4. Portfolio theory

Suppose there are $N$ risky assets, whose rates of returns are given by the random variables $R_1, \cdots, R_N$, where

$$R_n = \frac{S_n(1) - S_n(0)}{S_n(0)}, n = 1, 2, \cdots, N.$$ 

Let $w = (w_1 \cdots w_N)^T$, $w_n$ denotes the proportion of wealth invested in asset $i$, with $\sum_{n=1}^{N} w_n = 1$. The rate of return of the portfolio is

$$R_P = \sum_{n=1}^{N} w_n R_n.$$ 

Assumptions

1. There does not exist any asset that is a combination of other assets in the portfolio.

2. $\mu = (\bar{R}_1 \bar{R}_2 \cdots \bar{R}_N)$ and $\mathbf{1} = (1 \ 1 \cdots 1)$ are linearly independent.
The first two moments of $R_P$ are

$$
\mu_P = E[R_P] = \sum_{n=1}^{N} E[w_nR_n] = \sum_{n=1}^{N} w_n\mu_n, \text{ where } \mu_n = \bar{R}_n,
$$

and

$$
\sigma^2_P = \text{var}(R_P) = \sum_{i=1}^{N} \sum_{j=1}^{N} w_iw_j\text{cov}(R_i, R_j) = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i\sigma_{ij}w_j.
$$

Let $\Omega$ denote the covariance matrix so that

$$
\sigma^2_P = w^T\Omega w.
$$

Remark

The portfolio risk of return is quantified by $\sigma^2_P$. In mean-variance analysis, only the first two moments are considered in the portfolio model. Investment theory prior to Markowitz considered the maximization of $\mu_P$ but without $\sigma_P$. 
**Two-asset portfolio**

Consider two assets with known means $\bar{R}_1$ and $\bar{R}_2$, variances $\sigma_1^2$ and $\sigma_2^2$, of the expected rates of returns $R_1$ and $R_2$, together with the correlation coefficient $\rho$.

Let $1 - \alpha$ and $\alpha$ be the weights of assets 1 and 2 in this two-asset portfolio.

Portfolio mean: $\bar{R}_P = (1 - \alpha)\bar{R}_1 + \alpha\bar{R}_2, 0 \leq \alpha \leq 1$

Portfolio variance: $\sigma_P^2 = (1 - \alpha)^2\sigma_1^2 + 2\rho\alpha(1 - \alpha)\sigma_1\sigma_2 + \alpha^2\sigma_2^2$. 
We represent the two assets in a mean-standard deviation diagram (recall: standard deviation = $\sqrt{\text{variance}}$)

As $\alpha$ varies, $(\sigma_P, \overline{R}_P)$ traces out a conic curve in the $\sigma - \overline{R}$ plane. With $\rho = -1$, it is possible to have $\sigma = 0$ for some suitable choice of weight.
In particular, when $\rho = 1$,

\[
\sigma_P(\alpha; \rho = 1) = \sqrt{(1 - \alpha)^2 \sigma_1^2 + 2\alpha(1 - \alpha)\sigma_1\sigma_2 + \alpha^2\sigma_2^2}
\]

\[
= (1 - \alpha)\sigma_1 + \alpha\sigma_2.
\]

This is the straight line joining $P_1(\sigma_1, \bar{R}_1)$ and $P_2(\sigma_2, \bar{R}_2)$.

When $\rho = -1$, we have

\[
\sigma_P(\alpha; \rho = -1) = \sqrt{[(1 - \alpha)\sigma_1 - \alpha\sigma_2]^2} = |(1 - \alpha)\sigma_1 - \alpha\sigma_2|.
\]

When $\alpha$ is small (close to zero), the corresponding point is close to $P_1(\sigma_1, \bar{R}_1)$. The line $AP_1$ corresponds to

\[
\sigma_P(\alpha; \rho = -1) = (1 - \alpha)\sigma_1 - \alpha\sigma_2.
\]

The point $A$ corresponds to $\alpha = \frac{\sigma_1}{\sigma_1 + \sigma_2}$.

The quantity $(1 - \alpha)\sigma_1 - \alpha\sigma_2$ remains positive until $\alpha = \frac{\sigma_1}{\sigma_1 + \sigma_2}$.

When $\alpha > \frac{\sigma_1}{\sigma_1 + \sigma_2}$, the locus traces out the upper line $AP_2$. 
Suppose $-1 < \rho < 1$, the minimum variance point on the curve that represents various portfolio combinations is determined by

$$\frac{\partial \sigma_P^2}{\partial \alpha} = -2(1 - \alpha)\sigma_1^2 + 2\alpha\sigma_2^2 + 2(1 - 2\alpha)\rho\sigma_1\sigma_2 = 0$$

\[\uparrow\text{set}\]

giving

$$\alpha = \frac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}.$$
\[ P_1(\sigma_1, \bar{R}_1) \]

minimum-variance point

\[ P_2(\sigma_2, \bar{R}_2) \]
Mathematical formulation of Markowitz's mean-variance analysis

minimize \( \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} \)

subject to \( \sum_{i=1}^{N} w_i \bar{R}_i = \mu_P \) and \( \sum_{i=1}^{N} w_i = 1 \).

Solution

We form the Lagrangian

\[
L = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} - \lambda_1 \left( \sum_{i=1}^{N} w_i - 1 \right) - \lambda_2 \left( \sum_{i=1}^{N} w_i \bar{R}_i - \mu_P \right)
\]

where \( \lambda_1 \) and \( \lambda_2 \) are Lagrangian multipliers.

We then differentiate \( L \) with respect to \( w_i \) and set the derivative to zero.

\[
\frac{\partial L}{\partial w_i} = \sum_{j=1}^{N} \sigma_{ij} w_j - \lambda_1 - \lambda_2 \bar{R}_i = 0, \quad i = 1, 2, \ldots, N. \quad (1)
\]

\[
\frac{\partial L}{\partial \lambda_1} = \sum_{i=1}^{N} w_i - 1 = 0; \quad (2)
\]

\[
\frac{\partial L}{\partial \lambda_2} = \sum_{i=1}^{N} w_i \bar{R}_i - \mu_P = 0. \quad (3)
\]
From Eq. (1), the portfolio weight admits solution of the form

\[ w^* = \Omega^{-1}(\lambda_1 \mathbf{1} + \lambda_2 \mu) \]  

(4)

where \( \mathbf{1} = (1 \ 1 \cdots 1)^T \) and \( \mu = (\bar{R}_1 \ \bar{R}_2 \cdots \bar{R}_N)^T \).

To determine \( \lambda_1 \) and \( \lambda_2 \), we apply the two constraints

\[ 1 = \mathbf{1}^T \Omega^{-1} \Omega w^* = \lambda_1 \mathbf{1}^T \Omega^{-1} \mathbf{1} + \lambda_2 \mathbf{1}^T \Omega^{-1} \mu. \]  

(5)

\[ \mu_P = \mu^T \Omega^{-1} \Omega w^* = \lambda_1 \mu^T \Omega^{-1} \mathbf{1} + \lambda_2 \mu^T \Omega^{-1} \mu. \]  

(6)

Write \( a = \mathbf{1}^T \Omega^{-1} \mathbf{1}, b = \mathbf{1}^T \Omega^{-1} \mu \) and \( c = \mu^T \Omega^{-1} \mu \), we have

\[ 1 = \lambda_1 a + \lambda_2 b \]  

and \( \mu_P = \lambda_1 b + \lambda_2 c. \)

Solving for \( \lambda_1 \) and \( \lambda_2 \) :

\[ \lambda_1 = \frac{c - b \mu_P}{\Delta} \]  

and \( \lambda_2 = \frac{a \mu_P - b}{\Delta} \), where \( \Delta = ac - b^2 \).

