# Quantitative Modeling of Derivative Securities 

Homework Four

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1. Suppose the dividends and interest incomes are taxed at the rate $R$ but capital gains taxes are zero. Find the price formulas for the European put and call on an asset which pays a continuous dividend yield at the constant rate $q$, assuming that the riskless interest rate $r$ is also constant.

Hint: Explain why the riskless interest rate $r$ and dividend yield $q$ should be replaced by $r(1-R)$ and $q(1-R)$, respectively, in the Black-Scholes formulas.
2. Consider a futures on an underlying asset which pays $N$ discrete dividends between $t$ and $T$ and let $D_{i}$ denote the amount of the $i$ th dividend paid on the ex-dividend date $t_{i}$, $i=1,2, \ldots, N$. Show that the futures price is given by

$$
F(S, t)=S e^{r(T-t)}-\sum_{i=1}^{N} D_{i} e^{r\left(T-t_{i}\right)}
$$

where $S$ is the current asset price and $r$ is the riskless interest rate. Consider a European call option on the above futures. Show that the governing differential equation for the price of the call, $c_{F}(F, t)$, is given by

$$
\frac{\partial c_{F}}{\partial t}+\frac{\sigma^{2}}{2}\left[F+\sum_{i=1}^{N} D_{i} e^{r\left(T-t_{i}\right)}\right]^{2} \frac{\partial^{2} c_{F}}{\partial F^{2}}-r c_{F}=0
$$

3. A forward start option is an option which comes into existence at some future time $T_{1}$ and expires at $T_{2}\left(T_{2}>T_{1}\right)$. The strike price is set equal the asset price at $T_{1}$ such that the option is at-the-money at the future option's initiation time $T_{1}$. Consider a forward start call option whose underlying asset has value $S$ at current time $t$ and constant dividend yield $q$, show that the value of the forward start call is given by

$$
e^{-q\left(T_{1}-t\right)} c\left(S, T_{2}-T_{1} ; S\right)
$$

where $c\left(S, T_{2}-T_{1} ; S\right)$ is the value of an at-the-money call (strike price same as asset price) with time to expiry $T_{2}-T_{1}$.

Hint: The value of an at-the-money call option is proportional to the asset price.
4. Show that the theta may become positive for an in-the-money European call option on a continuous dividend paying asset when the dividend yield is sufficiently high.
5. Let $Q^{*}$ denote the equivalent martingale measure where the asset price $S_{t}$ is used as the numeraire. Suppose $S_{t}$ follows the lognormal distribution with drift rate $r$ and volatility $\sigma$ under $Q^{*}$, where $r$ is the riskless interest rate. Show that

$$
\frac{d Q^{*}}{d Q}=\frac{S_{T}}{S_{0}} e^{-r T}=e^{-\frac{\sigma^{2}}{2}+\sigma Z_{T}}
$$

where $Q$ is the martingale measure with the money market account as the numeraire and $Z_{T}$ is a Brownian motion under $Q$. Using the Girsanov Theorem, show that

$$
Z_{T}^{*}=Z_{T}-\sigma T
$$

is a Brownian motion under $Q^{*}$. Explain why

$$
E_{Q^{*}}\left[\mathbf{1}_{\left\{S_{T} \geq X\right\}}\right]=N\left(\frac{\ln \frac{S_{0}}{X}+\left(r+\frac{\sigma^{2}}{2}\right) T}{\sigma \sqrt{T}}\right)
$$

then deduce that

$$
E_{Q}\left[S_{T} \mathbf{1}_{\left\{S_{T} \geq X\right\}}\right]=e^{r T} S_{0} N\left(\frac{\ln \frac{S_{0}}{X}+\left(r+\frac{\sigma^{2}}{2}\right) T}{\sigma \sqrt{T}}\right) .
$$

6. Consider a contingent claim whose value at maturity $T$ is given by

$$
\min \left(S_{T_{0}}, S_{T}\right)
$$

where $T_{0}$ is some intermediate time before maturity, $T_{0}<T$, and $S_{T}$ and $S_{T_{0}}$ are the asset price at $T$ and $T_{0}$, respectively. Show that the value of the contingent claim at time $t$ is given by

$$
V_{t}=S_{t}\left[1-N\left(d_{1}\right)+e^{-r\left(T-T_{0}\right)} N\left(d_{2}\right)\right],
$$

where $S_{t}$ is the asset price at time $t$ and

$$
d_{1}=\frac{r-q+\frac{\sigma^{2}}{2}}{\sigma} \sqrt{T-T_{0}}, \quad d_{2}=d_{1}-\sigma \sqrt{T-T_{0}} .
$$

