3.7 If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Rudin's Ex. 8, this is the so-called Abel's Test.

Proof Suppose $\{b_n\}$ is increasing, otherwise we use $-b_n$ to replace b_n in the proof. Since $\{b_n\}$ is also bounded, by Theorem 3.14, it converges. Suppose the limit is b. Then the series $\sum a_n(b_n - b)$ satisfies the conditions in Theorem 3.42 (Dirichlet's Test). Hence $\sum a_n(b_n - b)$ coverges. The series $\sum a_n b_n$ converges follows from $\sum a_n b_n = \sum a_n(b_n - b) + b \sum a_n$. ∎

3.10 Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Rudin's Ex. 13

Proof For given two series $\sum a_n$ and $\sum b_n$, their Cauchy product is the series $\sum c_n$, where $c_n = \sum_{k=0}^{n} a_k b_{nk}$. Put $A_n = \sum_{k=0}^{n} |a_k|$ and $B_n = \sum_{k=0}^{n} |b_k|$. By the condition, we can assume that $A_n \to A$ and $B_n \to B$. Since A_n and B_n are increasing, we know $A_n \leq A$ and $B_n \leq n$ for all n. To see that $\sum c_n$ converges absolutely, we have

$$\begin{aligned}
|C_n| = \sum_{k=0}^{n} |c_n| &= |a_0 b_0| + |a_0 b_1 + a_1 b_0| + \cdots + |a_0 b_n + a_1 b_{n-1} + \cdots a_n b_0| \\
&\leq |a_0|\,|b_0| + (|a_0|\,|b_1| + |a_1|\,|b_0|) + \cdots \\
&\quad + (|a_0|\,|b_n + |a_1|\,|b_{n-1}| + \cdots |a_n|\,|b_0|) \\
&= |a_0| B_n + |a_1| B_{n-1} + \cdots + |a_n| B_0 \\
&\leq |a_0| B_n + |a_1| B_n + \cdots + |a_n| B_n = A_n B_n \leq AB.
\end{aligned}$$

Thus, $\sum c_n$ converges absolutely. ∎

3.11 Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, x_4, \ldots$, by the recursion formula

Rudin's Ex. 16

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

(a) Prove that $\{x_n\}$ decreasing monotonically and that $\lim x_n = \sqrt{\alpha}$.

(b) Put $\varepsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta \left(\frac{\varepsilon_1}{\beta}\right)^{2^n}, \qquad n = 1, 2, 3, \ldots.$$

(c) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\varepsilon_1/\beta = \frac{1}{10}$ and that therefore

$$\varepsilon_5 < 4 \cdot 10^{-16}, \qquad \varepsilon_6 = 4 \cdot 10^{-32}.$$

Proof It is clear that $x_n > 0$ for all n. We can prove that

$$x_n \geq \sqrt{\alpha}, \qquad n \geq 1,$$

by mathematical induction. In fact, $x_1 \geq \sqrt{\alpha}$ by the initial choice. If $x_n \geq \sqrt{\alpha}$, then

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) \geq \frac{1}{2} \cdot 2\sqrt{x_n \cdot \frac{\alpha}{x_n}} = \sqrt{\alpha}.$$

Moreover, the sequence $\{x_n\}$ is decreasing monotonically, since

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) = \frac{x_n^2 - \alpha}{2x_n} \geq 0.$$

Hence, by Theorem 3.14, $\lim x_n$ exists. Put $\lim x_n = x$. Then$x \geq \sqrt{\alpha} > 0$. Letting $n \to \infty$ in the formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right),$$

we have

$$x = \frac{1}{2}\left(x + \frac{\alpha}{x}\right),$$

which gives $x = \pm\sqrt{\alpha}$. We have $x = \sqrt{\alpha}$ because $\{x_n\}$ is bounded below by $\sqrt{\alpha} > 0$.

(b) By using $x_n > \sqrt{\alpha}$ in part (a), for $\varepsilon_n = x_n - \sqrt{\alpha}$, we have

$$\varepsilon_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} = \frac{(x_n - \sqrt{\alpha})^2}{2x_n} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}.$$

We prove that for $n \geq 1$, with $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n},$$

by mathematical induction.

In fact, for $n = 1$,

$$\varepsilon_2 < \frac{\varepsilon_1^2}{2\sqrt{\alpha}} = 2\sqrt{\alpha}\left(\frac{\varepsilon_1}{2\sqrt{\alpha}}\right)^2 = \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^1}.$$

Suppose the inequality holds for $n = k$:

$$\varepsilon_{k+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^k}.$$

Then,

$$\varepsilon_{(k+1)+1} < \frac{\varepsilon_{k+1}^2}{2\sqrt{\alpha}} < \frac{\left(\beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^k}\right)^2}{2\sqrt{\alpha}} = \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^{k+1}}.$$

This completes the proof of part (b).

(c) If take $\alpha = 3$ and $x_1 = 2$, then

$$\varepsilon_1/\beta = (x_1 - \sqrt{\alpha})/2\sqrt{\alpha} = (2 - \sqrt{3})/2\sqrt{3} = \frac{1}{6}(2\sqrt{3} - 3).$$

Use the fact that $\sqrt{3} < 1.8$, we have $\frac{1}{6}(2\sqrt{3} - 3) = 0.1$. Thus,

$$\varepsilon_1/\beta < \frac{1}{10}.$$

The error bounds for the computation are

$$\begin{aligned}
\varepsilon_5 &= x_5 - \sqrt{3} < 2\sqrt{3} \times 10^{-2^4} < 4 \cdot 10^{-16}; \\
\varepsilon_6 &= x_6 - \sqrt{3} < 2\sqrt{3} \times 10^{-2^5} < 4 \cdot 10^{-32}.
\end{aligned}$$