Note that \( \lambda_1 \) and \( \lambda_2 \) have dependence on \( \mu_P \), which is the target mean prescribed in the variance minimization problem.
Assume $\mu \neq h \mathbf{1}$, and $\Omega^{-1}$ exists. Since $\Omega$ is positive definite, so $a > 0$, $c > 0$. By virtue of the Cauchy-Schwarz inequality, $\Delta > 0$. The minimum portfolio variance for a given value of $\mu_P$ is given by

$$\sigma_P^2 = w^T \Omega w^* = w^T \Omega (\lambda_1 \Omega^{-1} \mathbf{1} + \lambda_2 \Omega^{-1} \mu) = \lambda_1 + \lambda_2 \mu_P = \frac{a\mu_P^2 - 2b\mu_P + c}{\Delta}.$$ 

The set of minimum variance portfolios is represented by a parabolic curve in the $\sigma_P^2 - \mu_P$ plane. The parabolic curve is generated by varying the value of the parameter $\mu_P$. 

![Diagram](image.png)
How about the asymptotic values of \( \lim_{\mu \to \pm \infty} \frac{d\mu_P}{d\sigma_P} \)?

\[
\frac{d\mu_P}{d\sigma_P} = \frac{d\mu_P}{d\sigma_P^2} \frac{d\sigma_P^2}{d\sigma_P} \\
= \frac{\Delta}{2a\mu_P - 2b} 2\sigma_P \\
= \frac{\sqrt{\Delta}}{a\mu_P - b} \sqrt{a\mu_P^2 - 2b\mu_P + c}
\]

so that

\[
\lim_{\mu \to \pm \infty} \frac{d\mu_P}{d\sigma_P} = \pm \sqrt{\frac{\Delta}{a}}.
\]
Summary

Given $\mu_P$, we obtain $\lambda_1 = \frac{c - b\mu_P}{\Delta}$ and $\lambda_2 = \frac{a\mu_P - b}{\Delta}$, and the optimal weight $w^* = \Omega^{-1}(\lambda_1 \mathbf{1} + \lambda_2 \mu)$.

To find the global minimum variance portfolio, we set

$$\frac{d\sigma_P^2}{d\mu_P} = \frac{2a\mu_P - 2b}{\Delta} = 0$$

so that $\mu_P = b/a$ and $\sigma_P^2 = 1/a$. Also, $\lambda_1 = 1/a$ and $\lambda_2 = 0$. We obtain

$$w_g = \frac{\Omega^{-1} \mathbf{1}}{a} = \frac{\Omega^{-1} \mathbf{1}}{\mathbf{1}^T \Omega^{-1} \mathbf{1}}.$$

Another portfolio that corresponds to $\lambda_1 = 0$ is obtained when $\mu_P$ is taken to be $\frac{c}{b}$. The value of the other Lagrangian multiplier is given by

$$\lambda_2 = \frac{a \left( \frac{c}{b} \right) - b}{\Delta} = \frac{1}{b}.$$

The optimal weight of this particular portfolio is

$$w_d^* = \frac{\Omega^{-1} \mu}{b} = \frac{\Omega^{-1} \mu}{\mathbf{1}^T \Omega^{-1} \mathbf{1}}.$$

Also, $\sigma_d^2 = \frac{a \left( \frac{c}{b} \right)^2 - 2b \left( \frac{c}{b} \right) + c}{\Delta} = \frac{c}{b^2}$.
Feasible set

Given $N$ risky assets, we form various portfolios from these $N$ assets. We plot the point $(\sigma_P, \bar{R}_P)$ representing the portfolios in the $\sigma - \bar{R}$ diagram. The collection of these points constitutes the feasible set or feasible region.
Consider a 3-asset portfolio, the various combinations of assets 2 and 3 sweep out a curve between them (the particular curve taken depends on the correlation coefficient $\rho_{12}$).

A combination of assets 2 and 3 (labelled 4) can be combined with asset 1 to form a curve joining 1 and 4. As 4 moves between 2 and 3, the curve joining 1 and 4 traces out a solid region.
Properties of feasible regions

1. If there are at least 3 risky assets (not perfectly correlated and with different means), then the feasible set is a solid two-dimensional region.

2. The feasible region is *convex to the left*. That is, given any two points in the region, the straight line connecting them does not cross the left boundary of the feasible region.
The left boundary of a feasible region is called the *minimum variance set*. The most left point on the minimum variance set is called the *minimum variance point*. The portfolios in the minimum variance set are called *frontier funds*.

For a given level of risk, only those portfolios on the *upper half* of the efficient frontier are desired by investors. They are called *efficient funds*.

A portfolio $w^*$ is said to be mean-variance efficient if there exists no portfolio $w$ with $\mu_P \geq \mu^*_P$ and $\sigma^2_P \leq \sigma^*_P$. That is, you cannot find a portfolio that has a higher return and lower risk than those for an efficient portfolio.
Two-fund theorem

Two frontier funds (portfolios) can be established so that any frontier portfolio can be duplicated, in terms of mean and variance, as a combination of these two. In other words, all investors seeking frontier portfolios need only invest in combinations of these two funds.

Remark

Any convex combination (that is, weights are non-negative) of efficient portfolios is an efficient portfolio. Let $\alpha_i \geq 0$ be the weight of Fund $i$ whose rate of return is $R_f^i$. Since $E[R_f^i] \geq \frac{b}{a}$ for all $i$, we have

$$\sum_{i=1}^{n} \alpha_i E[R_f^i] \geq \sum_{i=1}^{n} \alpha_i \frac{b}{a} = \frac{b}{a}.$$
Proof

Let \( w^1 = (w_1^1 \cdots w_n^1), \lambda_1^1, \lambda_2^1 \) and \( w^2 = (w_1^2 \cdots w_n^2)^T, \lambda^2, \mu^2 \) are two known solutions to the minimum variance formulation with expected rates of return \( \mu_P^1 \) and \( \mu_P^2 \), respectively.

\[
\sum_{j=1}^{n} \sigma_{ij} w_j - \lambda_1 - \lambda_2 \bar{R}_i = 0, \quad i = 1, 2, \cdots, n \tag{1}
\]

\[
\sum_{i=1}^{n} w_i \bar{r}_i = \mu_P \tag{2}
\]

\[
\sum_{i=1}^{n} w_i = 1. \tag{3}
\]

It suffices to show that \( \alpha w_1 + (1 - \alpha) w_2 \) is a solution corresponds to the expected rate of return \( \alpha \mu_P^1 + (1 - \alpha) \mu_P^2 \).
1. $\alpha w^1 + (1 - \alpha)w^2$ is a legitimate portfolio with weights that sum to one.

2. Eq. (1) is satisfied by $\alpha w^1 + (1 - \alpha)w^2$ since the system of equations is linear.

3. Note that

$$\sum_{i=1}^{n} \left[ \alpha w_i^1 + (1 - \alpha)w_i^2 \right] R_i$$

$$= \alpha \sum_{i=1}^{n} w_i^1 R_i + (1 - \alpha) \sum_{i=1}^{n} w_i^2 R_i$$

$$= \alpha \mu^1_P + (1 - \alpha) \mu^2_P.$$
**Proposition**

Any minimum variance portfolio with target mean $\mu_P$ can be uniquely decomposed into the sum of two portfolios

$$w^*_P = A w_g + (1 - A) w_d$$

where $A = \lambda_1 a = \frac{c - b\mu_P}{\Delta}a$.

