Numerical Differentiation & Integration

Numerical Differentiation I

Numerical Analysis (9th Edition)
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Beamer Presentation Slides
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Outline

1. Introduction to Numerical Differentiation
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2. General Derivative Approximation Formulas
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3. Some useful three-point formulas
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Introduction to Numerical Differentiation

Approximating a Derivative
Introduction to Numerical Differentiation

Approximating a Derivative

The derivative of the function $f$ at $x_0$ is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
Introduction to Numerical Differentiation

Approximating a Derivative

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This formula gives an obvious way to generate an approximation to $f'(x_0)$; simply compute

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

for small values of $h$. Although this may be obvious, it is not very successful, due to our old nemesis round-off error.
Introduction to Numerical Differentiation

Approximating a Derivative

The derivative of the function $f$ at $x_0$ is

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This formula gives an obvious way to generate an approximation to $f'(x_0)$; simply compute

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for small values of $h$. Although this may be obvious, it is not very successful, due to our old nemesis round-off error.

But it is certainly a place to start.
To approximate $f'(x_0)$, suppose first that $x_0 \in (a, b)$, where $f \in C^2[a, b]$, and that $x_1 = x_0 + h$ for some $h \neq 0$ that is sufficiently small to ensure that $x_1 \in [a, b]$. 
Introduction to Numerical Differentiation

Approximating a Derivative (Cont’d)

- To approximate \( f'(x_0) \), suppose first that \( x_0 \in (a, b) \), where \( f \in C^2[a, b] \), and that \( x_1 = x_0 + h \) for some \( h \neq 0 \) that is sufficiently small to ensure that \( x_1 \in [a, b] \).

- We construct the first Lagrange polynomial \( P_{0,1}(x) \) for \( f \) determined by \( x_0 \) and \( x_1 \), with its error term:

\[
f(x) = P_{0,1}(x) + \frac{(x - x_0)(x - x_1)}{2!} f''(\xi(x))
\]

\[
= \frac{f(x_0)(x - x_0 - h)}{-h} + \frac{f(x_0 + h)(x - x_0)}{h} + \frac{(x - x_0)(x - x_0 - h)}{2} f''(\xi(x))
\]

for some \( \xi(x) \) between \( x_0 \) and \( x_1 \).
Numerical Differentiation

\[ f(x) = \frac{f(x_0)(x - x_0 - h)}{-h} + \frac{f(x_0 + h)(x - x_0)}{h} + \frac{(x - x_0)(x - x_0 - h)}{2} f''(\xi(x)) \]

Differentiating gives

\[ f'(x) = \frac{f(x_0 + h) - f(x_0)}{h} + D_x \left[ \frac{(x - x_0)(x - x_0 - h)}{2} f''(\xi(x)) \right] \]
Numerical Differentiation

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\[ = \frac{f(x_0 + h) - f(x_0)}{h} + \frac{2(x - x_0) - h}{2} f''(\xi(x)) \]

\[ + \frac{(x - x_0)(x - x_0 - h)}{2} D_x(f''(\xi(x))) \]
Numerical Differentiation

\[ f(x) = \frac{f(x_0)(x - x_0 - h)}{-h} + \frac{f(x_0 + h)(x - x_0)}{h} + \frac{(x - x_0)(x - x_0 - h)}{2} f''(\xi(x)) \]

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\[ = \frac{f(x_0 + h) - f(x_0)}{h} + \frac{2(x - x_0) - h}{2} f''(\xi(x)) \]

\[ + \frac{(x - x_0)(x - x_0 - h)}{2} D_x(f''(\xi(x))) \]

Deleting the terms involving \( \xi(x) \) gives

\[ f'(x) \approx \frac{f(x_0 + h) - f(x_0)}{h} \]
Numerical Differentiation

\[ f'(x) \approx \frac{f(x_0 + h) - f(x_0)}{h} \]

Approximating a Derivative (Cont’d)

