Outline

1. Functional (Fixed-Point) Iteration

Sample Problem:
\[ f(x) = x^3 + 4x^2 - 10 = 0 \]
Outline

1. Functional (Fixed-Point) Iteration

2. Convergence Criteria for the Fixed-Point Method

Sample Problem:

$$f(x) = x^3 + 4x^2 - 10 = 0$$
Outline

1. Functional (Fixed-Point) Iteration
2. Convergence Criteria for the Fixed-Point Method
3. Sample Problem: \( f(x) = x^3 + 4x^2 - 10 = 0 \)
Outline

1. Functional (Fixed-Point) Iteration

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3. Sample Problem: \( f(x) = x^3 + 4x^2 - 10 = 0 \)
Now that we have established a condition for which \( g(x) \) has a unique fixed point in \( I \), there remains the problem of how to find it. The technique employed is known as fixed-point iteration.

**Basic Approach**

To approximate the fixed point of a function \( g \), we choose an initial approximation \( p_0 \) and generate the sequence \( \{p_n\} \) for each \( n \geq 1 \) by letting

\[
p_n = g(p_{n-1})
\]

If the sequence converges to \( p \) and \( g \) is continuous, then

\[
\lim_{n \to \infty} p_n = \lim_{n \to \infty} g(p_{n-1}) = g(\lim_{n \to \infty} p_{n-1}) = g(p),
\]

and a solution to \( x = g(x) \) is obtained.
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Now that we have established a condition for which $g(x)$ has a unique fixed point in $I$, there remains the problem of how to find it. The technique employed is known as fixed-point iteration.

**Basic Approach**

- To approximate the fixed point of a function $g$, we choose an initial approximation $p_0$ and generate the sequence $\{p_n\}_{n=0}^{\infty}$ by letting $p_n = g(p_{n-1})$, for each $n \geq 1$.
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$$p = \lim_{n \to \infty} p_n = \lim_{n \to \infty} g(p_{n-1}) = g \left( \lim_{n \to \infty} p_{n-1} \right) = g(p),$$

and a solution to $x = g(x)$ is obtained.
- This technique is called **fixed-point**, or **functional iteration**.
Functional (Fixed-Point) Iteration

- $p_2 = g(p_1)$
- $p_3 = g(p_2)$
- $p_1 = g(p_0)$

$y = x$

(a)

- $p_3 = g(p_2)$
- $p_2 = g(p_1)$
- $p_1 = g(p_0)$

$y = g(x)$

(b)
Functional (Fixed-Point) Iteration

To find the fixed point of $g$ in an interval $[a, b]$, given the equation $x = g(x)$ with an initial guess $p_0 \in [a, b]$:

1. $n = 1$;
2. $p_n = g(p_{n-1})$;
3. If $|p_n - p_{n-1}| < \epsilon$ then 5;
4. $n \rightarrow n + 1$; go to 2.
5. End of Procedure.
Fixed-Point Algorithm

To find the fixed point of $g$ in an interval $[a, b]$, given the equation $x = g(x)$ with an initial guess $p_0 \in [a, b]$: 

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Fixed-Point Iteration

Convergence Criteria

Sample Problem

Functional (Fixed-Point) Iteration

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3. If $|p_n - p_{n-1}| < \epsilon$ then 5;
4. $n \rightarrow n + 1$; go to 2.
5. End of Procedure.
A Single Nonlinear Equation

Example 1

The equation

\[ x^3 + 4x^2 - 10 = 0 \]

has a unique root in \([1, 2]\). Its value is approximately 1.365230013.
$f(x) = x^3 + 4x^2 - 10 = 0$ on $[1, 2]$
Fixed-Point Iteration

Convergence Criteria

Sample Problem

\[ f(x) = x^3 + 4x^2 - 10 = 0 \text{ on } [1, 2] \]

Possible Choices for \( g(x) \)

There are many ways to change the equation to the fixed-point form
\[ x = g(x) \]
using simple algebraic manipulation. For example, to obtain the function \( g \) described in part (c), we can manipulate the equation
\[ x^3 + 4x^2 - 10 = 0 \]
as follows:

\[ 4x^2 = 10 - x^3, \]
so
\[ x^2 = \frac{1}{4} (10 - x^3), \]
and
\[ x = \pm \sqrt[4]{10 - x^3}. \]