7. Suppose the dynamics of $X_{t}$ and $S_{t}$ under the risk neutral measure $Q$ are governed by

$$
\frac{d X_{t}}{X_{t}}=r d t+\sigma_{X} d Z_{X, t}^{Q} \quad \text { and } \quad \frac{d S_{t}}{S_{t}}=r d t+\sigma_{S} d Z_{S, t}^{Q}
$$

where $Z_{X, t}^{Q}$ and $Z_{S, t}^{Q}$ are $Q$-Brownian. Note that both $X_{t} / M_{t}$ and $S_{t} / M_{t}$ are martingales under $Q$, where $M_{t}=e^{r t}$. Show that $X_{t} / S_{t}$ is a martingale under $Q^{*}$, where $Q^{*}$ is the measure defined by

$$
L_{t}=\left.\frac{d Q^{*}}{d Q}\right|_{\mathcal{F}_{t}}=\frac{S_{t}}{S_{0}} / \frac{M_{t}}{M_{0}}, \quad t \in(0, T] .
$$

Hint: $Z_{X, t}^{Q}-\rho \sigma_{S} t$ and $Z_{S, t}^{Q}-\sigma_{S} t$ are $Q^{*}$-Brownian.
8. Consider the exchange option which entities the holder the right but not the obligation to exchange risky asset $S_{2}$ for another risky asset $S_{1}$. Let the price dynamics of $S_{1}$ and $S_{2}$ under the risk neutral measure be governed by

$$
\frac{d S_{i}}{S_{i}}=\left(r-q_{i}\right) d t+\sigma_{i} d Z_{i}, \quad i=1,2
$$

where $d Z_{1} d Z_{2}=\rho d t$. Let $V\left(S_{1}, S_{2}, \tau\right)$ denote the price function of the exchange option, whose terminal payoff takes the form

$$
V\left(S_{1}, S_{2}, 0\right)=\max \left(S_{1}-S_{2}, 0\right)
$$

Show that the governing equation for $V\left(S_{1}, S_{2}, \tau\right)$ is given by

$$
\begin{aligned}
\frac{\partial V}{\partial \tau}= & \frac{\sigma_{1}^{2}}{2} S_{1}^{2} \frac{\partial^{2} V}{\partial S_{1}^{2}}+\rho \sigma_{1} \sigma_{2} S_{1} S_{2} \frac{\partial^{2} V}{\partial S_{1} \partial S_{2}}+\frac{\sigma_{2}^{2}}{2} S_{2}^{2} \frac{\partial^{2} V}{\partial S_{2}^{2}} \\
& +\left(r-q_{1}\right) S_{1} \frac{\partial V}{\partial S_{1}}+\left(r-q_{2}\right) S_{2} \frac{\partial V}{\partial S_{2}}-r V
\end{aligned}
$$

By taking $S_{2}$ as the numeraire and defining the similarity variables

$$
x=\frac{S_{1}}{S_{2}} \quad \text { and } \quad W(x, \tau)=\frac{V\left(S_{1}, S_{2}, \tau\right)}{S_{2}}
$$

show that the governing equation for $W(x, \tau)$ becomes

$$
\frac{\partial W}{\partial \tau}=\frac{\sigma^{2}}{2} x^{2} \frac{\partial^{2} W}{\partial x^{2}}+\left(q_{2}-q_{1}\right) x \frac{\partial W}{\partial x}-q_{2} W,
$$

where $\sigma^{2}=\sigma_{1}^{2}-2 \rho \sigma_{1} \sigma_{2}+\sigma_{2}^{2}$. Show that the solution to $W(x, \tau)$ is given by

$$
W(x, \tau)=e^{-q_{1} \tau} x N\left(d_{1}\right)-e^{-q_{2} \tau} N\left(d_{2}\right),
$$

where

$$
\begin{aligned}
d_{1} & =\frac{\ln \frac{S_{1}}{S_{2}}+\left(q_{2}-q_{1}+\frac{\sigma^{2}}{2}\right) \tau}{\sigma \sqrt{\tau}}, \quad d_{2}=d_{1}-\sigma \sqrt{\tau}, \\
\sigma^{2} & =\sigma_{1}^{2}-2 \rho \sigma_{1} \sigma_{2}+\sigma_{2}^{2} .
\end{aligned}
$$

