MAFS5250 - Computational Methods for Pricing Structured Products

Solution to Homework One

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1. When the underlying asset pays a *continuous dividend yield* at the rate q, the expected rate of return of the asset is $r - q$ under the risk neutral measure. Under the continuous Geometric Brownian process model, the logarithm of the asset price ratio over Δt time interval is normally distributed with mean $\left(r - q - \dfrac{\sigma^2}{2}\right)\Delta t$ and variance $\sigma^2 \Delta t$. Accordingly, the mean and variance of $\dfrac{S_{t+\Delta t}}{S_t}$ are $e^{(r-q)\Delta t}$ and $e^{2(r-q)\Delta t}(e^{\sigma^2 \Delta t} - 1)$. By equating the mean and variance of the discrete binomial model and the continuous Geometric Brownian process model, we obtain

$$pu + (1-p)d = e^{(r-q)\Delta t}$$
$$pu^2 + (1-p)d^2 = e^{2(r-q)\Delta t} e^{\sigma^2 \Delta t}.$$

Also, we use the usual tree-symmetry condition: $u = 1/d$. Solving the system of 3 equations, we obtain

$$u = \frac{1}{d} = \frac{\widetilde{\sigma}^2 + 1 + \sqrt{(\widetilde{\sigma}^2 + 1)^2 - 4R^2}}{2R}, \quad p = \frac{R - d}{u - d},$$

where $R = e^{(r-q)\Delta t}$ and $\widetilde{\sigma}^2 = R^2 e^{\sigma^2 \Delta t}$. As an analytic approximation to u and d up to order Δt accuracy, we take

$$u = e^{\sigma\sqrt{\Delta t}} \quad \text{and} \quad d = e^{-\sigma\sqrt{\Delta t}}.$$

There is only one modification that occurs in the binomial parameter p, where

$$p = \frac{e^{(r-q)\Delta t} - d}{u - d},$$

while u and d remain the same. The binomial pricing formula takes a similar form (discounted expectation of the terminal payoff):

$$V = [pV_u^{\Delta t} + (1-p)V_d^{\Delta t}]e^{-r\Delta t}.$$

The discount factor $e^{-r\Delta t}$ remains the same while the risk neutral probability of up-move p is modified.

2. (a) With the usual notation

$$p = \frac{R - d}{u - d} \quad \text{and} \quad 1 - p = \frac{u - R}{u - d}.$$

If $R < d$ or $R > u$, then one of the above two probabilities becomes negative. This happens when either

$$e^{(r-q)\Delta t} < e^{-\sigma\sqrt{\Delta t}}$$

or

$$e^{(r-q)\Delta t} > e^{\sigma\sqrt{\Delta t}}.$$

The above two inequalities are equivalent to $(q-r)\sqrt{\Delta t} > \sigma$ or $(r-q)\sqrt{\Delta t} > \sigma$. Hence, negative probabilities occur when

$$\sigma < |(r-q)\sqrt{\Delta t}|.$$

To avoid the occurrence of negative probability values, the time step must be chosen to be sufficiently small such that

$$\Delta t < \frac{\sigma^2}{(r-q)^2}.$$

(b) We approximate $\ln \frac{S_{t+\Delta t}}{S_t}$ by a discrete random variable ζ^a, where

$$\zeta^a = \begin{cases} v_1 & \text{with probability equals } 0.5 \\ v_2 & \text{with probability equals } 0.5 \end{cases}.$$

Matching the mean and variance of the discrete and continuous distributions, we obtain

$$\begin{aligned} E[\zeta^a] &= \frac{v_1+v_2}{2} = \left(r-q-\frac{\sigma^2}{2}\right)\Delta t \\ \text{var}(\zeta^2) &= \frac{v_1^2+v_2^2}{2} = \sigma^2\Delta t \text{ [dropping } O((\Delta t)^2) \text{ term]}. \end{aligned}$$

Solving the pair of equations [up to $O(\Delta t)$ accuracy], we obtain

$$v_1 = \left(r-q-\frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t} \quad \text{and} \quad v_2 = \left(r-q-\frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}.$$