We will consider 5 such rearrangements and, later in this section, provide a brief analysis as to why some do and some not converge to \( p = \frac{3}{2} \).
Possible Choices for $g(x)$

There are many ways to change the equation to the fixed-point form $x = g(x)$ using simple algebraic manipulation.
$f(x) = x^3 + 4x^2 - 10 = 0$ on $[1, 2]$

Possible Choices for $g(x)$

- There are many ways to change the equation to the fixed-point form $x = g(x)$ using simple algebraic manipulation.
- For example, to obtain the function $g$ described in part (c), we can manipulate the equation $x^3 + 4x^2 - 10 = 0$ as follows:

$$4x^2 = 10 - x^3, \quad \text{so} \quad x^2 = \frac{1}{4}(10 - x^3), \quad \text{and} \quad x = \pm \frac{1}{2}(10 - x^3)^{1/2}.$$
\[ f(x) = x^3 + 4x^2 - 10 = 0 \] on \([1, 2]\)

### Possible Choices for \(g(x)\)

- There are many ways to change the equation to the fixed-point form \(x = g(x)\) using simple algebraic manipulation.
- For example, to obtain the function \(g\) described in part (c), we can manipulate the equation \(x^3 + 4x^2 - 10 = 0\) as follows:

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  \]

- We will consider 5 such rearrangements and, later in this section, provide a brief analysis as to why some do and some not converge to \(p = 1.365230013\).
Solving $f(x) = x^3 + 4x^2 - 10 = 0$

5 Possible Transpositions to $x = g(x)$

- $x = g_1(x) = x - x^3 - 4x^2 + 10$
- $x = g_2(x) = \sqrt{\frac{10}{x}} - 4x$
- $x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3}$
- $x = g_4(x) = \sqrt{\frac{10}{4 + x}}$
- $x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$
Numerical Results for $f(x) = x^3 + 4x^2 - 10 = 0$

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</table>
Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$x = g(x)$ with $x_0 = 1.5$

$x = g_1(x) = x - x^3 - 4x^2 + 10$  Does not Converge

$x = g_2(x) = \sqrt{\frac{10}{x}} - 4x$  Does not Converge

$x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3}$  Converges after 31 Iterations

$x = g_4(x) = \sqrt{\frac{10}{4 + x}}$  Converges after 12 Iterations

$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$  Converges after 5 Iterations
Outline

1. Functional (Fixed-Point) Iteration
2. Convergence Criteria for the Fixed-Point Method
3. Sample Problem: $f(x) = x^3 + 4x^2 - 10 = 0$
A Crucial Question

How can we find a fixed-point problem that produces a sequence that reliably and rapidly converges to a solution to a given root-finding problem?
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How can we find a fixed-point problem that produces a sequence that reliably and rapidly converges to a solution to a given root-finding problem?

The following theorem and its corollary give us some clues concerning the paths we should pursue and, perhaps more importantly, some we should reject.
Functional (Fixed-Point) Iteration

Convergence Result

Let $g \in C[a, b]$ with $g(x) \in [a, b]$ for all $x \in [a, b]$. Let $g'(x)$ exist on $(a, b)$ with

$$|g'(x)| \leq k < 1, \quad \forall x \in [a, b].$$
Convergence Result

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$$|g'(x)| \leq k < 1, \quad \forall \ x \in [a, b].$$

If $p_0$ is any point in $[a, b]$ then the sequence defined by

$$p_n = g(p_{n-1}), \quad n \geq 1,$$

will converge to the unique fixed point $p$ in $[a, b]$. 
Functional (Fixed-Point) Iteration

Proof of the Convergence Result

By the Uniqueness Theorem, a unique fixed point exists in $[a, b]$. Since $g$ maps $[a, b]$ into itself, the sequence $\{p_n\}_{n=0}^{\infty}$ is defined for all $n \geq 0$ and $p_n \in [a, b]$ for all $n$.

