4.11 Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$. Rudin's Ex. 14

Proof Put $g(x) = x - f(x)$ for $x \in I$. It is clear that g is continuous on I. If $g(0) = 0$ or $g(1) = 0$, the conclusion of the problem holds either for $x = 0$ or $x = 1$. Otherwise, we have $g(0) = -f(0) < 0$ and $g(1) = 1 - f(1) > 0$. By Theorem 4.23 (the Intermediate Value Theorem), there exists a $x \in (0, 1)$ such that $g(x) = 0$, since $g(0) < 0 < g(1)$. This gives $f(x) = x$ for this x. ∎

4.14 Let f be a real function defined on (a, b). Prove that the set of points at which f has a simple discontinuity is at most countable. Rudin's Ex. 17

f has a simple discontinuity at x if f is discontinuous at x with $f(x-)$ and $f(x+)$ existing.

Proof Put

$$D = \{x \in (a, b)\colon f \text{ is discontinuous at } x,\ f(x-) \text{ and } f(x+) \text{ exist}\},$$

and

$$\begin{aligned} E &= \{x \in (a, b)\colon f(x-) < f(x+)\}, \\ F &= \{x \in (a, b)\colon f(x-) > f(x+)\}, \\ G &= \{x \in (a, b)\colon f(x-) = f(x+) > f(x)\}, \\ H &= \{x \in (a, b)\colon f(x-) = f(x+) < f(x)\}. \end{aligned}$$

It is clear that $D = E \cup F \cup G \cup H$. We shall prove that E and G are at most countable. For F and H, we only need to replace f by $-f$.

To show that E is at most countable, for each $x \in E$, we associate it with a triple (p, q, r) of rational numbers such that $f(x-) < p < f(x+)$ and $a < q < x < r < b$. Since $\lim_{t \to x^-} f(t) = f(x-) < p$, we can further require that q is sufficient close to x such that $q < t < x$ implies $f(t) < p$. Similarly, we can require that r is sufficient close to x such that $x < t < r$ implies $p < f(t)$.

Conversely, we show that if a triple (p, q, r) of rational numbers associates with x_1 and x_2 in E, then $x_1 = x_2$. If fact, if $x_1 < x_2$, we take a point t such that $x_1 < t < x_2$. Hence, by the construction of the triple, we have $q < x_1 < t < x_2 < r$. This leads to a contradiction, since $q < t < x_2$ implies $f(t) < p$ while $x_1 < t < r$ implies $f(t) > p$. Similarly $x_1 > x_2$ is also impossible. Thus, $x_1 = x_2$.

Therefore, E is at most countable, by Theorem 2.13.

The proof of G being at most countable is similar to that of E. A triple (p, q, r) of rational numbers is associated with a point $x \in G$ such that $f(x-) = f(x+) > p > f(x)$ and $a < q < x < r < b$. We further require q and r are so close to x such that $q < t < x$ implies that $f(t) > p$ and $x < t < r$ implies $f(t) > p$.

We can similarly prove that a triple (p, q, r) of rational numbers associates with x_1 and x_2 in G, then $x_1 = x_2$. In fact, if $x_1 < x_2$, then $q < x_1 < x_2 < r$. The choice of q gives $f(x_1) > p$, but the choice of p gives $p > f(x_1)$, a contradiction. Similarly, $x_1 > x_2$ also leads to a contradiction. Thus, $x_1 = x_2$.

Since E, F, G, H are all at most countable, so is D. ∎

4.15 Every rational x can be written in the form $x = m/n$, where $n > 0$, and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on $\mathbb{R}$ by

Rudin's Ex. 18

This function is known as Thomae's function, named after Johannes Karl Thomae (1840-1921), also known as the popcorn function, the raindrop function, the ruler function, the Riemann function or the Stars over Babylon.

$$f(x) = \begin{cases} 0, & \text{if } x \text{ irrational}, \\ \dfrac{1}{n}, & \text{if } x = \dfrac{m}{n}. \end{cases}$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

Proof By the definition of f, the only numbers x satisfying $f(x) \geq \dfrac{1}{n}$ are those rational numbers $x = \dfrac{p}{q}$ with $1 \leq q \leq n$. For any $x \in \mathbb{R}$, if we put

$$S_q = \left\{ \frac{p}{q} : p \in \mathbb{Z}, \ \left| \frac{p}{q} - x \right| < 1 \right\},$$

then $|S_q| \leq 2q$, since $x = [x] + (x)$. Hence, for any fixed positive integer n, on any bounded interval, we have $0 \leq f(x) < \dfrac{1}{n}$ for all but finitely many rational x. For any irrational a, let $\delta > 0$ be the smallest distance between a and these rational numbers. Then, for $|x - a| < \delta$, we have

$[x]$ is the largest integer contained in x, and (x) is the fractional part of x.

$$|f(x) - f(a)| < \frac{1}{n},$$

which implies f is continuous at irrational a. On the other hand, for any rational $a = \dfrac{p}{q}$, since the set of irrational numbers in any segment uncountable, we can find a sequence of irrational numbers $\{x_n\}$ such that $x_n \to a$. Hence, for this sequence, we have

$$|f(x_n) - f(a)| = \frac{1}{q} \nrightarrow 0,$$

as $n \to \infty$. Hence, f is not continuous at rational a. ∎

4.16 Suppose f is a real function with domain $\mathbb{R}$ which has the intermediate value property: If $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b. Suppose also, for every rational r, that the set of all x with $f(x) = r$ is closed. Prove that f is continuous.