As a check, we consider

$$v_1^2+v_2^2 = 2\left[\left(r-q-\frac{\sigma^2}{2}\right)\Delta t\right]^2 + 2\sigma^2\Delta t$$

so that

$$\frac{v_1^2+v_2^2}{2} = \sigma^2\Delta t + O((\Delta t)^2).$$

3. For a n-step trinomial tree, the number of nodes at which we need to perform backward induction calculations is given by

$$\sum_{i=0}^{n-1}(2i+1) = n + \frac{n(n-1)}{2} \times 2 = n^2.$$

Each backward induction calculation involves 2 additions and 3 multiplications. Therefore, the number of multiplications is $3n^2$ and the number of additions is $2n^2$. In a similar manner, the number of nodes in a n-step binomial tree at which we need to perform backward induction calculations is given by

$$\sum_{i=0}^{n-1}(i+1) = n + \frac{n(n-1)}{2} = \frac{n^2+n}{2}.$$

Each backward induction calculation involves 1 addition and 2 multiplications. Therefore, the number of multiplications is n^2+n and the number of additions is $\frac{n^2+n}{2}$.

4. Unlike the derivation in the lecture note, we now keep all the terms that are $O((\Delta t)^2)$. From the second equation, we obtain

$$v = \sqrt{\left(r - \frac{\sigma^2}{2}\right)^2 \Delta t^2 + \sigma^2 \Delta t}.$$

Substituting v into the first equation: $(2p-1)v = \left(r - \frac{\sigma^2}{2}\right)\Delta t$, we have

$$p = \frac{1}{2}\left[1 + \frac{\left(r - \frac{\sigma^2}{2}\right)\Delta t}{\sqrt{\sigma^2\Delta t + \left(r - \frac{\sigma^2}{2}\right)^2 \Delta t^2}}\right].$$

5. Consider the system of equations for p_1, p_2 and p_3:

$$\begin{pmatrix} 1 & 1 & 1 \\ u & 1 & d \\ u^2 & 1 & d^2 \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 1 \\ R \\ W \end{pmatrix}.$$

Eliminating p_2 from the equations, we obtain

$$\begin{aligned}(u-1)p_1 + (d-1)p_3 &= R-1 \\ (u^2-1)p_1 + (d^2-1)p_3 &= W-1.\end{aligned}$$

Solving for p_1 and p_3 gives

$$p_1 = \frac{(W-R)u - (R-1)}{(u-1)(u^2-1)} \quad \text{and} \quad p_3 = \frac{(W-R)u^2 - (R-1)u^3}{(u-1)(u^2-1)}.$$

When $\lambda = 1$, the parameter u becomes $e^{\sigma\sqrt{\Delta t}}$, which agrees with that of the Cox-Rubinstein-Ross binomial scheme. One can show that

$$p_1 + p_3 = 1 + O(\Delta t),$$

or equivalently,

$$p_2 = O(\Delta t).$$

If we consider order of accuracy up to $O(\Delta t)$, then p_2 vanishes. As a result, the trinomial scheme reduces to a binomial scheme.

6. By equating the corresponding mean, variances and covariances [up to $O(\Delta t)$ accuracy], we have

$$
\begin{aligned}
E[\zeta_1^a] &= v_1(p_1+p_2+p_3+p_4-p_5-p_6-p_7-p_8) = \left(r-\frac{\sigma_1^2}{2}\right)\Delta t && \text{(i)}\\
E[\zeta_2^a] &= v_2(p_1+p_2-p_3-p_4+p_5+p_6-p_7-p_8) = \left(r-\frac{\sigma_2^2}{2}\right)\Delta t && \text{(ii)}\\
E[\zeta_3^a] &= v_3(p_1-p_2+p_3-p_4+p_5-p_6+p_7-p_8) = \left(r-\frac{\sigma_3^2}{2}\right)\Delta t && \text{(iii)}\\
\text{var}[\zeta_1^a] &= v_1^2(p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8) = \sigma_1^2\Delta t && \text{(iv)}\\
\text{var}[\zeta_2^a] &= v_2^2(p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8) = \sigma_2^2\Delta t && \text{(v)}\\
\text{var}[\zeta_3^a] &= v_3^2(p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8) = \sigma_3^2\Delta t && \text{(vi)}\\
E[\zeta_1^a\zeta_2^a] &= v_1v_2(p_1+p_2-p_3-p_4-p_5-p_6+p_7+p_8) = \sigma_1\sigma_2\rho_{12}\Delta t && \text{(vii)}\\
E[\zeta_1^a\zeta_3^a] &= v_1v_3(p_1-p_2-p_3-p_4-p_5-p_6+p_7+p_8) = \sigma_1\sigma_3\rho_{13}\Delta t && \text{(viii)}\\
E[\zeta_2^a\zeta_3^a] &= v_2v_3(p_1-p_2-p_3+p_4+p_5-p_6-p_7+p_8) = \sigma_2\sigma_3\rho_{23}\Delta t && \text{(ix)}
\end{aligned}
$$