Using the Mean Value Theorem (MVT) and the assumption that $|g'(x)| \leq k < 1$, $\forall x \in [a, b]$, we write

$$|p_n - p| = |g(p_n - 1) - g(p)| \leq |g'(\xi)| |p_n - 1 - p| \leq k |p_n - 1 - p|$$

where $\xi \in (a, b)$. 
Functional (Fixed-Point) Iteration

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Proof of the Convergence Result
Proof of the Convergence Result (Cont’d)

Applying the inequality of the hypothesis inductively gives

\[ |p_n - p_{n-1}| \leq k |p_{n-1} - p_{n-2}| \leq k^2 |p_{n-2} - p_{0}| \]

Since \( k < 1 \),

\[ \lim_{n \to \infty} |p_n - p_{n-1}| = \lim_{n \to \infty} k^n |p_0 - p| = 0, \]

and \( \{ p_n \} \) converges to \( p \).
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Corollary to the Convergence Result

If \( g \) satisfies the hypothesis of the Theorem, then

\[
|p_n - p| \leq \frac{k^n}{1 - k} |p_1 - p_0|.
\]
Functional (Fixed-Point) Iteration

Corollary to the Convergence Result

If $g$ satisfies the hypothesis of the Theorem, then

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Proof of Corollary (1 of 3)

For $n \geq 1$, the procedure used in the proof of the theorem implies that

$$|p_{n+1} - p_n| = |g(p_n) - g(p_{n-1})|$$.
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\leq \ldots \\
\leq k^n |p_1 - p_0|
\]
Functional (Fixed-Point) Iteration

\[ |p_{n+1} - p_n| \leq k^n |p_1 - p_0| \]

Proof of Corollary (2 of 3)
Functional (Fixed-Point) Iteration

\[ |p_{n+1} - p_n| \leq k^n |p_1 - p_0| \]

Proof of Corollary (2 of 3)

Thus, for \( m > n \geq 1 \),

\[ |p_m - p_n| = |p_m - p_{m-1} + p_{m-1} - p_{m-2} + \cdots + p_{n+1} - p_n| \]
Functional (Fixed-Point) Iteration

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Proof of Corollary (2 of 3)

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\leq k^{m-1} |p_1 - p_0| + k^{m-2} |p_1 - p_0| + \cdots + k^n |p_1 - p_0|
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Functional (Fixed-Point) Iteration

\[ |p_{n+1} - p_n| \leq k^n |p_1 - p_0| \]

Proof of Corollary (2 of 3)

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\leq |p_m - p_{m-1}| + |p_{m-1} - p_{m-2}| + \cdots + |p_{n+1} - p_n| \\
\leq k^{m-1} |p_1 - p_0| + k^{m-2} |p_1 - p_0| + \cdots + k^n |p_1 - p_0| \\
\leq k^n \left( 1 + k + k^2 + \cdots + k^{m-n-1} \right) |p_1 - p_0|.
\]
**Functional (Fixed-Point) Iteration**

\[ |p_m - p_n| \leq k^n \left(1 + k + k^2 + \cdots + k^{m-n-1}\right) |p_1 - p_0| . \]

**Proof of Corollary (3 of 3)**
Functional (Fixed-Point) Iteration

\[ |p_m - p_n| \leq k^n \left( 1 + k + k^2 + \cdots + k^{m-n-1} \right) |p_1 - p_0|. \]

Proof of Corollary (3 of 3)

However, since \( \lim_{m \to \infty} p_m = p \), we obtain

\[ |p - p_n| = \lim_{m \to \infty} |p_m - p_n| \]
Functional (Fixed-Point) Iteration

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\[
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Functional (Fixed-Point) Iteration

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\[ = \frac{k^n}{1 - k} |p_1 - p_0|. \]
Example: \( g(x) = 3^{-x} \)

Consider the iteration function \( g(x) = 3^{-x} \) over the interval \([\frac{1}{3}, 1]\) starting with \( p_0 = \frac{1}{3} \). Determine a lower bound for the number of iterations \( n \) required so that \( |p_n - p| < 10^{-5} \)?
Functional (Fixed-Point) Iteration

Example: $g(x) = g(x) = 3^{-x}$

Consider the iteration function $g(x) = 3^{-x}$ over the interval $[\frac{1}{3}, 1]$ starting with $p_0 = \frac{1}{3}$. Determine a lower bound for the number of iterations $n$ required so that $|p_n - p| < 10^{-5}$?