**Proof**

For a minimum-variance portfolio whose solution of the Lagrangian multipliers are $\lambda_1$ and $\lambda_2$, the optimal weight is

$$w^*_P = \lambda_1 (\Omega^{-1}1 + \Omega^{-1}\mu) = \lambda_1 (aw_g) + \lambda_2 (bw_d).$$

Observe that the sum of weights is

$$\lambda_1 a + \lambda_2 b = a \frac{c - \mu_P b}{\Delta} + b \frac{\mu_P a - b}{\Delta} = \frac{ac - b^2}{\Delta} = 1.$$  

We set $\lambda_1 a = A$ and $\lambda_2 b = 1 - A$. 
Indeed, any two minimum-variance portfolios can be used to substitute for $w_g$ and $w_d$. Suppose

$$w_u = (1 - u)w_g + uw_d$$

$$w_v = (1 - v)w_g + vw_d$$

we then solve for $w_g$ and $w_d$ in terms of $w_u$ and $w_v$. Then

$$w^*_P = \lambda_1aw_g + (1 - \lambda_1a)w_d$$

$$= \frac{\lambda_1a + v - 1}{v - u}w_u + \frac{1 - u - \lambda_1a}{v - u}w_v,$$

where sum of coefficients = 1.
Mean, variances, and covariances of the rates of return of 5 risky assets are listed:

<table>
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<th>covariance</th>
<th>$\bar{R}_i$</th>
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<td>1</td>
<td>2.30 0.93 0.62 0.74 −0.23</td>
<td>15.1</td>
</tr>
<tr>
<td>2</td>
<td>0.93 1.40 0.22 0.56 0.26</td>
<td>12.5</td>
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<tr>
<td>3</td>
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<td>14.7</td>
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<td>4</td>
<td>0.74 0.56 0.78 3.40 −0.56</td>
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<td>5</td>
<td>−0.23 0.26 −0.27 −0.56 2.60</td>
<td>17.68</td>
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</table>
Solution procedure to find the two funds in the minimum variance set:

1. Set $\lambda_1 = 1$ and $\lambda_2 = 0$; solve the system of equations

$$\sum_{j=1}^{5} \sigma_{ij} v_j^1 = 1, \quad i = 1, 2, \cdots, 5.$$ 

Normalize $v_k^1$'s so that they sum to one

$$w_i^1 = \frac{v_i^1}{\sum_{j=1}^{n} v_j^1}.$$ 

After normalization, this gives the solution to $w_g$, where $\lambda_1 = \frac{1}{a}$ and $\lambda_2 = 0$. 
2. Set $\lambda_1 = 0$ and $\lambda_2 = 1$; solve the system of equations:

$$
\sum_{j=1}^{5} \sigma_{ij} v_j^2 = R_i, \quad i = 1, 2, \ldots, 5.
$$

Normalize $v_i^2$'s to obtain $w_i^2$.

After normalization, this gives the solution to $w_d$, where $\lambda_1 = 0$ and $\lambda_2 = \frac{1}{b}$.

The above procedure avoids the computation of $a = \mathbf{1}^T \Omega^{-1} \mathbf{1}$ and $b = \mathbf{1}^T \Omega^{-1} \mu$. 
<table>
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<th>$v^1$</th>
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<th>$w^1$</th>
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<td>standard deviation</td>
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<td>0.791</td>
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* Note that $w^1$ corresponds to the global minimum variance point.
We know that $\mu_g = b/a$; how about $\mu_d$?

$$
\mu_d = \mu^T w_d = \mu^T \Omega^{-1} \mu = \frac{c}{b}.
$$

Difference in expected returns $= \mu_d - \mu_g = \frac{c}{b} - \frac{b}{a} = \frac{\Delta}{ab} > 0$.

Also, difference in variances $= \sigma_d^2 - \sigma_g^2 = \frac{c}{b^2} - \frac{1}{a} = \frac{\Delta}{ab^2} > 0$. 
How about the covariance of portfolio returns for any two minimum variance portfolios?

Write

\[ R_P^u = w_u^T \mathbf{R} \quad \text{and} \quad R_P^v = w_v^T \mathbf{R} \]

where \( \mathbf{R} = (R_1 \cdots R_N)^T \). Recall that

\[
\sigma_{gd} = \text{cov} \left( \frac{\Omega^{-1}}{a} \mathbf{1}, \frac{\Omega^{-1}}{b} \boldsymbol{\mu} \right) = \left( \frac{\Omega^{-1}}{a} \mathbf{1} \right)^T \Omega \left( \frac{\Omega^{-1}}{b} \boldsymbol{\mu} \right) = \frac{1}{ab} \Omega^{-1} \boldsymbol{\mu} = \frac{1}{a} \quad \text{since} \quad b = \mathbf{1} \Omega^{-1} \boldsymbol{\mu}.
\]
\[ \text{cov}(R_P^u, R_P^v) = (1 - u)(1 - v)\sigma_g^2 + uv\sigma_d^2 + [u(1 - v) + v(1 - u)]\sigma_{gd} \]

\[ = \frac{(1 - u)(1 - v)}{a} + \frac{uv}{b^2} + \frac{u + v - 2uv}{a} \]

\[ = \frac{1}{a} + \frac{uv\Delta}{ab^2}. \]

In particular,

\[ \text{cov}(R_g, R_P) = w_g^T\Omega w_P = \frac{1\Omega^{-1}\Omega w_P}{a} = \frac{1}{a} = \text{var}(R_g) \]

for any portfolio \( w_P \).

For any Portfolio \( u \), we can find another Portfolio \( v \) such that these two portfolios are uncorrelated. This can be done by setting

\[ \frac{1}{a} + \frac{uv\Delta}{ab^2} = 0. \]
Inclusion of a riskfree asset

Consider a portfolio with weight $\alpha$ for a risk free asset and $1 - \alpha$ for a risky asset. The mean of the portfolio is

$$R_P = \alpha R_f + (1 - \alpha) R_j$$

(note that $R_f = \overline{R}_f$).

The covariance $\sigma_{f,j}$ between the risk free asset and any risky asset is zero since

$$E[(R_j - \overline{R}_j)(R_f - \overline{R}_f)] = 0.$$

Therefore, the variance of portfolio $\sigma_P^2$ is

$$\sigma_P^2 = \alpha^2 \sigma_f^2 + (1 - \alpha)^2 \sigma_j^2 + 2\alpha(1 - \alpha) \sigma_{f,j}$$

so that $\sigma_P = |1 - \alpha| \sigma_j$. 

The points representing \((\sigma_P, \bar{R}_P)\) for varying values of \(\alpha\) lie on a straight line joining \((0, R_f)\) and \((\sigma_j, \bar{R}_j)\).

If borrowing of risk free asset is allowed, then \(\alpha\) can be negative. In this case, the line extends beyond the right side of \((\sigma_j, \bar{R}_j)\) (possibly up to infinity).
Consider a portfolio with $N$ risky assets originally, what is the impact of the inclusion of a risk free asset on the feasible region?

*Lending and borrowing of risk free asset is allowed*

For each original portfolio formed using the $N$ risky assets, the new combinations with the inclusion of the risk free asset trace out the infinite straight line originating from the risk free point and passing through the point representing the original portfolio.

The totality of these lines forms an infinite triangular feasible region bounded by the two tangent lines through the risk free point to the original feasible region.
No shorting of risk free asset

The line originating from the risk free point cannot be extended beyond points in the original feasible region (otherwise entail borrowing of the risk free asset). The new feasible region has straight line front edges.
The new efficient set is the single straight line on the top of the new triangular feasible region. This tangent line touches the original feasible region at a point $F$, where $F$ lies on the efficient frontier of the original feasible set.

Here, $R_f < \frac{b}{a}$.
One fund theorem

Any efficient portfolio (any point on the upper tangent line) can be expressed as a combination of the risk free asset and the portfolio (or fund) represented by $F$.