One difficulty with this formula is that we have no information about \( D_x f''(\xi(x)) \), so the truncation error cannot be estimated.
Numerical Differentiation

\[ f'(x) \approx \frac{f(x_0 + h) - f(x_0)}{h} \]

Approximating a Derivative (Cont’d)

- One difficulty with this formula is that we have no information about \( D_x f''(\xi(x)) \), so the truncation error cannot be estimated.
- When \( x \) is \( x_0 \), however, the coefficient of \( D_x f''(\xi(x)) \) is 0, and the formula simplifies to

\[ f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{h}{2} f''(\xi) \]
Numerical Differentiation

\[ f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{h}{2} f''(\xi) \]

Forward-Difference and Backward-Difference Formulae

- For small values of \( h \), the difference quotient
  \[ \frac{f(x_0 + h) - f(x_0)}{h} \]
  can be used to approximate \( f'(x_0) \) with an error bounded by \( M|h|/2 \), where \( M \) is a bound on \( |f''(x)| \) for \( x \) between \( x_0 \) and \( x_0 + h \).
Numerical Differentiation

\[ f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} - \frac{h}{2} f''(\xi) \]

Forward-Difference and Backward-Difference Formulae

- For small values of \( h \), the difference quotient

\[ \frac{f(x_0 + h) - f(x_0)}{h} \]

can be used to approximate \( f'(x_0) \) with an error bounded by \( M|h|/2 \), where \( M \) is a bound on \( |f''(x)| \) for \( x \) between \( x_0 \) and \( x_0 + h \).

- This formula is known as the **forward-difference formula** if \( h > 0 \) and the **backward-difference formula** if \( h < 0 \).
Forward-Difference Formula to Approximate $f'(x_0)$

\[ f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h} \]
Example 1: $f(x) = \ln x$

Use the forward-difference formula to approximate the derivative of $f(x) = \ln x$ at $x_0 = 1.8$ using $h = 0.1$, $h = 0.05$, and $h = 0.01$, and determine bounds for the approximation errors.
Example 1: $f(x) = \ln x$

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Solution (1/3)

The forward-difference formula

$$\frac{f(1.8 + h) - f(1.8)}{h}$$

with $h = 0.1$
Example 1: \( f(x) = \ln x \)

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Solution (1/3)

The forward-difference formula

\[
\frac{f(1.8 + h) - f(1.8)}{h}
\]

with \( h = 0.1 \) gives

\[
\frac{\ln 1.9 - \ln 1.8}{0.1} = \frac{0.64185389 - 0.58778667}{0.1} = 0.5406722
\]
Numerical Differentiation: Example 1

Solution (2/3)

Because \( f''(x) = -1/x^2 \) and \( 1.8 < \xi < 1.9 \), a bound for this approximation error is

\[
\frac{|hf''(\xi)|}{2} = \frac{|h|}{2\xi^2} < \frac{0.1}{2(1.8)^2} = 0.0154321
\]
Numerical Differentiation: Example 1

Solution (2/3)

Because \( f''(x) = -1/x^2 \) and \( 1.8 < \xi < 1.9 \), a bound for this approximation error is

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\frac{|hf''(\xi)|}{2} = \frac{|h|}{2\xi^2} < \frac{0.1}{2(1.8)^2} = 0.0154321
\]

The approximation and error bounds when \( h = 0.05 \) and \( h = 0.01 \) are found in a similar manner and the results are shown in the following table.
Numerical Differentiation: Example 1

Solution (3/3): Tabulated Results

| $h$   | $f(1.8 + h)$ | $\frac{f(1.8 + h) - f(1.8)}{h}$ | $\frac{|h|}{2(1.8)^2}$ |
|-------|--------------|---------------------------------|------------------------|
| 0.1   | 0.64185389   | 0.5406722                       | 0.0154321              |
| 0.05  | 0.61518564   | 0.5479795                       | 0.0077160              |
| 0.01  | 0.59332685   | 0.5540180                       | 0.0015432              |

Since $f'(x) = \frac{1}{x}$ The exact value of $f'(1.8)$ is 0.555, and in this case the error bounds are quite close to the true approximation error.
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General Derivative Approximation Formulas

Method of Construction

To obtain general derivative approximation formulas, suppose that \( \{x_0, x_1, \ldots, x_n\} \) are \((n + 1)\) distinct numbers in some interval \( I \) and that \( f \in C^{n+1}(I) \).

