Fractals and Fractal Dimension

Math 333 Tutorial 8
31-3-2000

- You can read Chapter 4 in the book **Chaos: Making a New Science** on this object.

Introduction

- 0-D in $\Re^3$: Point $x = a,\ y = b,\ z = c$
- 1-D in $\Re^3$: Line $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$
- 2-D in $\Re^3$: Plane $ax + by + cz + d = 0$
- 3-D: Whole real space $x, y, z \in \Re$

- However there are objects living in the real space having a non-integer dimension. For example

$$\frac{\log 2}{\log 3} \approx 0.6309,\ \frac{\log 4}{\log 3} \approx 1.2619,$$

$$\frac{\log 8}{\log 3} \approx 1.8928...$$

- Q: what is the dimension of a lump of cotton?
- A: It depends on what' s your point of view: if you were far enough from the object, it would be a 0-D object; if you walked closer, it became 3-D object; if looked more closer, then it is a 1-D object. But, how about using a microscope to see the cotton, it returns to a 3-D object again.

- This means the dimension of the object will depend on the relationship between it and the observer.
- Therefore, there are some space for us to generalize the definition of dimension.

Classical Fractals

- 1. Cantor Set:

C_1 —————————————

C_2 ———— ————

C_3 — — — —

$\vdots$ The remaining **dust** are infinite number of points with zero length

C_∞

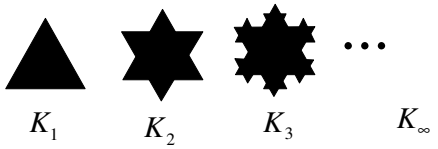
- Infinite number of points:
- Using base 3 notation, a point with no 1 will belong to the Cantor set.
- Zero length: $L_1 =$ length of the set $C_1 = 1$

$$L_2 = \frac{2}{3}, L_3 = \left(\frac{2}{3}\right)^2, ..., L_n = \left(\frac{2}{3}\right)^n$$

$$\therefore L = \lim_{n\to\infty} L_n = \lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 0$$

- Conclusion: what is the dimension of the Cantor set?
- Dimension 0? But there are infinite many elements in the set.
- Dimension 1? But the length of the set is zero.
- It seems that the Cantor set is having dimension somewhere between 0 and 1.

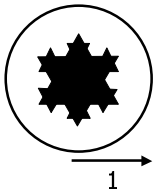
- 2. Koch Curve: Cantor set is **constructed** by removing something from the original set. Koch curve is **constructed** by adding something to the original curve.



K_1 K_2 K_3 K_∞

The finial shape is called the Koch Flake, constructed by Helge von Koch in 1904. Finite area is having an infinite circumference.

- Finite area: the flake can be enclosed by a circle, say with radius 1.



$$A = \lim_{n\to\infty} A_n < 2\pi$$

- Infinite circumference:

$$L_1 = 3,\ L_2 = 3\times\frac{4}{3},$$

$$L_3 = 3\times\left(\frac{4}{3}\right)^2,\ L_4 = 3\times\left(\frac{4}{3}\right)^3,\ldots,$$

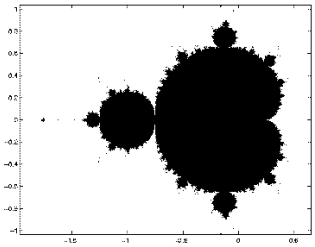
$$L_n = 3\times\left(\frac{4}{3}\right)^{n-1}$$

- Conclusion: what is the dimension of the Koch curve, one side of the boundary of the Koch flake?
- Dimension 1? But the infinite length curve is enclosed in the finite region.
- Dimension 2? But it should be a curve only.
- It seems that the dimension of the Koch curve is somewhere between 1 and 2.

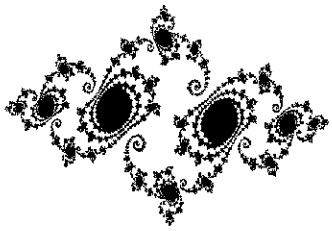
- Q: what is the length of the boundary between the ocean and the land?
- A: the answer is the same as that in the Koch curve: infinite.

Other fractals:

- 1. Mandelbrot Set:



- Julia Set:



Some Properties of Fractals

- 1. Structure of arbitrary small scales: no matter how close you are looking at, there are some shapes in the fractal.
- 2. Self-similar: the whole shape can be seen as the summation of several smaller copies of itself.
- 3. Non-integer dimension.

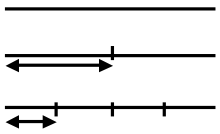
Fractal Dimension

- So, how can we give a definition of dimension to a fractal?
- 2 properties of this definition has to satisfy:
  - 1. Original idea of dimension has to be included.
  - 2. Larger the value of the dimension, the fractal is **living** in a higher dimensional space.

- If the fractal is composed of *m* copies when the size of the measure is scaled-down by factor *r*, the we define the dimension to be

$$d = \frac{\log m}{\log r}$$

- Example 1:

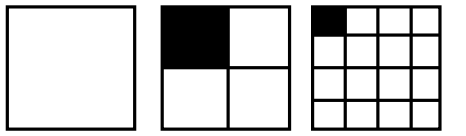


$m = 2, r = 2$

$m = 4, r = 4$

$\Rightarrow d = 1$

- Example 2:



$m = 4, r = 2$ $m = 16, r = 4$

$\Rightarrow d = 2$

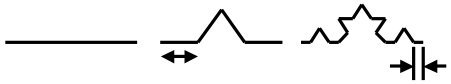
- Example 3:

$m = 2, r = 3$

$m = 4, r = 9$

$$d = \frac{\log 2}{\log 3} \approx 0.6309$$

- Example 4:



$m = 4, r = 3$ $m = 16, r = 9$

$$d = \frac{\log 4}{\log 3} \approx 1.2619$$