Lastly, the sum of probabilities must be one so that

$$p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8+p_9 = 1. \qquad \text{(x)}$$

Recall that $v_1 = \lambda_1\sigma\sqrt{\Delta t}, v_2 = \lambda_2\sigma\sqrt{\Delta t}$ and $v_3 = \lambda_3\sigma\sqrt{\Delta t}$. In order that Eqs (iv), (v) and (vi) are consistent, we must set $\lambda_1 = \lambda_2 = \lambda_3$. We write the common value as λ. These 3 equations then reduce to single equation:

$$p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8 = \frac{1}{\lambda^2}.$$

There are only 8 equations and 9 unknowns. We impose the last condition: $E[\zeta_1^a\zeta_2^a\zeta_3^a] = 0$ [up to $O(\Delta t)$ accuracy], which gives one additional equation:

$$E[\zeta_1^a\zeta_2^a\zeta_3^a] = v_1v_2v_3(p_1-p_2-p_3+p_4-p_5+p_6+p_7-p_8) = 0.$$

The probability values are obtained as follows:

$$
\begin{aligned}
p_2 &= \frac{1}{8}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{r-\frac{\sigma_1^2}{2}}{2} + \frac{r-\frac{\sigma_2^2}{2}}{2} - \frac{r-\frac{\sigma_3^2}{2}}{2}\right)\right.\\
&\qquad \left. + \frac{\rho_{12}-\rho_{13}-\rho_{23}}{\lambda^2}\right],\\
p_3 &= \frac{1}{8}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{r-\frac{\sigma_1^2}{2}}{\sigma_1} - \frac{r-\frac{\sigma_2^2}{2}}{\sigma_2} + \frac{r-\frac{\sigma_3^2}{2}}{2}\right)\right.\\
&\qquad \left. + \frac{\rho_{13}-\rho_{12}-\rho_{23}}{\lambda^2}\right],\\
p_4 &= \frac{1}{8}\left[\frac{1}{\lambda^2} + \frac{\sqrt{\Delta t}}{\lambda}\left(\frac{r-\frac{\sigma_1^2}{2}}{2} - \frac{r-\frac{\sigma_2^2}{2}}{2} - \frac{r-\frac{\sigma_3^2}{2}}{2}\right)\right.\\
&\qquad \left. + \frac{\rho_{23}-\rho_{13}-\rho_{12}}{\lambda^2}\right], \text{ etc.}
\end{aligned}
$$

7. Unlike the floating strike lookback call normalized by the asset price, where the exercise payoff is expressible as $\dfrac{M(t_j)}{S(t_j)} - 1 = Y_j - 1$, the exercise payoff in the fixed strike lookback

call normalized by the asset price is $\dfrac{M(t_j) - K}{S(t_j)}$. It cannot be expressed in terms of $\dfrac{\max(M(t_j), K)}{S(t_j)}$; so the dynamic programming procedure in pricing the corresponding American option cannot be implemented. Pricing of the European option counterpart causes no problem since the terminal payoff of the uncertain option component is zero while the accumulated payments of the certainty option component is homogeneous in $S(t_j)$.

8. (a) When $t_{j+1} \neq iZ$, for all i, there will be no monitoring of the realized maximum of the asset price in the next time step, so the usual backward induction procedure prevails. The usual discounted expectation procedure gives

$$C_X(S(t_j), K', t_j) = [pC_X(uS(t_j), K', t_{j+1}) + (1-p)C_X(dS(t_j), K', t_{j+1})]e^{-r\Delta t}.$$