From the interpolation error theorem, we have

\[
f(x) = \sum_{k=0}^{n} f(x_k) L_k(x) + \frac{(x - x_0) \cdots (x - x_n)}{(n + 1)!} f^{(n+1)}(\xi(x))
\]

for some \( \xi(x) \) in \( I \), where \( L_k(x) \) denotes the \( k \)th Lagrange coefficient polynomial for \( f \) at \( x_0, x_1, \ldots, x_n \).
General Derivative Approximation Formulas

\[ f(x) = \sum_{k=0}^{n} f(x_k) L_k(x) + \frac{(x - x_0) \cdots (x - x_n)}{(n + 1)!} f^{(n+1)}(\xi(x)) \]

Method of Construction (Cont’d)

Differentiating this expression gives

\[ f'(x) = \sum_{k=0}^{n} f(x_k) L'_k(x) + D_x \left[ \frac{(x - x_0) \cdots (x - x_n)}{(n + 1)!} \right] f^{(n+1)}(\xi(x)) \]

\[ + \frac{(x - x_0) \cdots (x - x_n)}{(n + 1)!} D_x \left[ f^{(n+1)}(\xi(x)) \right] \]
General Derivative Approximation Formulas

\[ f'(x) = \sum_{k=0}^{n} f(x_k)L'_k(x) + D_x \left[ \frac{(x - x_0) \cdots (x - x_n)}{(n + 1)!} \right] f^{(n+1)}(\xi(x)) \]

\[ + \frac{(x - x_0) \cdots (x - x_n)}{(n + 1)!} D_x[f^{(n+1)}(\xi(x))] \]

Method of Construction (Cont’d)

We again have a problem estimating the truncation error unless \( x \) is one of the numbers \( x_j \).
General Derivative Approximation Formulas

\[ f'(x) = \sum_{k=0}^{n} f(x_k) L'_k(x) + D_x \left[ \frac{(x - x_0) \cdots (x - x_n)}{(n + 1)!} \right] f^{(n+1)}(\xi(x)) \]

\[ + \frac{(x - x_0) \cdots (x - x_n)}{(n + 1)!} D_x[f^{(n+1)}(\xi(x))] \]

Method of Construction (Cont’d)

We again have a problem estimating the truncation error unless \( x \) is one of the numbers \( x_j \). In this case, the term multiplying \( D_x[f^{(n+1)}(\xi(x))] \) is 0, and the formula becomes

\[ f'(x_j) = \sum_{k=0}^{n} f(x_k) L'_k(x_j) + \frac{f^{(n+1)}(\xi(x_j))}{(n + 1)!} \prod_{k=0}^{n} (x_j - x_k) \]
General Derivative Approximation Formulas

\[
f'(x) = \sum_{k=0}^{n} f(x_k) L'_k(x) + D_x \left[ \frac{(x - x_0) \cdots (x - x_n)}{(n + 1)!} \right] f^{(n+1)}(\xi(x)) + \frac{(x - x_0) \cdots (x - x_n)}{(n + 1)!} D_x[f^{(n+1)}(\xi(x))]\]

Method of Construction (Cont’d)

We again have a problem estimating the truncation error unless \( x \) is one of the numbers \( x_j \). In this case, the term multiplying \( D_x[f^{(n+1)}(\xi(x))] \) is 0, and the formula becomes

\[
f'(x_j) = \sum_{k=0}^{n} f(x_k) L'_k(x_j) + \frac{f^{(n+1)}(\xi(x_j))}{(n + 1)!} \prod_{\substack{k=0 \atop k \neq j}}^{n} (x_j - x_k)\]

which is called an \((n + 1)\)-point formula to approximate \( f'(x_j) \).
General Derivative Approximation Formulas

\[ f'(x_j) = \sum_{k=0}^{n} f(x_k) L'_k(x_j) + \frac{f^{(n+1)}(\xi(x_j))}{(n+1)!} \prod_{k=0, k\neq j}^{n} (x_j - x_k) \]