Determine the Parameters of the Problem
Example: \( g(x) = g(x) = 3^{-x} \)

Consider the iteration function \( g(x) = 3^{-x} \) over the interval \([\frac{1}{3}, 1]\) starting with \( p_0 = \frac{1}{3} \). Determine a lower bound for the number of iterations \( n \) required so that \( |p_n - p| < 10^{-5} \)?

Determine the Parameters of the Problem

Note that \( p_1 = g(p_0) = 3^{-\frac{1}{3}} = 0.6933612 \) and, since \( g'(x) = -3^{-x} \ln 3 \), we obtain the bound

\[
|g'(x)| \leq 3^{-\frac{1}{3}} \ln 3 \leq 0.7617362 \approx 0.762 = k.
\]
Therefore, we have

\[ |p_n - p| \leq k |p_0 - p_1| \]

\[ \leq 0.762 \]

\[ |p_n - p| \leq 0.762 |p_{13} - p_{12}| \leq 1.513 \times 0.762 \]

We require that

\[ 1.513 \times 0.762 n < 10^{-5} \text{ or } n > 43.88 \]
Use the Corollary

Therefore, we have

\[ |p_n - p| \leq \frac{k^n}{1 - k} |p_0 - p_1| \]
Use the Corollary

Therefore, we have

\[ |p_n - p| \leq \frac{k^n}{1 - k} |p_0 - p_1| \]

\[ \leq \frac{.762^n}{1 - .762} |\frac{1}{3} - .6933612| \]
Use the Corollary

Therefore, we have

$$|p_n - p| \leq \frac{k^n}{1 - k} |p_0 - p_1|$$

$$\leq \frac{.762^n}{1 - .762} \left| \frac{1}{3} - .6933612 \right|$$

$$\leq 1.513 \times 0.762^n$$
Using the corollary, we have

\[ |p_n - p| \leq \frac{k^n}{1 - k} |p_0 - p_1| \]

\[ \leq \frac{.762^n}{1 - .762} \left| \frac{1}{3} - .6933612 \right| \]

\[ \leq 1.513 \times 0.762^n \]

We require that

\[ 1.513 \times 0.762^n < 10^{-5} \quad \text{or} \quad n > 43.88 \]
Footnote on the Estimate Obtained

It is important to realise that the estimate for the number of iterations required given by the theorem is an upper bound. In the previous example, only 21 iterations are required in practice, i.e. \( p = 0.54781 \) is accurate to \( 10^{-5} \).

The reason, in this case, is that we used \( g'(1) = 0.762 \) whereas \( g'(0.54781) = 0.602 \). If one had used \( k = 0.602 \) (were it available) to compute the bound, one would obtain \( N = 23 \) which is a more accurate estimate.
Footnote on the Estimate Obtained

- It is important to realise that the estimate for the number of iterations required given by the theorem is an upper bound.

In the previous example, only 21 iterations are required in practice, i.e. \( p_{21} = 0.54781 \) is accurate to \( 10^{-5} \).

The reason, in this case, is that we used \( g'(1) = 0.762 \) whereas \( g'(0.54781) = 0.602 \). If one had used \( k = 0.602 \) (were it available) to compute the bound, one would obtain \( N = 23 \) which is a more accurate estimate.
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- If one had used \( k = 0.602 \) (were it available) to compute the bound, one would obtain \( N = 23 \) which is a more accurate estimate.
Outline

1. Functional (Fixed-Point) Iteration
2. Convergence Criteria for the Fixed-Point Method
3. Sample Problem: $f(x) = x^3 + 4x^2 - 10 = 0$
A Single Nonlinear Equation

Example 2

We return to Example 1 and the equation

\[ x^3 + 4x^2 - 10 = 0 \]

which has a unique root in \([1, 2]\). Its value is approximately 1.365230013.
$f(x) = x^3 + 4x^2 - 10 = 0$ on $[1, 2]$
Solving \( f(x) = x^3 + 4x^2 - 10 = 0 \)