Since K' does not change, so k increases (decreases) by one when $S(t_j)$ moves up (down). Upon normalizing $C_X(S(t_j), K', t_j)$ by $S(t_j)$, we obtain

$$X^Z(k, t_j) = [pX^Z(k+1, t_{j+1})u + (1-p)X^Z(k-1, t_{j+1})d]e^{-r\Delta t}.$$

(b) When $k \geq 1$ and $t_{j+1} = iZ$, the next time step is a monitoring instant for the new realized maximum. Since $k \geq 1$, an updated realized K' will be recorded at $t_{j+1} = iZ$, so the index k becomes zero at t_{iZ}, irrespective of either an up-move or down-move of the asset price. Recall that $S(t_j) = K'u^k$, $k \geq 1$, given that an up-move of the asset price occurs in the next time step, the guaranteed payment at maturity is $uS(t_j) - K'$. When normalized by $S(t_j)$, it becomes $u - u^{-k}$. This occurs with probability p and it is paid $N - (iz - 1)$ time steps later. Similarly, for a down-move, the normalized guaranteed payment at maturity is $[dS(t_j) - K']/S(t_j) = u^{-1} - u^{-k}$. Combining all these results together, we obtain

$$X^Z(k, t_{iZ-1}) \quad = \quad X^Z(0, t_{iZ}) + [p(u - u^{-k}) + (1-p)(u^{-1} - u^{-k})]e^{-(N-iz+1)\Delta t}.$$

(c) When $k = 0$ and $t_{j+1} = iz$, we have $S(t_j) = K'$. A newly realized maximum is recorded if the asset price has an up-move in the next time step and k becomes zero. Otherwise, k becomes -1 for a down-move of the asset price. For an up-move with probability p, since $S(t_j) = K'$, the normalized guaranteed payment at maturity is $[uS(t_j) - K']/S(t_j) = u - 1$. Combining all these results, we obtain

$$\begin{aligned} X^Z(0, T_{iZ-1}) \quad &= \quad [pX^Z(0, t_{iZ})u + (1-p)X^Z(-1, t_{iZ})d]e^{-r\Delta t} \\ &\quad + p(u-1)e^{-(N-iZ+1)\Delta t} \end{aligned}$$

9. If m is set equal to $\widehat{m}$, then the window Parisian feature reduces to the consecutive Parisian feature. We define a binary string $A = (a_1, a_2, \cdots, a_{N_w})$ of size N_w to represent the history of the asset price path falling inside or outside the knock-out region at the previous N_w consecutive monitoring instants prior to the current time. By convention, the value of a_p is set to be 1 if the asset price falls on or below the down barrier B at the p-th monitoring instant counting backward from the current time; and it is set to be 0 if otherwise.

There are altogether 2^{N_w} different binary strings to represent all possible breaching history of asset price path at the previous N_w monitoring instants. The number of states that have to be recorded is $C_0^{N_w} + C_1^{N_w} + \cdots + C_{N-1}^{N_w}$, where $C_i^{N_w}$ denotes the combination of N_w strings taken i strings at a time. We sum from $i = 0$ to $i = N - 1$ since the window Parisian option value becomes zero when the number of breaches reaches N, so those states with N or more "1" in the string are irrelevant.

Let $V_{win}[m, j; A]$ denote the value of a window Parisian option at the (m, j)-th node, arugmented with the asset price path history represented by the binary string A. The binary string A has to be modified according to the event of either breaching or no breaching at a monitoring instant. The corresponding numerical scheme can be succinctly represented by

$$V_{win}[m-1, j; A] = \begin{cases} \{p_u V_{win}[m, j+1; A] \\ + p_0 V_{win}[m, j; A] \\ + p_d V_{win}[m, j-1; A]\} e^{-r\Delta t} \quad \text{if } m\Delta t \neq t_\ell^* \end{cases}$$

where

$$g_{win}(A, j) = \begin{cases} (1, a_1, a_2, \cdots, a_{N_w - 1}) & \text{if } x_j \leq \ln B \\ (0, a_1, a_2, \cdots, a_{N_w - 1}) & \text{if } x_j > \ln B \end{cases}.$$

Note that $V_{win}[m, j; A] = 0$ at a monitoring instant when the string A has N or more "1". Due to the higher level of path dependence exhibited by the window feature, the operation counts of the window Parisian option calculations are roughly $C_0^{N_w} + C_1^{N_w} + \cdots + C_{N-1}^{N_w}$ times of those of the plain vanila option calculations.

10. The payoff of a floating strike lookback call at time t, $t \in (0, T]$, is given by

$$\max_{\tau \in [0,t]} S_\tau - S_t,$$

where $\max_{\tau \in [0,t]} S_\tau$ denotes the realized *maximum* of the asset price over $[0, t]$. The corresponding grid function at the $(n, j)^{th}$ node with asset price $S_j^n = Su^j$ is given by

$$g_{lookback}(k, j) = \max(k, j).$$

Here, k is the numbering index for the lookback state variable and $S_k^{\max} = Su^k$. The FSG algorithm is given by

$$V_{j,k}^n = [p_u V_{j+1, g_{lookback}(k,j+1)}^{n+1} + p_0 V_{j, g_{lookback}(k,j)}^{n+1} + p_d V_{j-1, g_{lookback}(k,j-1)}^{n+1}] e^{-r\Delta t}.$$

At the terminal nodes in the N-step trinomial tree, the terminal payoff dictates

$$V_{j,k}^N = Su^k - Su^j,$$

where $j = -N, -N+1, \cdots, N-1, N$, and $k = -N, -N+1, \cdots, N-1, N$.