“There is a single fund $F$ of risky assets such that any efficient portfolio can be constructed as a combination of the fund $F$ and the risk free asset.”

Remark Under the assumptions that

- every investor is a mean-variance optimizer
- they all agree on the probabilistic structure of assets
- unique risk free asset

Then everyone will purchase a single fund, market portfolio.
New Lagrangian formulation

minimize \[ \frac{\sigma_P^2}{2} = \frac{1}{2} w^T \Omega w \]

subject to \[ w^T \mu + (1 - w^T 1)r = \mu_P. \]

Define \[ L = \frac{1}{2} w^T \Omega w + \lambda[\mu_P - r - (\mu - r 1)^T w] \]

\[ \frac{\partial L}{\partial w_i} = \sum_{j=1}^{N} \sigma_{ij} w_j - \lambda(\mu - r 1) = 0, \quad i = 1, 2, \ldots, N \] (1)

\[ \frac{\partial L}{\partial \lambda} = 0 \quad \text{giving} \quad (\mu - r 1)^T w = \mu_P - r. \] (2)

Solving (1): \[ w^* = \lambda \Omega^{-1}(\mu - r 1). \] Substituting into (2)

\[ \mu_P - r = \lambda(\mu - r 1)^T \Omega^{-1}(\mu - r 1) = \lambda(c - 2rb + r^2a). \]

Lastly, the relation between \( \mu_P \) and \( \sigma_P \) is given by the following pair of half lines

\[ \sigma_P^2 = w^*^T \Omega w^* = \lambda(w^*^T \mu - rw^*^T 1) \]

\[ = \lambda(\mu_P - r) = (\mu_P - r)^2/(c - 2rb + r^2a). \]
With the inclusion of the riskfree asset, the set of minimum variance portfolios are represented by portfolios on the two half lines

\begin{align*}
L_{up} & : \mu_P - r = \sigma_P \sqrt{ar^2 - 2br + c} \\
L_{low} & : \mu_P - r = -\sigma_P \sqrt{ar^2 - 2br + c}.
\end{align*} \tag{1a, 1b}

Recall that \(ar^2 - 2br + c > 0\) for all values of \(r\) since \(\Delta = ac - b^2 > 0\). The minimum variance portfolios without the riskfree asset lie on the hyperbola

\[
\sigma_P^2 = \frac{a\mu_P^2 - 2b\mu_P + c}{\Delta}.
\]
When \( r < \mu_g = \frac{b}{a} \), the upper half line is a tangent to the hyperbola. The tangency portfolio is the tangent point to the efficient frontier (upper part of the hyperbolic curve) through the point \((0, r)\).
The tangency portfolio $M$ is represented by the point $(\sigma_{P,M}, \mu_M^P)$, and the solution to $\sigma_{P,M}$ and $\mu_M^P$ are obtained by solving simultaneously

\[
\sigma_P^2 = \frac{a\mu_P^2 - 2b\mu_P + c}{\Delta}
\]

\[
\mu_P = \frac{r + \sigma_P \sqrt{c - 2rb + r^2a}}{\Delta}
\]

Once $\mu_P$ is obtained, we solve for $\lambda$ and $w^*$ from

\[
\lambda = \frac{\mu_P - r}{c - 2rb + r^2a}
\]

and

\[
w^* = \lambda \Omega^{-1}(\mu - r \mathbf{1}).
\]

The tangency portfolio $M$ is shown to be

\[
w_M^* = \frac{\Omega^{-1}(\mu - r \mathbf{1})}{b - ar}, \quad \mu_M^P = \frac{c - br}{b - ar} \quad \text{and} \quad \sigma_{P,M}^2 = \frac{c - 2rb + r^2a}{(b - ar)^2}.
\]
When \( r < \frac{b}{a} \), it can be shown that \( \mu^M_P > r \). Note that

\[
\left( \mu^M_P - \frac{b}{a} \right) \left( \frac{b}{a} - r \right) = \left( \frac{c - br}{b - ar} - \frac{b}{a} \right) \frac{b - ar}{a} \\
= \frac{c - br}{a} - \frac{b^2}{a^2} + \frac{br}{a} \\
= \frac{ca - b^2}{a^2} = \frac{\Delta}{a^2} > 0,
\]

so we deduce that \( \mu^M_P > \frac{b}{a} > r \), where \( \mu_g = \frac{b}{a} \). Indeed, we can deduce \((\sigma_{P,M}, \mu^M_P)\) does not lie on the upper half line if \( r \geq \frac{b}{a} \).
When \( r < \frac{b}{a} \), we have the following properties on the minimum variance portfolios.

1. *Efficient portfolios*

   Any portfolio on the upper half line
   \[
   \mu_P = r + \sigma_P \sqrt{ar^2 - 2br + c}
   \]
   within the segment \( FM \) joining the two points \((0, r)\) and \( M \) involves long holding of the market portfolio and riskfree asset, while those outside \( FM \) involves short selling of the riskfree asset and long holding of the market portfolio.

2. Any portfolio on the lower half line
   \[
   \mu_P = r - \sigma_P \sqrt{ar^2 - 2br + c}
   \]
   involves short selling of the market portfolio and investing the proceeds in the riskfree asset. This represents non-optimal investment strategy since the investor faces risk but gains no extra expected return above \( r \).
What happens when $r = b/a$? The half lines become

$$
\mu_P = r \pm \sigma_P \sqrt{c - 2 \left( \frac{b}{a} \right) b \pm \frac{b^2}{a}} = r \pm \sigma_P \sqrt{\frac{\Delta}{a}},
$$

which correspond to the asymptotes of the feasible region with risky assets only.

When $r = \frac{b}{a}$, $\mu_P^M$ does not exist. Recall that

$$
w^* = \lambda \Omega^{-1}(\mu - r \mathbf{1}) \text{ so that } \mathbf{1}^T w = \lambda (\mathbf{1} \Omega^{-1} \mu - r \mathbf{1} \Omega^{-1} \mathbf{1}) = \lambda (b - ra).
$$

When $r = b/a$, $\mathbf{1}^T w = 0$ as $\lambda$ is finite. Any minimum variance portfolio involves investing everything in the riskfree asset and holding a portfolio of risky assets whose weights sum to zero.
When $r > \frac{b}{a}$, only the lower half line touches the feasible region with risky assets only.

Any portfolio on the upper half line involves short selling of the tangency portfolio and investing the proceeds in the riskfree asset.
Alternative approach

Given a point \((\sigma_P, \mu_P)\) in the feasible region, we draw a line joining this point and the risk free asset. Let \(\theta\) be the angle of inclination of this line, where \(\tan \theta = \frac{\mu_P - r_f}{\sigma_P}\). The tangency portfolio is the feasible point that maximizes \(\theta\) or \(\tan \theta\).