Comment on the \((n + 1)\)-point formula

In general, using more evaluation points produces greater accuracy, although the number of functional evaluations and growth of round-off error discourages this somewhat.
General Derivative Approximation Formulas

\[ f'(x_j) = \sum_{k=0}^{n} f(x_k) L'_k(x_j) + \frac{f^{(n+1)}(\xi(x_j))}{(n+1)!} \prod_{k=0}^{n} (x_j - x_k) \]

Comment on the \((n + 1)\)-point formula

- In general, using more evaluation points produces greater accuracy, although the number of functional evaluations and growth of round-off error discourages this somewhat.
- The most common formulas are those involving three and five evaluation points.
General Derivative Approximation Formulas

\[ f'(x_j) = \sum_{k=0}^{n} f(x_k) L'_k(x_j) + \frac{f^{(n+1)}(\xi(x_j))}{(n+1)!} \prod_{\substack{k=0 \atop k \neq j}}^{n} (x_j - x_k) \]

Comment on the \((n + 1)\)-point formula

- In general, using more evaluation points produces greater accuracy, although the number of functional evaluations and growth of round-off error discourages this somewhat.
- The most common formulas are those involving three and five evaluation points.

We first derive some useful three-point formulas and consider aspects of their errors.
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Some useful three-point formulas

Important Building Blocks

Since

\[ L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \]
Some useful three-point formulas

Important Building Blocks

Since

\[ L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \]

we obtain

\[ L'_0(x) = \frac{2x - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} \]
Some useful three-point formulas

Important Building Blocks

Since

\[ L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \]

we obtain

\[ L'_0(x) = \frac{2x - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} \]

In a similar way, we find that

\[ L'_1(x) = \frac{2x - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} \]

\[ L'_2(x) = \frac{2x - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} \]
Some useful three-point formulas

Important Building Blocks (Cont’d)

Using these expressions for $L_j'(x)$, $1 \leq j \leq 2$, the $n + 1$-point formula

$$f'(x_j) = \sum_{k=0}^{n} f(x_k)L_k'(x_j) + \frac{f^{(n+1)}(\xi(x_j))}{(n + 1)!} \prod_{\substack{k=0 \atop k \neq j}}^{n} (x_j - x_k)$$
Some useful three-point formulas

Important Building Blocks (Cont’d)

Using these expressions for $L'_j(x)$, $1 \leq j \leq 2$, the $n + 1$-point formula

$$f'(x_j) = \sum_{k=0}^{n} f(x_k)L'_k(x_j) + \frac{f^{(n+1)}(\xi(x_j))}{(n + 1)!} \prod_{k=0}^{n}^{k \neq j} (x_j - x_k)$$

becomes for $n = 2$:

$$f'(x_j) = f(x_0) \left[ \frac{2x_j - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} \right] + f(x_1) \left[ \frac{2x_j - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} \right]$$

$$+ f(x_2) \left[ \frac{2x_j - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} \right] + \frac{1}{6} f^{(3)}(\xi_j) \prod_{k=0}^{2}^{k \neq j} (x_j - x_k)$$

for each $j = 0, 1, 2$, where $\xi_j = \xi_j(x)$. 