To incorporate the American early exercise feature, we simply incorporate the dynamic programming procedure at each node and for each number index:

$$V_{j,k}^n = \max \left\{ \left[p_u V_{j+1, g_{lookback}(k,j+1)}^{n+1} + p_0 V_{j, g_{lookback}(k,j)}^{n+1} + p_d V_{j-1, g_{lookback}(k,j-1)}^{n+1} \right] e^{-r\Delta t}, Su^k - Su^j \right\}.$$

11. We let the current time be time 0 for convenience, t_i be the i^{th} observation date, T_i be the settlement date of stocks based on the i^{th} observation, $T_i > t_i$. Let S denote the asset value at the current time and define $M = \ln \frac{H}{S}$. We write X_{t_i} as the log asset price ratio $\ln \frac{S_{t_i}}{S}$ on the i^{th} observation date. According to eq.(4.1.27) in Kwok's text, the restricted density function of the log asset price ratio with an upstream barrier M is given by

$$f_{\text{up}}(x, t_i; M) = \frac{1}{\sigma\sqrt{t_i}} \left[n\left(\frac{x - \mu t_i}{\sigma\sqrt{t_i}}\right) - e^{\frac{2\mu M}{\sigma^2}} n\left(\frac{x - 2M - \mu t_i}{\sigma\sqrt{t_i}}\right) \right],$$

where $\mu = r - q - \frac{\sigma^2}{2}$. The up-and-out call option value is given by

$$c_{\text{uo}}(S, t_i; K, H) = e^{-rt_i} \int_{\ln \frac{K}{S}}^{\ln \frac{H}{S}} (Se^x - K) f_{\text{up}}(x, t_i; M)\, dx.$$

We take advantage of the well known down-and-out call option value function $c_{\text{do}}(S, t_i; H)$, where the strike price K is larger than the down barrier H. From eq.(4.1.40) in Kwok's text [also refer to eq.(4.1.10)], we obtain

$$\begin{aligned} c_{\text{do}}(S, t_i; K, H) &= e^{-rt_i} \int_{\ln K}^{\infty} (Se^x - K) f_{\text{down}}(x, t_i; M)\, dx \\ &= c_E(S, t_i; K) - \left(\frac{H}{S}\right)^{\lambda - 1} c_E(\frac{H^2}{S}, t_i; K), \end{aligned}$$

where $\lambda = \frac{2(r-q)}{\sigma^2}$ and $f_{\text{down}}(x, t_i; M)$ has the same analytic form as that of $f_{\text{up}}(x, t_i; M)$, $M = \ln \frac{H}{S}$. Here, $c_E(S, t_i; K)$ is the value function of the vanilla European call option as given by

$$c_E(S, t_i; K) = Se^{-qt_i} N(d_1^{(i)}) - Ke^{-rt_i} N(d_2^{(i)}),$$

where

$$d_1^{(i)} = \frac{\ln \frac{S}{K} + \left(r - q + \frac{\sigma^2}{2}\right) t_i}{\sigma\sqrt{t_i}} \quad \text{and} \quad d_2^{(i)} = d_1^{(i)} - \sigma\sqrt{t_i}.$$

Since $f_{\text{up}}(x, t_i; M)$ and $f_{\text{down}}(x, t_i; M)$ share the same analytic form, we deduce that [see eq.(4.1.41) in Kwok's text]

$$\begin{aligned} c_{\text{uo}}(S, t_i; K, H) &= c_{\text{do}}(S, t_i; K, H) - c_{\text{do}}(S, t_i; H, H) \\ &= \left[c_E(S, t_i; K) - \left(\frac{H}{S}\right)^{\lambda - 1} c_E(\frac{H^2}{S}, t_i; K) \right] \\ &\quad - \left[c_E(S, t_i; H) - \left(\frac{H}{S}\right)^{\lambda - 1} c_E(\frac{H^2}{S}, t_i; H) \right]. \end{aligned}$$

