5).

4. CONCLUDING REMARKS

In Theorem A, a solution of (1.1) with \( \lambda(f) < \infty \) exists for each possible \( n \). The situation in Theorem 1 is different. In fact, the tridiagonal determinant condition (2.3) seems to be equivalent to \( n \neq 3k + 2, k = 0, 1, 2, \ldots \). This has been verified numerically up to \( n = 100 \). Unfortunately, we have been unable to find a general proof. As the referee has pointed out, the condition (2.3) in fact implies that \( n \neq 3k + 2, k = 0, 1, 2, \ldots \), by applying a simple congruence argument on the formulae below. The converse conclusion seems to be a non-trivial problem. By elementary linear algebra, the tridiagonal matrix \( A \) in (2.3) can be expressed as the product of three matrices \( (\beta_{ij}), (\alpha_{ij}) \) and \( (\gamma_{ij}) \), where \( (\alpha_{ij}) \) is a diagonal matrix, while \( (\beta_{ij}) \) is a lower triangular matrix such that \( \beta_{ij} = 1 \) for all \( i \), and \( (\gamma_{ij}) \) an upper triangular matrix such that \( \gamma_{ii} = 1 \) for all \( i \). Therefore, it suffices to consider the vanishing of \( \det(\alpha_{ij}) \). Now, it is easy to see that \( \alpha_{0,0} = a_{0,0} \) and that the recursion formula

\[
\alpha_{j+1,j+1} = a_{j+1,j+1} - \frac{a_{j+1,j}a_{j,j+1}}{\alpha_{j,j}}, \quad j = 0, \ldots, n - 1
\]

holds. By (2.4), this results in a continued fractional representation

\[
\alpha_{j+1,j+1} = A_{j,j} + \frac{B_{j,j}}{\alpha_{j,j}}, \quad j = 0, \ldots, n - 1,
\]
where

\[ A_{j,j} = -3(j + 1)(j + 2) + 2n + n^2, \]
\[ B_{j,j} = -3(j + 1)(j(j + 1)(j + 2) - 2(j + 1)n - (j + 1)n^2), \]

for the diagonal elements of \((\alpha_{ij})\). Hence, the determinant condition (2.3) reduces to the question whether at least one of the diagonal elements \(\alpha_{j,j}, \ j = 0, \ldots, n\), vanishes.

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**References**