Earlier, we listed 5 possible transpositions to \( x = g(x) \)

\[
x = g_1(x) = x - x^3 - 4x^2 + 10
\]

\[
x = g_2(x) = \sqrt{\frac{10}{x}} - 4x
\]

\[
x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3}
\]

\[
x = g_4(x) = \sqrt{\frac{10}{4 + x}}
\]

\[
x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}
\]
Solving \( f(x) = x^3 + 4x^2 - 10 = 0 \)

Results Observed for \( x = g(x) \) with \( x_0 = 1.5 \)

\[
x = g_1(x) = x - x^3 - 4x^2 + 10 \quad \text{Does not Converge}
\]

\[
x = g_2(x) = \sqrt{\frac{10}{x}} - 4x \quad \text{Does not Converge}
\]

\[
x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3} \quad \text{Converges after 31 Iterations}
\]

\[
x = g_4(x) = \sqrt{\frac{10}{4 + x}} \quad \text{Converges after 12 Iterations}
\]

\[
x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} \quad \text{Converges after 5 Iterations}
\]
Solving \( f(x) = x^3 + 4x^2 - 10 = 0 \)

\[ x = g(x) \text{ with } x_0 = 1.5 \]

\[ x = g_1(x) = x - x^3 - 4x^2 + 10 \quad \text{Does not Converge} \]

\[ x = g_2(x) = \sqrt{\frac{10}{x}} - 4x \quad \text{Does not Converge} \]

\[ x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3} \quad \text{Converges after 31 Iterations} \]

\[ x = g_4(x) = \sqrt{\frac{10}{4 + x}} \quad \text{Converges after 12 Iterations} \]

\[ x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} \quad \text{Converges after 5 Iterations} \]
Solving \( f(x) = x^3 + 4x^2 - 10 = 0 \)

\[ x = g_1(x) = x - x^3 - 4x^2 + 10 \]

Iteration for \( x = g_1(x) \) Does Not Converge
Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$x = g_1(x) = x - x^3 - 4x^2 + 10$

Iteration for $x = g_1(x)$ Does Not Converge

Since

$g_1'(x) = 1 - 3x^2 - 8x$

$g_1'(1) = -10$

$g_1'(2) = -27$

there is no interval $[a, b]$ containing $p$ for which $|g_1'(x)| < 1$. 
Solving \( f(x) = x^3 + 4x^2 - 10 = 0 \)

\[
x = g_1(x) = x - x^3 - 4x^2 + 10
\]

Iteration for \( x = g_1(x) \) Does Not Converge

Since

\[
g_1'(x) = 1 - 3x^2 - 8x \quad g_1'(1) = -10 \quad g_1'(2) = -27
\]

there is no interval \([a, b]\) containing \( p \) for which \( |g_1'(x)| < 1 \). Also, note that

\[
g_1(1) = 6 \text{ and } g_2(2) = -12
\]

so that \( g(x) \notin [1, 2] \) for \( x \in [1, 2] \).
Iteration Function: \( x = g_1(x) = x - x^3 - 4x^2 + 10 \)

| \( n \) | \( p_{n-1} \)     | \( p_n \)         | \( |p_n - p_{n-1}| \) |
|------|-----------------|------------------|----------------------|
| 1    | 1.50000000      | -0.87500000      | 2.37500000           |
| 2    | -0.87500000     | 6.7324219        | 7.6074219            |
| 3    | 6.7324219       | -469.7200120     | 476.4524339          |

\[ p_4 \approx 1.03 \times 10^8 \]
$g_1$ Does Not Map $[1, 2]$ into $[1, 2]$

$$g_1(x) = x - x^3 - 4x^2 + 10$$
$$|g'_1(x)| > 1 \text{ on } [1, 2]$$

$$g'_1(x) = 1 - 3x^2 - 8x$$
Solving \( f(x) = x^3 + 4x^2 - 10 = 0 \)

