Formulas

Trigonometric formulas

$\sin a \pm \sin b = 2\sin\frac{1}{2}(a\pm b)\cos\frac{1}{2}(a\mp b)$

$\cos a + \cos b = 2\cos\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b)$ $\qquad$ $\cos a - \cos b = 2\sin\frac{1}{2}(a+b)\sin\frac{1}{2}(b-a)$

$\sin a\ \cos b = \frac{1}{2}(\sin(a-b)+\sin(a+b))$

$\sin a\ \sin b = \frac{1}{2}(\cos(a-b)-\cos(a+b))$ $\qquad$ $\cos a\ \cos b = \frac{1}{2}(\cos(a-b)+\cos(a+b))$

$\sin(a\pm b) = \sin a\cos b \pm \cos a \sin b$ $\qquad$ $\cos(a\pm b) = \cos a\cos b \mp \sin a\sin b$

Fourier series

Fourier series of $f(x)$ defined on $[-L,-L]$: $\quad \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos(n\pi x/L) + b_n\sin(n\pi/L))$

where $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)dx,\ \ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)dx.$

Fourier cosine series ($x\in[0,L]$): $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(n\pi x/L)$ where $a_n = \frac{2}{L}\int_0^L f(x)\cos(n\pi x/L)dx.$

Fourier sine series ($x\in[0,L]$): $\sum_{n=1}^{\infty} b_n\sin(n\pi x/L)$ where $b_n = \frac{2}{L}\int_0^L f(x)\sin(n\pi x/L)dx.$

Differentiation of a parameter in an integral

$$\frac{dI}{dt} = \int_{\alpha(t)}^{\beta(t)}\frac{\partial f}{\partial t}(x,t)dx + f(\beta(t),t)\beta'(t) - f(\alpha(t),t)\alpha'(t).$$

Solution for wave equation with a source on $\mathbb{R}$ $\quad u_{tt} - c^2u_{xx} = f$

$$u(x,t) = \frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds + \frac{1}{2c}\iint_{\Delta_{(x,t)}} f$$

where $\phi(x) = u(x,0)$, $\psi(x) = u_t(x,0)$, and $\Delta_{(x,t)}$ is the characteristic triangle for (x,t).

The Kirchhoff-Poisson solution for wave equation $\quad u_{tt} - c^2\Delta u = 0$

In 3D: $u(\vec{x},t) = \frac{1}{4\pi c^2}\frac{\partial}{\partial t}\left[\frac{1}{t}\iint_{S_t}\phi(\vec{\xi})d\sigma\right] + \frac{1}{4\pi c^2 t}\iint_{S_t}\psi(\vec{\xi})d\sigma$ $\quad$ where S_t is a sphere of radius ct.

In 2D: $u(x,y,t) = \frac{1}{2\pi c}\frac{\partial}{\partial t}\left[\iint_{D_t}\frac{\phi(\xi,\eta)}{\sqrt{(ct)^2-(\xi-x)^2-(\eta-y)^2}}d\xi d\eta\right]$

$+ \frac{1}{2\pi c}\iint_{D_t}\frac{\psi(\xi,\eta)}{\sqrt{(ct)^2-(\xi-x)^2-(\eta-y)^2}}d\xi d\eta$ $\quad$ where D_t is a disk of radius ct.

Solution for the diffusion equation with a source on $\mathbb{R}$ $\quad u_t - ku_{xx} = f(x,t) \quad (t>0)$

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy + \int_0^t\int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds \quad (t>0)$$

where $S(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-x^2/4kt}$ and $\phi(x) = u(x,0)$.

The error function: $\quad \mathcal{E}rf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-p^2}dp;\quad \mathcal{E}rf(\infty) = 1.$

$\underline{\Delta \text{ in polar coordinates:}}$ $\quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

$\underline{\Delta \text{ in spherical coordinates:}}$ $\quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$

Poisson's formula for harmonic function in a disk: $u(\mathbf{x}) = \frac{1}{2\pi a}\int_{|\mathbf{x}'|=a} \frac{a^2 - |\mathbf{x}|^2}{|\mathbf{x} - \mathbf{x}'|^2}\, u(\mathbf{x}')\, ds'$, or

$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\phi) + r^2}\, h(\phi)d\phi$ where $h(\phi)$ is the value of u on the circle $r = a$.