In a similar manner, one can show that the value of the up-and-out put option with $K < H$ is given by (see Problem 4.5 in Kwok's text)

$$\begin{aligned} p_{\text{uo}}(S, t_i; K, H) &= Xe^{-rt_i} N(-d_2^{(i)}) - Se^{-qt_i} N(-d_1^{(i)}) \\ &\quad - \left[\left(\frac{H}{S}\right)^{\lambda - 1} Xe^{-rt_i} N(-d_4^{(i)}) - \left(\frac{H}{S}\right)^{\lambda + 1} Se^{-qt_i} N(-d_3^{(i)}) \right], \end{aligned}$$

where

$$d_3^{(i)} = \frac{2\ln\frac{H}{S}}{\sigma\sqrt{t_i}} + d_1^{(i)} \quad \text{and} \quad d_4^{(i)} = d_3^{(i)} - \sigma\sqrt{t_i}.$$

According to Lam *et al.* (2009), we may express $c_{\text{uo}}(S, t_i; K, H)$ and $p_{\text{uo}}(S, t_i; K, H)$ as

$$\begin{aligned} c_{\text{uo}}(S, t_i; K, H) &= e^{-rt_i}\left\{E_Q[e^{X_{t_i}}\mathbf{1}_A] - KE_Q[\mathbf{1}_A]\right\} \\ p_{\text{uo}}(S, t_i; K, H) &= e^{-rt_i}\left\{KE_Q[\mathbf{1}_B] - E_Q[e^{X_{t_i}}\mathbf{1}_B]\right\}, \end{aligned}$$

where Q is a risk neutral measure, and

$$\begin{aligned} A &= \left\{\omega \in \Omega \,\middle|\, X_{t_i} \geq \ln\frac{K}{S}, M_t < \ln\frac{H}{S}\right\}, \\ B &= \left\{\omega \in \Omega \,\middle|\, X_{t_i} < \ln\frac{K}{S}, M_t < \ln\frac{H}{S}\right\}, \\ M_t &= \max_{0\leq u\leq t_i} X_u. \end{aligned}$$

With the delay settlement on T_i, the value function of the up-and-out-call and up-and-out-put with delay settlement are given by

$$\begin{aligned} c_{\text{uo}}^{(\text{d})}(S, T_i; K, H) &= e^{-rT_i}\left\{E_Q[e^{X_{t_i}}\mathbf{1}_A]E_Q[e^{X_{T_i}-X_{t_i}}] - KE_Q[\mathbf{1}_A]\right\} \\ p_{\text{uo}}^{(\text{d})}(S, T_i; K, H) &= e^{-rT_i}\left\{KE_Q[\mathbf{1}_B] - E_Q[e^{X_{t_i}}\mathbf{1}_B]E_Q[e^{X_{T_i}-X_{t_i}}]\right\}. \end{aligned}$$

Since $E_Q[e^{X_{T_i}-X_{t_i}}] = e^{-q(T_i-t_i)}$, we can deduce easily that-

$$\begin{aligned} p_{\text{uo}}^{(\text{d})}(S, T_i; K, H) \;=\; & Xe^{-rT_i}N(-d_2^{(i)}) - Se^{-qT_i}N(-d_1^{(i)}) \\ & - \left[\left(\frac{H}{S}\right)^{\lambda-1} Xe^{-rT_i}N(-d_4^{(i)}) - \left(\frac{H}{S}\right)^{\lambda+1} Se^{-qT_i}N(-d_3^{(i)})\right]. \end{aligned}$$

Comparing $p_{\text{uo}}^{(\text{d})}(S, T_i; K, H)$ and $p_{\text{uo}}(S, t_i; K, H)$, the up-and-out put price function with and without delay settlement, we observe that the discount factors differ while the expectation terms $N(-d_j^{(i)})$, $j = 1, 2, 3, 4$, stay the same. This is not surprising since the shares of stock and strike price are delivered on T_i after the i^{th} observation date t_i while the deciding criterion on the delivery of one or two units of stock remains to be determined by $S_{t_i} \geq K$ or $S_{t_i} < K$. In a similar manner, we can deduce $c_{\text{uo}}^{(\text{d})}(S, T_i; K, H)$ from $c_{\text{uo}}(S, t_i; K, H)$ by modifying the discount factors while the expectation terms $N(d_j^{(i)})$, $j = 1, 2, 3, 4$, stay the same.