Write \(R_P = \sum_{i=1}^{n} w_i R_i\), where \(w_i\) is the weight associated with the risky asset \(i\). Since \(\mu_P = \sum_{i=1}^{n} w_i \bar{R}_i\) and \(r_f = \sum_{i=1}^{n} w_i r_f\), we have

\[
\tan \theta = \frac{\sum_{i=1}^{n} w_i (\bar{R}_i - r_f)}{\left( \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} w_i w_j \right)^{1/2}}.
\]
Set the derivative of $\tan \theta$ with respect to each $w_k$ equal to zero:–

$$\frac{\partial}{\partial w_i} \tan \theta = \left( \bar{R}_i - r_f \right) \left( \sum_{i,j=1}^{n} \sigma_{ij} w_i w_j \right)^{1/2} - \left[ \sum_{i=1}^{n} w_i (\bar{R}_i - r_f) \right] \left( \sum_{i,j=1}^{n} \sigma_{ij} w_i w_j \right)^{-1/2} \sum_{j=1}^{n} \sigma_{ij} w_j$$

$$= 0.$$  

This leads to the following system of equations:–

$$\sum_{j=1}^{n} \sigma_{ij} w_j = \lambda (\bar{R}_i - r_f), \quad \lambda \text{ is some constant, } i = 1, 2, \cdots, n.$$

**Hint** Use the following relation

$$\frac{\partial}{\partial w_i} \left( \sum_{i,j} w_i \sigma_{ij} w_j \right)^{1/2} = \left( \sum_{i,j} w_i \sigma_{ij} w_j \right)^{-1/2} \sum_{j=1}^{n} \sigma_{ij} w_j.$$
We write $\lambda v_j = w_j$ for each $j$, the above system becomes

$$\sum_{j=1}^{n} \sigma_{ij} v_j = \bar{r}_i - r_f, \quad i = 1, 2, \cdots, n.$$ 

We then solve for $v_j$ by a linear system solver.

Finally, we normalize $w_j$'s by $w_j = \frac{v_j}{\sum_{j=1}^{n} v_k}, \quad j = 1, \cdots, n.$
Example (5 risky assets and one risk free asset)

Data of the 5 risky assets are given in the earlier example, and $r_f = 10\%$.

The system of linear equations to be solved is

$$\sum_{j=1}^{5} \sigma_{ij} v_j = \bar{R}_i - r_f = 1 \times \bar{R}_i - r_f \times 1, \quad i = 1, 2, \ldots, 5.$$ 

Recall that $v^1$ and $v^2$ in the earlier example are solutions to

$$\sum_{j=1}^{5} \sigma_{ij} v_j^1 = 1 \quad \text{and} \quad \sum_{j=1}^{5} \sigma_{ij} v_j^2 = \bar{R}_i, \quad \text{respectively.}$$

Hence, $v_j = v_j^2 - r_f v_j^1, j = 1, 2, \ldots, 5$ (numerically, we take $r_f = 10\%$).
Addition of risk tolerance factor

Maximize $2\tau \mu_P - \sigma_P^2$, with $\tau \geq 0$, where $\tau$ is the risk tolerance.

Optimization problem: $\max_{w \in \mathbb{R}^N} 2\tau \mu_P - \sigma_P^2$ subject to $1 \cdot w = 1$.

Remark

$\tau$ is closely related to the relative risk aversion coefficient. Given an initial wealth $W_0$ and under a portfolio choice $w$, the end-of-period wealth is $W_0(1 + R_P)$. Let $\mu_P = E[R_P]$ and $\sigma_P^2 = \text{var}(R_P)$.

Consider the Taylor expansion

$$u[W_0(1 + R_P)] \approx u(W_0) + W_0 u'(W_0) R_P + \frac{W_0^2}{2} u''(W_0) R_P^2 + \cdots.$$ 

Neglecting third and higher order moments and noting $E[R_P^2] = \sigma_P^2 + \mu_P^2$. 
\[ E[u(W_0(1 + \mu_P))] \approx u(W_0) + W_0u'(W_0)\mu_P + \frac{W_0^2}{2}u''(W_0)(\sigma_P^2 + \mu_P^2) + \cdots \]
\[ = u(W_0) - \frac{W_0^2}{2}u''(W_0) \left[ -\frac{2u'(W_0)}{W_0u''(W_0)}\mu_P - (\sigma_P^2 + \mu_P^2) \right] \]
\[ + \cdots \]

Neglecting \( \mu_P^2 \) compared to \( \sigma_P^2 \) and letting \( R_R = -\frac{W_0u''(W_0)}{u'(W_0)} \), we have the objective function: \[ \frac{2}{R_R}\mu_P - \sigma_P^2. \]

Note that the expected utility can be expressed solely in terms of mean \( \mu_P \) and variance \( \sigma_P^2 \) when

(i) \( u \) is a quadratic function, or

(ii) \( R_P \) is normal.
Quadratic optimization problem

\[ \max_{w \in \mathbb{R}^N} [2\tau \mu^T w - w^T \Omega w] \text{ subject to } 1 \cdot w = 1. \]

The Lagrangian formulation becomes:

\[ L(w; \lambda) = 2\tau \mu^T w - w^T \Omega w + \lambda (w^T 1 - 1). \]

The first order conditions are

\[
\begin{cases}
2\tau \mu - 2\Omega w^* + \lambda 1 = 0 \\
1 \cdot w^* = 1 
\end{cases}.
\]
Express the optimal solution $w^*$ as $w_g + \tau z^*, \tau \geq 0$.

1. When $\tau = 0$, $2\Omega w = \lambda_0 1$ and $1^T w_g = 1$

$$w_g = \frac{\lambda_0}{2} \Omega^{-1} 1 \quad \text{and} \quad 1 = 1^T w_g = \frac{\lambda_0}{2} 1^T \Omega^{-1} 1$$

hence

$$w_g = \frac{\Omega^{-1} 1}{1^T \Omega^{-1} 1} \quad \text{(independent of } \mu)$$
2. When $\tau \geq 0$, $w = \tau \Omega^{-1} \mu + \frac{\lambda}{2} \Omega^{-1} \mathbf{1}$.

$$1 = \mathbf{1}^T w = \tau \mathbf{1}^T \Omega^{-1} \mu + \frac{\lambda}{2} \mathbf{1}^T \Omega^{-1} \mathbf{1}$$

so that

$$\frac{\lambda}{2} = \frac{1 - \tau \mathbf{1}^T \Omega^{-1} \mu}{\mathbf{1}^T \Omega^{-1} \mathbf{1}}.$$

$$w = \tau \Omega^{-1} \mu + \frac{1 - \tau \mathbf{1}^T \Omega^{-1} \mu}{\mathbf{1}^T \Omega^{-1} \mathbf{1}} \Omega^{-1} \mathbf{1} = \tau \left( \Omega^{-1} \mu - \frac{\mathbf{1}^T \Omega^{-1} \mu}{\mathbf{1}^T \Omega^{-1} \mathbf{1}} \Omega^{-1} \mathbf{1} \right) + w_g.$$  

We obtain

$$z^* = \Omega^{-1} \mu - \frac{\mathbf{1}^T \Omega^{-1} \mu}{\mathbf{1}^T \Omega^{-1} \mathbf{1}} \Omega^{-1} \mathbf{1} \quad \text{and} \quad \mathbf{1}^T z^* = 0.$$  

Observe that $\text{cov}(Rw_g, Rz^*) = z^*^T \Omega w_g = 0.$
Financial interpretation

\( w_g \) leads to a minimum risk position. This position is modified by investing in the self-financing portfolio \( z^* \) so as to maximize
\[
2\tau \mu^T w - w^T \Omega^{-1} w.
\]

Efficient frontier

Consider

\[
\begin{align*}
\mu_P &= \mu^T (w_g + \tau z^*) = \mu_g + \tau \mu_{P,z^*} \\
\sigma_P^2 &= \sigma_g^2 + 2\tau \text{cov}(R_{w_g}, R_{z^*}) + \tau^2 \sigma_{z^*}^2.
\end{align*}
\]

By eliminating \( \tau, \sigma_P^2 = \sigma_g^2 + \left( \frac{\mu_P - \mu_g}{\mu_{P,z^*}} \right)^2 \sigma_{z^*}^2. \) Hence, the frontier is parabolic in the \((\mu_P, \sigma_P^2)\)-diagram and hyperbolic in the \((\mu_P, \sigma_P)\)-diagram.
Inclusion of riskfree asset (deterministic rate of return \( R_0 = r \))

Let \( w = (w_0 \cdots w_N)^T \) and \( \sum_{i=0}^{N} w_i = 1 \).