\( x = g(x) \) with \( x_0 = 1.5 \)

\[ x = g_1(x) = x - x^3 - 4x^2 + 10 \]
Does not Converge

\[ x = g_2(x) = \sqrt{\frac{10}{x}} - 4x \]
Does not Converge

\[ x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3} \]
Converges after 31 Iterations

\[ x = g_4(x) = \sqrt{\frac{10}{4 + x}} \]
Converges after 12 Iterations

\[ x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} \]
Converges after 5 Iterations
Solving \( f(x) = x^3 + 4x^2 - 10 = 0 \)

\[
x = g_2(x) = \sqrt{\frac{10}{x} - 4x}
\]

Iteration for \( x = g_2(x) \) is Not Defined
Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$x = g_2(x) = \sqrt{\frac{10}{x} - 4x}$

Iteration for $x = g_2(x)$ is Not Defined

It is clear that $g_2(x)$ does not map $[1, 2]$ onto $[1, 2]$ and the sequence $\{p_n\}_{n=0}^{\infty}$ is not defined for $p_0 = 1.5$. 

$g'(x) \approx -3.43$ and $g'(x)$ is not defined for $x > 1.58$. 

$g''(1) \approx -2.86$. 

Also, there is no interval containing $p$ such that $|g'(p)| < 1$ since $g'(1) \approx -2$. 

Numerical Analysis (Chapter 2)
Solving \( f(x) = x^3 + 4x^2 - 10 = 0 \)

\[ x = g_2(x) = \sqrt{\frac{10}{x} - 4x} \]

**Iteration for \( x = g_2(x) \) is Not Defined**

It is clear that \( g_2(x) \) does not map \([1, 2]\) onto \([1, 2]\) and the sequence \( \{p_n\}_{n=0}^{\infty} \) is not defined for \( p_0 = 1.5 \). Also, there is no interval containing \( p \) such that

\[ |g_2'(x)| < 1 \]

since

\[ g'(1) \approx -2.86 \]
\[ g'(p) \approx -3.43 \]

and \( g'(x) \) is not defined for \( x > 1.58 \).
Iteration Function: \( x = g_2(x) = \sqrt{\frac{10}{x}} - 4x \)

Iterations starting with \( p_0 = 1.5 \)

| \( n \) | \( p_{n-1} \) | \( p_n \) | \( |p_n - p_{n-1}| \) |
|--------|---------------|---------------|------------------|
| 1      | 1.5000000     | 0.8164966     | 0.6835034        |
| 2      | 0.8164966     | 2.9969088     | 2.1804122        |
| 3      | 2.9969088     | \( \sqrt{-8.6509} \) | —                |
Solving \( f(x) = x^3 + 4x^2 - 10 = 0 \)

\( x = g(x) \) with \( x_0 = 1.5 \)

\[ \begin{align*}
    x &= g_1(x) = x - x^3 - 4x^2 + 10 & \text{Does not Converge} \\
    x &= g_2(x) = \sqrt{\frac{10}{x}} - 4x & \text{Does not Converge} \\
    x &= g_3(x) = \frac{1}{2} \sqrt{10 - x^3} & \text{Converges after 31 Iterations} \\
    x &= g_4(x) = \sqrt{\frac{10}{4 + x}} & \text{Converges after 12 Iterations} \\
    x &= g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} & \text{Converges after 5 Iterations}
\end{align*} \]
Solving \( f(x) = x^3 + 4x^2 - 10 = 0 \)

\[
x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3}
\]

Iteration for \( x = g_3(x) \) Converges (Slowly)
Solving \( f(x) = x^3 + 4x^2 - 10 = 0 \)

\[
x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3}
\]

Iteration for \( x = g_3(x) \) Converges (Slowly)

By differentiation,

\[
g'_3(x) = -\frac{3x^2}{4\sqrt{10 - x^3}} < 0 \quad \text{for } x \in [1, 2]
\]

and so \( g = g_3 \) is strictly decreasing on \([1, 2]\).
Solving \( f(x) = x^3 + 4x^2 - 10 = 0 \)

\[ x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3} \]

Iteration for \( x = g_3(x) \) Converges (Slowly)