Some useful three-point formulas

\[
f'(x_j) = f(x_0) \left[ \frac{2x_j - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} \right] + f(x_1) \left[ \frac{2x_j - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} \right] \\
+ f(x_2) \left[ \frac{2x_j - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} \right] + \frac{1}{6} f^{(3)}(\xi_j) \prod_{k=0}^{2} (x_j - x_k)
\]

Assumption
Some useful three-point formulas

\[
f'(x_j) = f(x_0) \left[ \frac{2x_j - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} \right] + f(x_1) \left[ \frac{2x_j - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} \right] \\
+ f(x_2) \left[ \frac{2x_j - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} \right] + \frac{1}{6} f^{(3)}(\xi_j) \prod_{k=0}^{2} (x_j - x_k)
\]

Assumption

The 3-point formulas become especially useful if the nodes are equally spaced, that is, when

\[x_1 = x_0 + h \quad \text{and} \quad x_2 = x_0 + 2h, \quad \text{for some } h \neq 0\]
Some useful three-point formulas

\[ f'(x_j) = f(x_0) \left[ \frac{2x_j - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} \right] + f(x_1) \left[ \frac{2x_j - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} \right] + f(x_2) \left[ \frac{2x_j - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} \right] + \frac{1}{6} f^{(3)}(\xi_j) \prod_{k=0, k\neq j}^{2} (x_j - x_k) \]

Assumption

The 3-point formulas become especially useful if the nodes are equally spaced, that is, when

\[ x_1 = x_0 + h \quad \text{and} \quad x_2 = x_0 + 2h, \quad \text{for some} \ h \neq 0 \]

We will assume equally-spaced nodes throughout the remainder of this section.
Some useful three-point formulas

\[ f'(x_j) = f(x_0) \left[ \frac{2x_j - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} \right] + f(x_1) \left[ \frac{2x_j - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} \right] \]

\[ + f(x_2) \left[ \frac{2x_j - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} \right] + \frac{1}{6} f^{(3)}(\xi_j) \prod_{k=0}^{2} (x_j - x_k) \]

Three-Point Formulas (1/3)

With \( x_j = x_0 \), \( x_1 = x_0 + h \), and \( x_2 = x_0 + 2h \), the general 3-point formula becomes

\[ f'(x_0) = \frac{1}{h} \left[ - \frac{3}{2} f(x_0) + 2 f(x_1) - \frac{1}{2} f(x_2) \right] + \frac{h^2}{3} f^{(3)}(\xi_0) \]
Some useful three-point formulas

\[ f'(x_j) = f(x_0) \left[ \frac{2x_j - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} \right] + f(x_1) \left[ \frac{2x_j - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} \right] + f(x_2) \left[ \frac{2x_j - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} \right] + \frac{1}{6} f^{(3)}(\xi_j) \prod_{\substack{k=0 \atop k \neq j}}^{2} (x_j - x_k) \]

Three-Point Formulas (2/3)

Doing the same for \( x_j = x_1 \) gives

\[ f'(x_1) = \frac{1}{h} \left[ -\frac{1}{2} f(x_0) + \frac{1}{2} f(x_2) \right] - \frac{h^2}{6} f^{(3)}(\xi_1) \]
Some useful three-point formulas

\[ f'(x_j) = f(x_0) \left[ \frac{2x_j - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} \right] + f(x_1) \left[ \frac{2x_j - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} \right] \\
+ f(x_2) \left[ \frac{2x_j - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} \right] + \frac{1}{6} f^{(3)}(\xi_j) \prod_{k=0}^{2} (x_j - x_k) \]

Three-Point Formulas (3/3)

... and for \( x_j = x_2 \), we obtain

\[ f'(x_2) = \frac{1}{h} \left[ \frac{1}{2} f(x_0) - 2f(x_1) + \frac{3}{2} f(x_2) \right] + \frac{h^2}{3} f^{(3)}(\xi_2) \]
Some useful three-point formulas

Three-Point Formulas: Further Simplification
Some useful three-point formulas

Three-Point Formulas: Further Simplification

Since \( x_1 = x_0 + h \) and \( x_2 = x_0 + 2h \), these formulas can also be expressed as

\[
f'(x_0) = \frac{1}{h} \left[ -\frac{3}{2} f(x_0) + 2f(x_0 + h) - \frac{1}{2} f(x_0 + 2h) \right] + \frac{h^2}{3} f^{(3)}(\xi_0)
\]

\[
f'(x_0 + h) = \frac{1}{h} \left[ -\frac{1}{2} f(x_0) + \frac{1}{2} f(x_0 + 2h) \right] - \frac{h^2}{6} f^{(3)}(\xi_1)
\]

\[
f'(x_0 + 2h) = \frac{1}{h} \left[ \frac{1}{2} f(x_0) - 2f(x_0 + h) + \frac{3}{2} f(x_0 + 2h) \right] + \frac{h^2}{3} f^{(3)}(\xi_2)
\]
Some useful three-point formulas