Lagrangian formulation becomes

\[
L = 2\tau \hat{\mu}^T \hat{w} - \hat{w}^T \Omega \hat{w} + \lambda (\hat{w}^T \mathbf{1} - 1) + 2\tau w_0 r + \lambda w_0
\]

where \( \hat{w} = (w_1 \cdots w_n)^T \in \mathbb{R}^N, \hat{\mu} = (\mu_1 \cdots \mu_N)^T \in \mathbb{R}^N, \mathbf{1}^T = (1 \cdots 1)^T \in \mathbb{R}^N \).

The optimality conditions become

\[
2\tau r + \lambda = 0 \quad (i)
\]
\[
2\tau \hat{\mu} - 2\Omega \hat{w}^* + \lambda \mathbf{1} = 0 \quad (ii)
\]
\[
\sum_{i=0}^{N} w_i^* = 1 \quad (iii)
\]
Estimation of risk tolerance (inverse problem)

Reverse optimization: given an efficient portfolio \( w^* \), it is possible to express \( \tau \) in terms of \( \text{var}(R^*_P) \) and \( \mu^*_P \). Taking the inner product of \( \hat{w}^* \) with (ii), we obtain

\[
2\tau(rw^*_0 + \hat{\mu}^T\hat{w}^*) - 2\hat{w}^T\hat{\Omega}\hat{w} + \lambda = 0.
\]

By eliminating \( \lambda \) using (i), we obtain the implied risk tolerance as follows:

\[
\tau = \frac{\text{var}(R^*_P)}{\mu^*_P - r}.
\]
Marginal utilities

With $w_0 = 1 - \hat{1} \cdot \hat{w}$, we obtain the objective function

$$F(\hat{w}) = 2\tau [r + (\hat{\mu} - r\hat{1})^T \hat{w}] - \hat{w}^T \Omega \hat{w}$$

$$\nabla F = 2\tau (\hat{\mu} - r\hat{1}) - 2\Omega \hat{w}$$

so that

$$(\nabla F)_i = 2\tau [\mu_i - r] - 2\text{cov}(R_P, R_i), \quad i = 1, 2, \ldots, N.$$ 

An increase of amount $dw_i$ in the weight of asset $i$ and a corresponding reduction of the riskless asset leads to a marginal change $(\nabla F)_i \, dw_i$ of the objective function. By increasing (decreasing) the positions with high (low) marginal utilities, an efficient portfolio $\hat{w}$ can be considerably improved.
Summary

1. The objective function \(2\tau \mu^T w - w^T \Omega w\) represents a balance of maximizing return \(2\tau \mu^T w\) against risk \(w^T \Omega w\), where the weighing factor \(\tau\) is related to the reciprocal of relative risk aversion coefficient \(R_R\).

2. The optimal solution takes the form

\[ w^* = w_g + \tau z^* \]

where \(w_g\) is the portfolio weight of the global minimum variance portfolio and the weights in \(z^*\) are summed to zero.
3. The additional variance above $\sigma_g^2$ is given by

$$\tau^2 \sigma_{z^*}^2.$$  

This is obvious since $\text{cov}(Rw_g, Rz^*) = 0$, that is, $Rw_g$ and $Rz^*$ are uncorrelated.

4. With the inclusion of riskfree asset, the marginal utility $(\nabla F)_i$ of the $i^{th}$ asset can be increased by

(i) higher value of $\tau$, $\tau \geq 0$,
(ii) higher positive value of $\mu_i - r$
(iii) higher negative correlation between portfolio’s rate of return $R_P$ and asset’s rate of return $R_i$. 
Asset-Liability Model

Liabilities of a pension fund = future benefits − future contributions

Market value can hardly be determined since liabilities are not readily marketable. Assume that some specific accounting rules are used to calculate an initial value $L_0$. If the same rule is applied one period later, a value $L_1$ results.

Growth rate of the liabilities $= R_L = \frac{L_1 - L_0}{L_0}$, where $R_L$ is expected to depend on the changes of interest rate structure, mortality and other stochastic factors. Let $A_0$ be the initial value of assets. The investment strategy of the pension fund is given by the portfolio choice $w$. 


**Surplus optimization**

Depending on the portfolio choice \( w \), the surplus gain after one period

\[
S_1 - S_0 = A_0 R_w - L_0 R_L, \quad \text{where } S_0 = A_0 - L_0.
\]

The return on surplus is defined by

\[
R_S = \frac{S_1 - S_0}{A_0} = R_w - \frac{1}{f_0} R_L
\]

where \( f_0 = A_0/L_0 \) is the initial funding ratio.

Maximization formulation:-

\[
\max_{w \in \mathbb{R}^N} \left\{ 2\tau E \left[ R_w - \frac{1}{f_0} R_L \right] - \text{var} \left( R_w - \frac{1}{f_0} R_L \right) \right\}
\]

subject to \( \sum_{i=1}^{N} w_i = 1 \). Note that \( \frac{1}{f_0} R_L \) and \( \text{var}(R_L) \) are independent of \( w \) so that they do not enter into the objective function.
\[
\max_{\mathbf{w} \in \mathbb{R}^N} \left\{ 2\tau E[R\mathbf{w}] - \text{var}(R\mathbf{w}) + \frac{2}{f_0}\text{cov}(R\mathbf{w}, R_L) \right\}
\]

subject to \( \sum_{i=1}^{N} w_i = 1 \). Recall that

\[
\text{cov}(R\mathbf{w}, R_L) = \text{cov} \left( \sum_{i=1}^{N} w_i R_i, R_L \right) = \sum_{i=1}^{N} w_i \text{cov}(R_i, R_L).
\]

\[
\max_{\mathbf{w} \in \mathbb{R}^N} \{ 2\tau \mu^T \mathbf{w} + 2\gamma^T \mathbf{w} - \mathbf{w}^T \Omega \mathbf{w} \} \quad \text{subject to} \quad \mathbf{1}^T \mathbf{w} = 1,
\]

where \( \gamma^T = (\gamma_1 \cdots \gamma_N) \) with \( \gamma_i = \frac{1}{f_0} \text{cov}(R_i, R_L) \),

\( \mu^T = (\mu_1 \cdots \mu_N) \) with \( \mu_i = E[R_i], \sigma_{ij} = \text{cov}(R_i, R_j) \).
Remarks

1. The additional term $2\gamma^T w$ in the objective function arises from the correlation $\text{cov}(R_i, R_L)$ between return of risky asset $i$ and return of liability multiplied by the factor $L_0/A_0$.

2. Compared to the earlier model, we just need to replace $\mu^T$ by $\mu^T + \frac{1}{\tau} \gamma^T$. The efficient portfolios are of the form

$$w^* = w_g + z^L + \tau z^*, \quad \tau \geq 0,$$

where $z^L = \Omega^{-1} \gamma - \frac{1^T \Omega^{-1} \gamma}{1^T \Omega^{-1} 1} \Omega^{-1} 1$ with $\sum_{i=1}^N z_i^L = 0$.

The occurrence of liabilities leads only to parallel shifts of the set of efficient portfolios.
The mean-variance criterion can be reconciled with the expected utility approach in either of two ways: (1) using a quadratic utility function, or (2) making the assumption that the random returns are normal variables.