Rudin's Ex. 19

Proof If f is not continuous at some point $a \in \mathbb{R}$, then there is a sequence $\{x_n\}$ satisfying $x_n \to x_0$, such that $f(x_n) \nrightarrow f(a)$. This implies that for some $\epsilon_0 > 0$, there is a subsequence $\{x_{n_k}\}$ such that $|f(x_{n_k}) - f(a)| \geq \epsilon_0$. Without loss of generality, we assume that $\{x_{n_k}\}$ is monotonic and

$$f(x_{n_k}) \geq f(a) + \epsilon_0.$$

Take a rational number r such that

$$f(x_{n_k}) > r > f(a), \qquad k = 1, 2, 3, \ldots.$$

By the intermediate value property, there is a sequence $\{t_k\}$ between a and x_{n_k} such that $f(t_k) = r$ for each k. By $x_{n_k} \to a$, we know that $t_k \to a$. By the hypothesis, the set

$$S_r = \{x\colon f(x) = r\}$$

is closed, we conclude that $a \in S_r$, i.e., $f(a) = r$. This contradicts to $r > f(a)$. Therefore, f is continuous. ∎

4.17 If E is nonempty subst of a metric space X, define the distance from $x \in X$ to E by Rudin's Ex. 20

$$\rho_E(x) = \inf_{y \in E} d(x, y).$$

(a) Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.

(b) Prove that ρ_E is a uniformly continuous function on X, by showing that

$$|\rho_E(x) - \rho_E(y)| \le d(x, y)$$

for all $x \in X$, $y \in X$.

Proof (a) It is obvious that $\rho_E(x) \ge 0$. If $x \in \overline{E}$, then either $x \in E$ or $x \in E' - E$. If $x \in E$, from

$$0 = d(x, x) \ge \inf_{y \in E} d(x, y) = \rho_E(x),$$

we have $\rho_E(x) = 0$. If $x \in E' - E$, then x is a limit point of E. Hence any neighborhood $N_{1/n}(x)$ of x contains a point $y_n \neq x$ such that $y_n \in E$. Hence, we have

$$\rho_E(x) = \inf_{y \in E} d(x, y) \le \inf_{n} d(x, y_n) < 1/n \to 0,$$

which also implies $\rho_E(x) = 0$.

Conversely, if $\rho_E(x) = 0$, and if $x \notin E$, then, by the definition of inf, for any positive integer n, there is a point y_n in E such that

$$d(x, y_n) < \inf_{y \in E} d(x, y) + 1/n = 1/n.$$

Hence $y_n \to x$, so x is a limit point of E, i.e., $x \in \overline{E}$.

(b) Let $x \in X$, $y \in X$. For any $z \in E$, we have

$$\rho_E(x) = \inf_{z' \in E} d(x, z') \le d(x, z) \le d(x, y) + d(y, z).$$

Since z is arbitrary in E, the last inequality implies

$$\rho_E(x) \le d(x, y) + \inf_{z \in E} d(y, z) = d(x, y) + \rho_E(y).$$

Hence $\rho_E(x) - \rho_E(y) \le d(x, y)$. Reversing the roles of x and y, we similarly have $\rho_E(y) - \rho_E(x) \le d(x, y)$. Hence

$$|\rho_E(x) - \rho_E(y)| \le d(x, y).$$

This inequality implies uniform continuity of the function ρ_E. In fact, for any $\epsilon > 0$, if we take $\delta = \epsilon$, then $d(x, y) < \delta$ implies

$$|\rho_E(x) - \rho_E(y)| < \epsilon.$$

4.18 Suppose K and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K$, $q \in F$. Show that the conclusion may fail for two disjoint closed sets if neither is compact. Rudin's Ex. 21

Proof Suppose there exists no such δ. Then, for any positive integer n, there exist $x_n \in K$ and $y_n \in F$ such that $d(x_n, y_n) < 1/n$. If $\{x_n\}$ is eventually constant, that is, there is a N such that $x_n = a$ for $n \geq N$, then the last inequality implies that $y_n \to a$. Since F is closed, we have $a \in F$. Hence $a \in K \cap F$, which contradicts to the hypothesis $K \cap F = \emptyset$. If $\{x_n\}$ is a infinite set, by Theorem 2.37, it has a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to a$ for some a in K.

Let $\epsilon > 0$ be given. By $x_{n_k} \to a$, there exists K_1 such that $k \geq K_1$ implies

$$d(x_{n_k}, a) < \epsilon/2.$$

Since $d(x_n, y_n) \to 0$, there exists K_2 such that $k \geq K_2$ implies

$$d(x_{n_k}, y_{n_k}) < \epsilon/2.$$

Hence, if $k \geq \max\{K_1, K_2\}$,

$$d(y_{n_k}, a) \leq d(x_{n_k}, y_{n_k}) + d(x_{n_k}, a) < \epsilon.$$

This means that $y_{n_k} \to a \in F$, since F is closed. We again have $a \in K \cap F$ that contradicts to $K \cap F = \emptyset$.

Consider two disjoint subsets in $\mathbb{R}$:

$$F_1 = \left\{ n - \frac{1}{n} : n \in \mathbb{Z} \right\} \quad \text{and} \quad F_2 = \left\{ n + \frac{1}{n} : n \in \mathbb{Z} \right\}.$$

Both are closed, since they have no limit points. Since

$$\left| \left(n - \frac{1}{n} \right) - \left(n + \frac{1}{n} \right) \right| = \frac{2}{n} \to 0,$$

there exists no $\delta > 0$ such that $|p - q| > \delta$ for $p \in F_1$, $q \in F_2$. ∎