By differentiation,

\[ g'_3(x) = -\frac{3x^2}{4\sqrt{10 - x^3}} < 0 \quad \text{for} \quad x \in [1, 2] \]

and so \( g=g_3 \) is strictly decreasing on \([1, 2]\). However, \( |g'_3(x)| > 1 \) for \( x > 1.71 \) and \( |g'_3(2)| \approx -2.12 \).
Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3}$

Iteration for $x = g_3(x)$ Converges (Slowly)

By differentiation,

$g'_3(x) = -\frac{3x^2}{4\sqrt{10 - x^3}} < 0$ for $x \in [1, 2]$ and so $g=g_3$ is strictly decreasing on $[1, 2]$. However, $|g'_3(x)| > 1$ for $x > 1.71$ and $|g'_3(2)| \approx -2.12$. A closer examination of $\{p_n\}_{n=0}^{\infty}$ will show that it suffices to consider the interval $[1, 1.7]$ where $|g'_3(x)| < 1$ and $g(x) \in [1, 1.7]$ for $x \in [1, 1.7]$. 

Numerical Analysis (Chapter 2)
**Iteration Function**: \( x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3} \)

**Iterations starting with** \( p_0 = 1.5 \)

| \( n \) | \( p_{n-1} \) | \( p_n \) | \( |p_n - p_{n-1}| \) |
|---|---|---|---|
| 1 | 1.500000000 | 1.286953768 | 0.213046232 |
| 2 | 1.286953768 | 1.402540804 | 0.115587036 |
| 3 | 1.402540804 | 1.345458374 | 0.057082430 |
| 4 | 1.345458374 | 1.375170253 | 0.029711879 |
| 5 | 1.375170253 | 1.360094193 | 0.015076060 |
| 6 | 1.360094193 | 1.367846968 | 0.007752775 |
| 30 | 1.365230013 | 1.365230014 | 0.000000001 |
| 31 | 1.365230014 | 1.365230013 | 0.000000000 |
$g_3$ Maps $[1, 1.7]$ into $[1, 1.7]$

$g_3(x) = \frac{1}{2} \sqrt{10 - x^3}$
Fixed-Point Iteration

Convergence Criteria

Sample Problem

\[ |g_3'(x)| < 1 \text{ on } [1, 1.7] \]

\[ g_3'(x) = -\frac{3x^2}{4\sqrt{10-x^3}} \]
Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$x = g(x)$ with $x_0 = 1.5$

$x = g_1(x) = x - x^3 - 4x^2 + 10$  
Does not Converge

$x = g_2(x) = \sqrt{\frac{10}{x}} - 4x$  
Does not Converge

$x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3}$  
Converges after 31 Iterations

$x = g_4(x) = \sqrt{\frac{10}{4 + x}}$  
Converges after 12 Iterations

$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$  
Converges after 5 Iterations
Solving \( f(x) = x^3 + 4x^2 - 10 = 0 \)

\[
x = g_4(x) = \sqrt{\frac{10}{4 + x}}
\]

Iteration for \( x = g_4(x) \) Converges (Moderately)
Solving $f(x) = x^3 + 4x^2 - 10 = 0$

\[ x = g_4(x) = \sqrt{\frac{10}{4 + x}} \]

Iteration for $x = g_4(x)$ Converges (Moderately)

By differentiation,

\[ g'_4(x) = -\sqrt{\frac{10}{4(4 + x)^3}} < 0 \]

and it is easy to show that

\[ 0.10 < |g'_4(x)| < 0.15 \quad \forall x \in [1, 2] \]
Solving \( f(x) = x^3 + 4x^2 - 10 = 0 \)

\[
x = g_4(x) = \sqrt{\frac{10}{4 + x}}
\]

Iteration for \( x = g_4(x) \) Converges (Moderately)

By differentiation,

\[
g'_4(x) = -\sqrt{\frac{10}{4(4 + x)^3}} < 0
\]

and it is easy to show that

\[
0.10 < |g'_4(x)| < 0.15 \quad \forall \ x \in [1, 2]
\]

The bound on the magnitude of \( |g'_4(x)| \) is much smaller than that for \( |g'_3(x)| \) and this explains the reason for the much faster convergence.
**Iteration Function:** \( x = g_4(x) = \sqrt{\frac{10}{4+x}} \)