**Quadratic utility**

The quadratic utility function can be defined as \( U(x) = ax - \frac{b}{2}x^2 \), where \( a > 0 \) and \( b > 0 \). This utility function is really meaningful only in the range \( x \leq a/b \), for it is in this range that the function is increasing. Note also that for \( b > 0 \) the function is strictly concave everywhere and thus exhibits risk aversion.
mean-variance analysis ⇔ maximum expected utility criterion
based on quadratic utility

Suppose that a portfolio has a random wealth value of \( y \). Using the expected utility criterion, we evaluate the portfolio using

\[
E[U(y)] = E\left[ ay - \frac{b}{2}y^2 \right]
\]

\[
= aE[y] - \frac{b}{2}E[y^2]
\]

\[
= aE[y] - \frac{b}{2}(E[y])^2 - \frac{b}{2}\text{var}(y).
\]

The optimal portfolio is the one that maximizes this value with respect to all feasible choices of the random wealth variable \( y \).
Normal Returns

When all returns are normal random variables, the mean-variance criterion is also equivalent to the expected utility approach for any risk-averse utility function.

To deduce this, select a utility function $U$. Consider a random wealth variable $y$ that is a normal random variable with mean value $M$ and standard deviation $\sigma$. Since the probability distribution is completely defined by $M$ and $\sigma$, it follows that the expected utility is a function of $M$ and $\sigma$. If $U$ is risk averse, then

$$E[U(y)] = f(M, \sigma), \quad \text{with} \quad \frac{\partial f}{\partial M} > 0 \quad \text{and} \quad \frac{\partial f}{\partial \sigma} < 0.$$
• Now suppose that the returns of all assets are normal random variables. Then any linear combination of these asset is a normal random variable. Hence any portfolio problem is therefore equivalent to the selection of combination of assets that maximizes the function $f(M, \sigma)$ with respect to all feasible combinations.

• For a risky-averse utility, this again implies that the variance should be minimized for any given value of the mean. In other words, the solution must be mean-variance efficient.

• Portfolio problem is to find $w^*$ such that $f(M, \sigma)$ is maximized with respect to all feasible combinations.
Two fund monetary separation

Consider a financial market with the riskfree asset and several risky assets, suppose the utility function satisfies

$$\frac{-u'(z)}{u''(z)} = a + bz, \quad \text{valid for all } z,$$

then the optimal portfolio at different wealth levels is given by the combination of the riskfree asset and market fund consisting of the risky assets. The relative proportions of risky assets in the market fund remain the same, irrespective of $W_0$.

Remark

The class of utility functions include

(i) quadratic utility
(ii) log utility: $a = 0$
(iii) exponential utility: $b = 0$
(iv) power utility: $a = 0$. 
Let $W_0$ be the initial wealth, then the wealth amount $a_j^*(W_0)$ of risky asset $j$ in the optimal portfolio satisfies

$$a_j^*(W_0) = \alpha_j h(W_0) \quad j = 1, 2, \ldots, n,$$

and $\alpha_j$ is independent of $W_0$, so that the relative proportion $b_j$ is given by

$$b_j = \frac{a_j^*(W_0)}{\sum_{k=1}^{n} a_k^*(W_0)} = \frac{\alpha_j}{\sum_{k=1}^{n} \alpha_k},$$

independent of $W_0$. 
Lemma

1. Suppose the utility function satisfies

\[- \frac{u'(W_1)}{u''(W_1)} = a + bW_1, \quad \text{for all } W_1,\]

then the optimal portfolio is given by

\[a_j^*(W_0) = \alpha_j(a + bRW_0), \quad j = 1, 2, \cdots, n, \quad (A)\]

where \( R = 1 + r_f \) and \( \tilde{R}_j = 1 + \tilde{r}_j, j = 1, 2, \cdots, n. \)

2. Define \( V(W_0) = \max_{\{a_j\}_{j=1}^n} E[u(\tilde{W}_1)], \)

where \( \tilde{W}_1 = \left( W_0 - \sum_{j=1}^m a_j \right) R + \sum_{j=1}^n a_j \tilde{R}_j = RW_0 + \sum_{j=1}^n a_j(\tilde{R}_j-R), \)

then

\[- \frac{V'(W_0)}{V''(W_0)} = \frac{a}{R} + bW_0, \quad \text{for all initial wealth } W_0. \quad (B)\]
Proof

Assume that

\[ a_j^*(W_0) = \alpha_j(W_0)(a + bRW_0) \]

where \( \alpha_j(W_0), \ j = 1, \cdots, n \), is a differentiable function.

For any value of \( W_0 \), from the optimality property of \( a_j^*(W_0) \), we deduce that

\[
\frac{\partial E[u(\tilde{W}_1)\big|a]}{\partial a_k} = E \left[ u' \left( RW_0 + \sum_{j=1}^{N} (\tilde{R}_j - R)\alpha_j(W_0)(a + bRW_0) \right) (\tilde{R}_k - R) \right] = 0,
\]

\[ k = 1, 2, \cdots, N. \]
Next, we differentiate eq (1) with respect to $W_0$. First, we observe that

$$\frac{d\tilde{W}_1}{dW_0} = R \left[ 1 + \sum_{j=1}^{N} (\tilde{R}_j - R)\alpha_j(W_0)b \right]$$

$$+ \sum_{j=1}^{N} \frac{d\alpha_j(W_0)}{dW_0} (\tilde{R}_j - R)(a + bRW_0).$$

Hence, for the $k^{th}$ component, we obtain

$$\sum_{j=1}^{N} E[u''(\tilde{W}_1)(\tilde{R}_j - R)(\tilde{R}_k - R)(a + bRW_0)] \frac{d\alpha_j(W_0)}{dW_0}$$

$$= -E \left[ u''(\tilde{W}_1)(R_k - R)R \left[ 1 + \sum_{j=1}^{N} (\tilde{R}_j - R)\alpha_j(W_0)b \right] \right],$$

$k = 1, 2, \cdots, N.$
In matrix form, we have

\[
E \left[ \begin{pmatrix}
(\tilde{R}_1 - R)^2 & \cdots & (\tilde{R}_1 - R)(\tilde{R}_N - R) \\
(\tilde{R}_2 - R)(\tilde{R}_1 - R) & \cdots & (\tilde{R}_2 - R)(\tilde{R}_N - R) \\
\vdots & \ddots & \vdots \\
(\tilde{R}_N - R)(\tilde{R}_1 - R) & \cdots & (\tilde{R}_N - R)^2 \\
\end{pmatrix}
\right]
\begin{pmatrix}
\frac{d\alpha_1(W_0)}{dW_0} \\
\frac{d\alpha_2(W_0)}{dW_0} \\
\vdots \\
\frac{d\alpha_N(W_0)}{dW_0}
\end{pmatrix}

= - E \left[ \begin{pmatrix}
\{ u''(\tilde{W}_1)(\tilde{R}_1 - R)R \left[ 1 + \sum_{j=1}^{N}(\tilde{R}_j - R)\alpha_j(W_0)b \right] \} \\
\{ u''(\tilde{W}_1)(\tilde{R}_2 - R)R \left[ 1 + \sum_{j=1}^{N}(\tilde{R}_j - R)\alpha_j(W_0)b \right] \} \\
\vdots \\
\{ u''(\tilde{W}_1)(\tilde{R}_N - R)R \left[ 1 + \sum_{j=1}^{N}(\tilde{R}_j - R)\alpha_j(W_0)b \right] \}
\end{pmatrix}
\right]

(2)
From the assumption

\[
- \frac{u'(W_1)}{u''(W_1)} = a + bW_1,
\]

we obtain

\[
u''(\tilde{W}_1) = -\frac{u'(\tilde{W}_1)}{a + b \left[ RW_0 + \sum_{j=1}^{N} (\tilde{R}_j - R) \alpha_j(W_0)(a + bRW_0) \right]}
= -\frac{u'(\tilde{W}_1)}{(a + bRW_0) \left[ 1 + \sum_{j=1}^{N} (\tilde{R}_j - R) \alpha_j(W_0)b \right]}.
\]