Show that the price function $V\left(S_{1}, S_{2}, \tau\right)$ can be expressed as

$$
V\left(S_{1}, S_{2}, \tau\right)=e^{-r \tau}\left[S_{1} e^{\left(r-q_{1}\right) \tau} N\left(d_{1}\right)-S_{2} e^{\left(r-q_{2}\right) \tau} N\left(d_{2}\right)\right]
$$

where

$$
\begin{aligned}
& d_{1}=\frac{\ln \frac{S_{1}}{S_{2}}+\left[\left(r-q_{1}\right)-\left(r-q_{2}\right)+\frac{\sigma^{2}}{2}\right] \tau}{\sigma \sqrt{\tau}}, \\
& d_{2}=-\frac{\ln \frac{S_{2}}{S_{1}}+\left[\left(r-q_{2}\right)-\left(r-q_{1}\right)+\frac{\sigma^{2}}{2}\right] \tau}{\sigma \sqrt{\tau}}=d_{1}-\sigma \sqrt{\tau} .
\end{aligned}
$$

9. Let $F_{S \backslash U}$ denote the Singaporean currency price of one unit of US currency and $F_{H \backslash S}$ denote the Hong Kong currency price of one unit of Singaporean currency. Suppose we assume $F_{S \backslash U}$ to be governed by the following dynamics under the risk neutral measure $Q_{S}$ in the Singaporean currency world:

$$
\frac{d F_{S \backslash U}}{F_{S \backslash U}}=\left(r_{S G D}-r_{U S D}\right) d t+\sigma_{F_{S \backslash U}} d Z_{F_{S \backslash U}}^{S},
$$

where $r_{S G D}$ and $r_{U S D}$ are the Singaporean and US riskless interest rates, respectively. Similar Geometric Brownian motion assumption is made for other exchange rate processes. The digital quanto option pays one US dollar at maturity if $F_{S \backslash U}$ is above $\alpha F_{H \backslash U}$ for some constant value $\alpha$. Find the value of the digital quanto option in Hong Kong dollar in terms of the exchange rates, the riskless interest rates of the different currency worlds and volatility values.
10. Derive the Dupire equation in terms of local volatility as a conditional expected value, where

$$
\frac{\partial c}{\partial T}=-K\left(r_{T}-q_{T}\right) \frac{\partial c}{\partial K}+\frac{K^{2}}{2} E\left[\sigma_{T}^{2} \mid S_{T}=K\right] \frac{\partial^{2} c}{\partial K^{2}}-q_{T} c .
$$

Here, $T$ is the maturity date, $K$ is the strike price and $\sigma_{t}=\sigma\left(S_{t}, t\right)$ is the local volatility function. The riskless interest rate $r_{t}$ and dividend yield $q_{t}$ are deterministic functions in time $t$.
11. Let $v_{L}$ be the local variance and $\Sigma=\Sigma(K, T)$ be the Black-Scholes implied volatility. Let $w=\Sigma(K, T)^{2} T$ be the Black-Scholes total implied variance and $y=\ln \frac{K}{F_{T}}$, where $F_{T}=\exp \left(\int_{0}^{T} r_{t}-q_{t} d t\right)$. Show that

$$
\begin{aligned}
v_{L} & =\frac{\frac{\partial w}{\partial T}}{1-\frac{y}{w} \frac{\partial w}{\partial y}+\frac{1}{2} \frac{\partial^{2} w}{\partial y^{2}}+\frac{1}{4}\left(-\frac{1}{4}-\frac{1}{w}+\frac{y^{2}}{w}\right)\left(\frac{\partial w}{\partial y}\right)^{2}} \\
& =\frac{\Sigma^{2}+2 \Sigma T\left[\frac{\partial \Sigma}{\partial T}+\left(r_{T}-q_{T}\right) K \frac{\partial \Sigma}{\partial K}\right]}{\left(1+\frac{K y}{\Sigma} \frac{\partial \Sigma}{\partial K}\right)^{2}+K \Sigma T\left[\frac{\partial \Sigma}{\partial K}-\frac{K \Sigma T}{4}\left(\frac{\partial \Sigma}{\partial K}\right)^{2}+K \frac{\partial^{2} \Sigma}{\partial K^{2}}\right]} .
\end{aligned}
$$