**Iterations starting with \( p_0 = 1.5 \)**

| \( n \) | \( p_{n-1} \)   | \( p_n \)   | \( |p_n - p_{n-1}| \)       |
|-------|-----------------|-------------|----------------------------|
| 1     | 1.50000000000  | 1.348399725 | 0.151600275               |
| 2     | 1.348399725     | 1.367376372 | 0.018976647               |
| 3     | 1.367376372     | 1.364957015 | 0.002419357               |
| 4     | 1.364957015     | 1.365264748 | 0.000307733               |
| 5     | 1.365264748     | 1.365225594 | 0.000039154               |
| 6     | 1.365225594     | 1.365230576 | 0.000004982               |
| 11    | 1.365230014     | 1.365230013 | 0.000000000               |
| 12    | 1.365230013     | 1.365230013 | 0.000000000               |
$g_4$ Maps $[1, 2]$ into $[1, 2]$
$|g_4'(x)| < 1$ on $[1, 2]$
Solving \( f(x) = x^3 + 4x^2 - 10 = 0 \)

**x = g(x) with \( x_0 = 1.5 \)**

- \( x = g_1(x) = x - x^3 - 4x^2 + 10 \) \( \text{Does not Converge} \)
- \( x = g_2(x) = \sqrt{\frac{10}{x}} - 4x \) \( \text{Does not Converge} \)
- \( x = g_3(x) = \frac{1}{2} \sqrt{10 - x^3} \) \( \text{Converges after 31 Iterations} \)
- \( x = g_4(x) = \sqrt{\frac{10}{4 + x}} \) \( \text{Converges after 12 Iterations} \)
- \( x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} \) \( \text{Converges after 5 Iterations} \)
Solving \( f(x) = x^3 + 4x^2 - 10 = 0 \)

\[
x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}
\]

Iteration for \( x = g_5(x) \) Converges (Rapidly)
Solving $f(x) = x^3 + 4x^2 - 10 = 0$

$x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$

Iteration for $x = g_5(x)$ Converges (Rapidly)

For the iteration function $g_5(x)$, we obtain:

$g_5(x) = x - \frac{f(x)}{f'(x)} \Rightarrow g'_5(x) = \frac{f(x)f''(x)}{[f'(x)]^2} \Rightarrow g'_5(p) = 0$
Solving \( f(x) = x^3 + 4x^2 - 10 = 0 \)

\[ x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} \]

Iteration for \( x = g_5(x) \) Converges (Rapidly)

For the iteration function \( g_5(x) \), we obtain:

\[ g_5(x) = x - \frac{f(x)}{f'(x)} \Rightarrow g'_5(x) = \frac{f(x)f''(x)}{[f'(x)]^2} \Rightarrow g'_5(p) = 0 \]

It is straightforward to show that \( 0 \leq |g'_5(x)| < 0.28 \) \( \forall x \in [1, 2] \) and the order of convergence is quadratic since \( g'_5(p) = 0 \).
Iteration Function: \( x = g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x} \)

| \( n \) | \( p_{n-1} \) | \( p_n \) | \( |p_n - p_{n-1}| \) |
|-----|------|------|-----------------|
| 1   | 1.500000000 | 1.373333333 | 0.126666667 |
| 2   | 1.373333333 | 1.365262015 | 0.008071318 |
| 3   | 1.365262015 | 1.365230014 | 0.000032001 |
| 4   | 1.365230014 | 1.365230013 | 0.000000001 |
| 5   | 1.365230013 | 1.365230013 | 0.000000000 |
$g_5$ Maps $[1, 2]$ into $[1, 2]$

$$g_5(x) = x - \frac{x^3 + 4x^2 - 10}{3x^2 + 8x}$$
\[ |g'_5(x)| < 1 \text{ on } [1, 2] \]

The function is defined as:

\[ g'_5(x) = \frac{(x^3 + 4x^2 - 10)(6x + 8)}{(3x^2 + 8x)^2} \]
Questions?
Reference Material
If \( f \in C[a, b] \) and \( f \) is differentiable on \((a, b)\), then a number \( c \) exists such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]