Three-Point Formulas: Further Simplification

Since \( x_1 = x_0 + h \) and \( x_2 = x_0 + 2h \), these formulas can also be expressed as

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f'(x_0) = \frac{1}{h} \left[ -\frac{3}{2} f(x_0) + 2f(x_0 + h) - \frac{1}{2} f(x_0 + 2h) \right] + \frac{h^2}{3} f^{(3)}(\xi_0)
\]

\[
f'(x_0 + h) = \frac{1}{h} \left[ -\frac{1}{2} f(x_0) + \frac{1}{2} f(x_0 + 2h) \right] - \frac{h^2}{6} f^{(3)}(\xi_1)
\]

\[
f'(x_0 + 2h) = \frac{1}{h} \left[ \frac{1}{2} f(x_0) - 2f(x_0 + h) + \frac{3}{2} f(x_0 + 2h) \right] + \frac{h^2}{3} f^{(3)}(\xi_2)
\]

As a matter of convenience, the variable substitution \( x_0 \) for \( x_0 + h \) is used in the middle equation to change this formula to an approximation for \( f'(x_0) \). A similar change, \( x_0 \) for \( x_0 + 2h \), is used in the last equation.
Three-Point Formulas: Further Simplification (Cont’d)

This gives three formulas for approximating $f'(x_0)$:
Some useful three-point formulas

Three-Point Formulas: Further Simplification (Cont’d)

This gives three formulas for approximating $f'(x_0)$:

$$f'(x_0) = \frac{1}{2h}[-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)] + \frac{h^2}{3}f^{(3)}(\xi_0)$$

$$f'(x_0) = \frac{1}{2h}[-f(x_0 - h) + f(x_0 + h)] - \frac{h^2}{6}f^{(3)}(\xi_1), \quad \text{and}$$

$$f'(x_0) = \frac{1}{2h}[f(x_0 - 2h) - 4f(x_0 - h) + 3f(x_0)] + \frac{h^2}{3}f^{(3)}(\xi_2)$$
Some useful three-point formulas

Three-Point Formulas: Further Simplification (Cont’d)

This gives three formulas for approximating $f'(x_0)$:

$$f'(x_0) = \frac{1}{2h} \left[ -3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h) \right] + \frac{h^2}{3} f^{(3)}(\xi_0)$$

$$f'(x_0) = \frac{1}{2h} \left[ -f(x_0 - h) + f(x_0 + h) \right] - \frac{h^2}{6} f^{(3)}(\xi_1), \quad \text{and}$$

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Finally, note that the last of these equations can be obtained from the first by simply replacing $h$ with $-h$, so there are actually only two formulas.
Some useful three-point formulas

Three-Point Endpoint Formula

\[ f'(x_0) = \frac{1}{2h} \left[ -3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h) \right] + \frac{h^2}{3} f^{(3)}(\xi_0) \]

where \( \xi_0 \) lies between \( x_0 \) and \( x_0 + 2h \).
Some useful three-point formulas

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\[ f'(x_0) = \frac{1}{2h} \left[ -3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h) \right] + \frac{h^2}{3} f^{(3)}(\xi_0) \]

where \( \xi_0 \) lies between \( x_0 \) and \( x_0 + 2h \).

