PART I: APPLICATION OF VECTORS

AND VECTOR-VALUED FUNCTIONS

Lines in Space

Vectors that connect any pair of points on a line $l$ are $\parallel$.

Vector description of a line

Consider the locus of the terminal points of the vectors (first in 2D)

$$t\vec{A} \quad -\infty < t < \infty$$

It is a straight line. This line can be displaced to describe any straight line $\parallel$ to $\vec{A}$.

The general vector expression for a line in space is:

$$\vec{r} = \vec{r}_0 + t\vec{A} \quad -\infty < t < \infty$$

where $\vec{r}_0$ is an arbitrary point on the line and $\vec{A}$ is a vector $\parallel$ to the line.

Example Find a vector equation of the line that contains $(-1, 3, 0)$ and is parallel to $2\vec{i} - 3\vec{j} - \vec{k}$

Example Find a vector equation of the line that passes through the two points: $(1, 2, 3)$ and $(4, 5, 6)$. 
Note $\vec{r}(t)$ gives the location of a point on the line. It is not parallel to the line. $\vec{A}$ is. Namely, if $\vec{P}$ and $\vec{Q}$ are any two different points on the line, then $\vec{P}\vec{Q} \parallel \vec{A}$.

**Parametric equations**

$$
\begin{align*}
x &= x_0 + tA_1 \\
y &= y_0 + tA_2 \\
z &= z_0 + tA_3
\end{align*}
$$

**Example** Find a set of parametric equations for the line of the previous example.

**Analytic geometry (AG) description of a line**

$$
\begin{align*}
a_1x + b_1y + c_1z &= d_1 \\
a_2x + b_2y + c_2z &= d_2
\end{align*}
$$

**Example** Find an AG description of the line in the previous example.
Degrees of Freedom / Dimensions

\[ \equiv \text{number of free parameters} \] used to describe the geometrical object

**Example** Let \( \vec{A} \) be a vector in space, the object \( t\vec{A} \) \((-\infty < t < \infty)\) has one free parameter, therefore the object has dimension 1.

**Example** If \( \vec{A} \) and \( \vec{B} \) are not parallel, and both \( \neq \vec{0} \), what is the object \( t\vec{A} + s\vec{B} \) \((-\infty < t < \infty \text{ and } -\infty < s < \infty)\)?

Planes in 3D Space

**Basic geometrical facts**

(i) Two different intersecting lines \( l_1, l_2 \) (directions given by \( \vec{A}, \vec{B} \) respectively) define a plane through them.

(ii) There is a unique line \( l_3 \) that passes through the point of intersection and is \( \perp \) to both lines. A vector \( \vec{N} \parallel l_3 \) defines a normal to the plane that contains \( l_1, l_2 \). \((\vec{N} \cdot \vec{A} = \vec{N} \cdot \vec{B} = 0)\)

(iii) There is only one plane that contains a given point and is \( \perp \) to a given nonzero vector.

(i) can be viewed as the definition of a plane; (ii) & (iii) come from the three-dimensional nature of space.
Vector description of a plane

A plane through two intersecting lines \( l_1 \) and \( l_2 \) can be generated as:

\[
\vec{r} = \vec{r}_0 + t\vec{A} + s\vec{B} \quad -\infty < t < \infty, \quad -\infty < s < \infty
\]

where \( \vec{r}_0 \) is the point of intersection of \( l_1, l_2 \), and \( \vec{A}, \vec{B} \) are vectors along \( l_1, l_2 \) respectively.

A normal is perpendicular to all directed line segments on the plane as \( \vec{N} \cdot (t\vec{A} + s\vec{B}) = 0 \).

Recall (ii) above for the definition of a normal. A vector is a normal to a plane iff it is perpendicular to all vectors associated with the directed line segments on the plane.

Parametric description of a plane:

Decompose the vector equation into components, as (linear) functions of the parameters.

\[
\begin{align*}
x &= x_0 + tA_1 + sB_1 \\
y &= y_0 + tA_2 + sB_2 \\
z &= z_0 + tA_3 + sB_3
\end{align*}
\]

This approach is not used much for planes, but very useful for general surfaces. The coordinates of a point on a surface can be specified as:
\[ x = f_1(t, s), \]
\[ y = f_2(t, s), \]
\[ z = f_3(t, s). \]

These are two-variable functions which make components of a function

\[ f : \mathbb{R}^2 \to \mathbb{R}^3. \]

**AG description of a plane:**

Let \( P \) be a fixed point on the plane, \( Q \) be any point on the plane, and \( \vec{N} \) be a normal. Then

\[ \vec{N} \cdot \vec{PQ} = 0 \]

Writing \( \vec{P} = (x_0, y_0, z_0) \), \( \vec{Q} = (x, y, z) \), and \( \vec{N} = (a, b, c) \), one has

\[ (a, b, c) \cdot [(x, y, z) - (x_0, y_0, z_0)] = \]
\[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \]

The resultant equation is of the form

\[ ax + by + cz = d. \]

A normal to the plane can be found easily as

\[ \vec{N} = a\vec{i} + b\vec{j} + c\vec{k}. \]
Example: Find an equation of the plane that contains the point \((-2, 4, 5)\) and has normal vector \(7\vec{i} - 6\vec{k}\)

Example: Find a unit normal to the plane 
\[x + y + z = 1.\]

Example: Find the plane through the three points 
\((0, 0, 1), (1, 0, 1), (0, 1, 1)\).

Quadric Surfaces

Definition: A quadric surface is a surface which contains points that satisfies the 2nd degree polynomial equation (AG description)

\[Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + J = 0\]

where \(A, \ldots, J\) are constants.
Quadric surfaces fall into nine major classes. Examples of each are provided as following \((a, b, c > 0)\):

1. Ellipsoid \(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\)

2. Hyperboloid of one sheet \(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1\)

3. Hyperboloid of two sheets \(\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1\)

4. Elliptic double cone \(\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0\)

5. Elliptic paraboloid \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}\)

6. Hyperbolic paraboloid \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}\)

7. Elliptic cylinder \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\)

8. Hyperbolic cylinder \(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\)

9. Parabolic cylinder \(x^2 = \frac{z}{c}\)

—— Problem Set 1 ——