Green's first identity: $\iint\limits_{\partial\mathcal{D}} v\frac{\partial u}{\partial n}dS = \iiint\limits_{\mathcal{D}} \nabla v\cdot\nabla u\, dV + \iiint\limits_{\mathcal{D}} v\Delta u\, dV$

Green's second identity: $\iiint\limits_{\mathcal{D}} (u\Delta v - v\Delta u)dV = \iint\limits_{\partial\mathcal{D}} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS$

Representation formula for a harmonic function u

$u(\mathbf{x}_0) = \iint\limits_{\partial\mathcal{D}} \left[u(\mathbf{x})\frac{\partial}{\partial n}K(\mathbf{x},\mathbf{x}_0) - K(\mathbf{x},\mathbf{x}_0)\frac{\partial u}{\partial n}\right] dS$ where $K(\mathbf{x},\mathbf{x}_0) \equiv -1/(4\pi|\mathbf{x}-\mathbf{x}_0|)$ for $\mathbb{R}^3$.

Solution of the Dirichlet problem for $\Delta u = f$ using Green's function

$u(\mathbf{x}_0) = \iint\limits_{\partial\mathcal{D}} u(\mathbf{x})\frac{\partial G(\mathbf{x},\mathbf{x}_0)}{\partial n}\, dS + \iiint\limits_{\mathcal{D}} f(\mathbf{x})G(\mathbf{x},\mathbf{x}_0)\, dV.$

On the plane, dS and dV are replaced by dl (length element) and dA (area element), respectively.

Bessel's differential equation of order s: $\quad \frac{d^2}{dr^2}u + \frac{1}{r}\frac{d}{dr}u + (1 - \frac{s^2}{r^2})u = 0$

Solutions of Bessel's equation

For a non-integer s, the two independent solutions are

$J_{\pm s}(r) = \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)\Gamma(\pm s + j + 1)}\left(\frac{r}{2}\right)^{\pm s + 2j}$ where $\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt$ is the Γ function.

For $s = n$, a non-negative integer, the solutions are

$$J_n(r) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!}\left(\frac{r}{2}\right)^{n+2j}$$

$$Y_n(r) = \frac{2}{\pi}\left(\gamma + \ln\frac{r}{2}\right)J_n(r) - \frac{(1-\delta_{n0})}{\pi}\sum_{j=0}^{n-1}\frac{(n-j-1)!}{j!}\left(\frac{2}{r}\right)^{n-2j} - \frac{1}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j(H_j + H_{n+j})}{j!(n+j)!}\left(\frac{r}{2}\right)^{n+2j}$$

Bessel functions of order $s = \pm 1/2$: $\quad J_{1/2}(r) = \sqrt{\frac{2}{\pi r}}\sin r \qquad J_{-1/2}(r) = \sqrt{\frac{2}{\pi r}}\cos r$

For $\beta_{sm}\, a$, $\beta_{sm'}\, a$ being roots of the Bessel function J_s

$$\int_0^a J_s(\beta_{sm}\, r)J_s(\beta_{sm'}\, r)rdr = \delta_{mm'}\, \frac{1}{2}a^2[J_s'(\beta_{sm}\, a)]^2 = \delta_{mm'}\, \frac{1}{2}a^2[J_{s\pm 1}(\beta_{sm}\, a)]^2$$

Spherical harmonics: $\quad Y_l^m(\theta,\phi) = P_l^{|m|}(\cos\theta)\cdot e^{im\phi}$

$$\int_0^{2\pi}\int_0^{\pi} Y_l^m(\theta,\phi)\overline{Y_{l'}^{m'}(\theta,\phi)}\sin\theta d\theta d\phi = \delta_{ll'}\delta_{mm'}\, \frac{4\pi}{2l+1}\frac{(l+m)!}{(l-m)!}$$