(3)

observe that

\[
-u''(\tilde{W}_1)(\tilde{R}_k - R)R \left[ 1 + \sum_{j=1}^{N} (\tilde{R}_j - R) \alpha_j(W_0)b \right]
= u'(\tilde{W}_1)(\tilde{R}_k - R)\frac{R}{a + bRW_0}, \quad k = 1, 2, \cdots N.
\]

(4)

Recall the first order condition:

\[
E[u'(\tilde{W}_1)(\tilde{R}_k - R)] = 0 \quad k = 1, 2, \cdots, N.
\]
Combining eqs (1) and (4), and knowing that the column vector on the right hand side of eq (2) is a zero vector, we deduce that

$$\frac{d\alpha_j}{dW_0}(W_0) = 0, \quad j = 1, 2, \cdots, n,$$

provided that the matrix in eq (2) is non-singular. We then have

$$\alpha_j(W_0) = \alpha_j, \quad \text{independent of } W_0.$$

Now, $$a_j^* = \alpha_j(a + bW_0), \quad a > 0.$$ When $$b = 0, a_j^*$$ is independent of the initial wealth $$W_0$$.

The portfolio $$(a_1^*(W_0) \cdots a_N^*(W_0))$$ is said to be partially separated if $$a_j^*(W_0)/a_j^*(W_0)$$ is independent of $$W_0$$, and it is said to be completely separated if $$a_j^*$$ is independent of $$W_0$$. 
To show eq (B), we start from the optimality condition on

\[ a_j^*(W_0) = \alpha_j(a + bRW_0) \]

to obtain

\[
V(W_0) = E \left[ u \left( RW_0 + \sum_{j=1}^{N} (\tilde{R}_j - R)\alpha_j(a + bRW_0) \right) \right]
\]

\[
= E \left[ u \left( \left( 1 + \sum_{j=1}^{N} (\tilde{R}_j - R)\alpha_j b \right) RW_0 + \sum_{j=1}^{N} (\tilde{R}_j - R)\alpha_j a \right) \right].
\]
Differentiate $V(W_0)$ twice with respect to $W_0$

$$V'(W_0) = E \left[ u'(\tilde{W}_1)R \left( 1 + \sum_{j=1}^{n} (\tilde{R}_j - R)\alpha_j b \right) \right]$$

$$V''(W_0) = E \left[ u''(\tilde{W}_1)R^2 \left( 1 + \sum_{j=1}^{n} (\tilde{R}_j - R)\alpha_j b \right)^2 \right].$$

Relating $u''(\tilde{W}_1)$ with $u'(\tilde{W}_1)$ using eq (3), we obtain

$$V''(W_0) = -\frac{RE \left[ u'(\tilde{W}_1)R \left( 1 + \sum_{j=1}^{n} (\tilde{R}_j - R)\alpha_j b \right) \right]}{a + bRW_0} = -\frac{R}{a + bRW_0}V'(W_0).$$

Combining the results

$$-\frac{V'(W_0)}{V''(W_0)} = \frac{a}{R} + bW_0.$$
Formulation for finding the optimal portfolio

Let $\alpha$ be the weight of the riskfree asset so that the wealth invested in risky assets is $W_0(1 - \alpha)$. Let $b_j$ be the weight of risky asset $j$ within $W_0(1 - \alpha)$ so that $\sum_{j=1}^{n} b_j = 1$. The random wealth $\tilde{W}$ at the end of the investment period is

$$\max_{\{\alpha, b_j\}} E[u(\tilde{W})]$$

where

$$\tilde{W} = W_0\alpha(1 + r_f) + \sum_{j=1}^{n} W_0(1 - \alpha) b_j(1 + \tilde{r}_j)$$

$$= W_0 \left[ 1 + \alpha r_f + (1 - \alpha) \sum_{j=1}^{n} b_j \tilde{r}_j \right]$$

subject to

$$\sum_{j=1}^{n} b_j = 1.$$
Lagrangian formulation

$$\max_{\{\alpha, b_j, \lambda\}} E[u(\tilde{W})] + \lambda \left( 1 - \sum_{j=1}^{n} b_j \right).$$

First order conditions give

$$E \left[ u'(\tilde{W}) W_0 \left( r_f - \sum_{j=1}^{n} b_j \tilde{r}_j \right) \right] = 0 \quad (1)$$

$$E \left[ u'(\tilde{W}) W_0 (1 - \alpha) \tilde{r}_j \right] = \lambda, \quad j = 1, 2, \ldots, n, \quad (2)$$

$$\sum_{j=1}^{n} b_j = 1. \quad (3)$$

From eq. (1),

$$E[u'(\tilde{W}) r_f] = E \left[ u'(\tilde{W}) \sum_{j=1}^{n} b_j \tilde{r}_j \right],$$

and from eqs (2) and (3), we have

$$\lambda = E \left[ u'(\tilde{W}) W_0 (1 - \alpha) \sum_{j=1}^{n} b_j \tilde{r}_j \right].$$
Substituting into eq (2)

\[ E[u'(\tilde{W})\tilde{r}_j] = E \left[ u'(\tilde{W}) \sum_{j=1}^{n} b_j \tilde{r}_j \right], \quad j = 1, 2, \ldots, n, \]

and using eq (1), we obtain

\[ E[u'(\tilde{W})(\tilde{r}_j - r_f)] = 0 \]

or equivalently,

\[ E \left[ u' \left( W_0 \left[ 1 + r_f + (1 - \alpha) \sum_{\ell=1}^{n} b_\ell (\tilde{r}_\ell - r_f) \right] \right) (\tilde{r}_j - r_f) \right] = 0, \quad j = 1, 2, \ldots, n. \]
Exponential utility

Consider \( u'(z) = Ae^{-az}, \quad a > 0, \) substituting into eq. (4)

\[
E \left[ A \exp \left( -a \left\{ W_0 \left[ (1 + r_f) + (1 - \alpha) \sum_{\ell=1}^{n} b_\ell (\tilde{r}_\ell - r_f) \right] \right\} \right) (\tilde{r}_j - r_f) \right] = 0
\]

and since \( A \exp(-aW_0(1 + r_f)) \) is non-random, we have

\[
E \left[ e^{-a \sum_{\ell=1}^{n} W_0 (1 - \alpha) b_\ell (\tilde{r}_\ell - r_f)} (\tilde{r}_j - r_f) \right] = 0, \quad j = 1, 2, \ldots, n. \quad (5a)
\]

For another initial wealth \( W'_0 \), we have similar result

\[
E \left[ e^{-a \sum_{\ell=1}^{n} W'_0 (1 - \alpha') b'_\ell (\tilde{r}_\ell - r_f)} (\tilde{r}_j - r_f) \right] = 0, \quad j = 1, 2, \ldots, n. \quad (5b)
\]
Suppose we **postulate** that the solution to the system of equations
\[
E \left[ e^{-a \sum_{\ell=1}^{n} \beta_{\ell}(\tilde{r}_\ell - r_f)(\tilde{r}_j - r_f)} \right] = 0, \quad j = 1, 2, \cdots, n,
\]
is unique, then by comparing eqs (5a,b), we obtain
\[
W_0(1 - \alpha)b_\ell = W'_0(1 - \alpha')b'_\ell.
\]
Summing \(\ell\) from 1 to \(n\), we obtain
\[
W_0(1 - \alpha) = W'_0(1 - \alpha'),
\]
hence
\[
b_\ell = b'_\ell, \quad \ell = 1, 2, \cdots, n.
\]
The total wealth amount \(W_0(1 - \alpha)\) invested in risky assets and the wealth amount in each asset are independent of \(W_0\).