Three-Point Midpoint Formula

\[ f'(x_0) = \frac{1}{2h} \left[ f(x_0 + h) - f(x_0 - h) \right] - \frac{h^2}{6} f^{(3)}(\xi_1) \]

where \( \xi_1 \) lies between \( x_0 - h \) and \( x_0 + h \).
Some useful three-point formulas

\begin{align*}
(1) \quad f'(x_0) &= \frac{1}{2h} \left[ -3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h) \right] + \frac{h^2}{3} f^{(3)}(\xi_0) \\
(2) \quad f'(x_0) &= \frac{1}{2h} \left[ f(x_0 + h) - f(x_0 - h) \right] - \frac{h^2}{6} f^{(3)}(\xi_1)
\end{align*}
Some useful three-point formulas

\[(1) \quad f'(x_0) = \frac{1}{2h} \left[ -3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h) \right] + \frac{h^2}{3} f^{(3)}(\xi_0) \]

\[(2) \quad f'(x_0) = \frac{1}{2h} \left[ f(x_0 + h) - f(x_0 - h) \right] - \frac{h^2}{6} f^{(3)}(\xi_1) \]

Comments

Although the errors in both Eq. (1) and Eq. (2) are \(O(h^2)\), the error in Eq. (2) is approximately half the error in Eq. (1).
Some useful three-point formulas

(1) \[ f'(x_0) = \frac{1}{2h} \left[ -3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h) \right] + \frac{h^2}{3} f^{(3)}(\xi_0) \]

(2) \[ f'(x_0) = \frac{1}{2h} \left[ f(x_0 + h) - f(x_0 - h) \right] - \frac{h^2}{6} f^{(3)}(\xi_1) \]

Comments

- Although the errors in both Eq. (1) and Eq. (2) are \( O(h^2) \), the error in Eq. (2) is approximately half the error in Eq. (1).
- This is because Eq. (2) uses data on both sides of \( x_0 \) and Eq. (1) uses data on only one side.
Some useful three-point formulas

(1) \[ f'(x_0) = \frac{1}{2h}[-3f(x_0) + 4f(x_0 + h) - f(x_0 + 2h)] + \frac{h^2}{3}f^{(3)}(\xi_0) \]

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Comments

- Although the errors in both Eq. (1) and Eq. (2) are \( O(h^2) \), the error in Eq. (2) is approximately half the error in Eq. (1).
- This is because Eq. (2) uses data on both sides of \( x_0 \) and Eq. (1) uses data on only one side.
- Note also that \( f \) needs to be evaluated at only two points in Eq. (2), whereas in Eq. (1) three evaluations are needed.
Three-Point Midpoint Formula

\[
f'(x_0) = \frac{1}{2h} [f(x_0 + h) - f(x_0 - h)] - \frac{h^2}{6} f^{(3)}(\xi_1)
\]

where \( \xi_1 \) lies between \( x_0 - h \) and \( x_0 + h \).
Examples of five-point formulas

Five-Point Midpoint Formula

\[
f'(x_0) = \frac{1}{12h} \left[ f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h) \right] + \frac{h^4}{30} f^{(5)}(\xi)
\]

where \( \xi \) lies between \( x_0 - 2h \) and \( x_0 + 2h \).
Examples of five-point formulas

**Five-Point Midpoint Formula**

\[
f'(x_0) = \frac{1}{12h} [f(x_0 - 2h) - 8f(x_0 - h) + 8f(x_0 + h) - f(x_0 + 2h)] + \frac{h^4}{30} f^{(5)}(\xi)
\]

where \( \xi \) lies between \( x_0 - 2h \) and \( x_0 + 2h \).

**Five-Point Endpoint Formula**

\[
f'(x_0) = \frac{1}{12h} [-25f(x_0) + 48f(x_0 + h) - 36f(x_0 + 2h) + 16f(x_0 + 3h) - 3f(x_0 + 4h)] + \frac{h^4}{5} f^{(5)}(\xi)
\]

where \( \xi \) lies between \( x_0 \) and \( x_0 + 4h \).
Questions?
Reference Material
Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each $x$ in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, x_1, \ldots, x_n$, and hence in $(a, b)$, exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

where $P(x)$ is the interpolating polynomial given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$