MAFS5250 - Computational Methods for Pricing Structured Products

Computer Assignment Two

Work in a group of two *Instructor: Prof. Y.K. Kwok*

Pricing of Participating Life Insurance Policies

Product nature

A contract of nominal value P_0 is issued by the company at time zero. The contract is immediately acquired by an investor for a single premium of V_0. We treat P_0 as exogenously given whereas V_0 is determined by the pricing model. We refer V_0 as the *fair value* of the contract.

The benefit from the contract at the maturity date is denoted by $P(T)$ and we shall generally refer to $\{P(t)\}_{0 \leq t \leq T}$ as the policy account balance process of the contract.

The evolution of $P(\cdot)$ between successive time points in the set $\Upsilon \equiv \{1, 2, \ldots, T\}$ is determined by the discretely compounded policy interest rate process, $\{r_P(t)\}_{t \in \Upsilon}$. Specifically, we have

$$P(t) = P_0 \Pi_{i=1}^{t} [1 + r_P(i)], \quad t \in \Upsilon.$$

Time is measured in years, given that $P(\cdot)$ is updated annually. The modeling of the annualized interest rate crediting $r_P(\cdot)$ is crucial, the details of which are presented below.

We use $A(t)$ to denote the market value of the *asset base* backing the contract. Since the pension and life insurance companies typically invest largely in highly liquid assets such as bonds and stocks for which market prices are easily observable, we can safely assume that A is tradeable. The policy account balance, $P(t)$, which was introduced above, is a book value. Alternatively, we can think of $P(t)$ as the funds set aside to cover the contract liability – a *distributed reserve*. $B(t)$ denotes the *undistributed reserve* or simply *the buffer*, which is used to partly protect the policy reserve, $P(t)$, (in some sense company solvency) from unfavorable fluctuations in the asset base.

To model the dynamics of the asset side, we specify the following stochastic differential equation under the risk neutral measure Q for the evolution of the market value of the tradeable asset base through time:

$$dA(t) = rA(t)\ dt + \sigma A(t)\ dW^Q(t), \quad A(0) = A_0.$$

Here, r, σ and A_0 are positive constants and $W^Q(t)$ is a standard Brownian motion defined on the filtered probability space $(\Omega, \Im, Q)$ in the finite time interval $[0, T]$. The asset base thus evolves through time according to the geometric Brownian motion (GBM).

Balance sheet

Assets	Liabilities
$A(t)$	$P(t)$
	$B(t)$
$\sum = A(t)$	$\sum = A(t)$

This balance sheet is *not* the company balance sheet but rather a snap-shot of the asset and liability situation in relation to a given contract.

For the liability side of the balance sheet, the interest rate crediting mechanism is modeled by specifying the policy interest rate (compounded annually) as follows

$$r_P(t) = \max\left\{r_G,\ \alpha\left[\frac{B(t^-)}{P(t^-)} - \gamma\right]\right\},$$

where r_G, α and γ are positive constants. We call γ as the target budget ratio and α as the distribution ratio. Note that $r_P(\cdot)$ is a discretely updated process and that $r_P(t)$ is fixed for the year beginning at time $t-1$. The investor is always guaranteed an annualized policy interest rate of at least r_G. Since r_G is assumed to be positive, $P(t)$ will be a strictly increasing process. This again implies that the buffer, $B(t)$, may in fact become (temporarily) negative – a situation that can be interpreted as insolvency with respect to the individual contract. The probability of this happening can be controlled mainly via the parameters α and γ. If the actual/observed buffer relative to the policy account balance exceeds the desired level, γ, of that ratio, the company will attempt to distribute a fraction, α, of the surplus. Provided that $\alpha\left[\frac{B(t^-)}{P(t^-)} - \gamma\right] > r_G$, we have

$$\begin{aligned}
P(t^+) &= P(t^-)\left\{1 + \alpha\left[\frac{B(t^-)}{P(t^-)} - \gamma\right]\right\} \\
&= P(t^-) + \alpha[B(t^-) - \gamma P(t^-)] \\
&= P(t^-) + \alpha[B(t^-) - B^*(t^-)],
\end{aligned}$$

where $B^*(t^-) = \gamma P(t^-)$ denotes the optimal buffer at time t^-. Finally, we have the following relation between account balances across the sampling time point t

$$\begin{aligned}
P(t) &= P(t^-)\left\{1 + \max\left(r_G, \alpha\left[\frac{B(t^-)}{P(t^-)} - \gamma\right]\right)\right\} \\
&= P(t^-)\left\{1 + r_G + \max\left(0, \alpha\left[\frac{A(t^-) - P(t^-)}{P(t^-)} - \gamma\right] - r_G\right)\right\}.
\end{aligned}$$

Numerical algorithms

The fully implicit finite difference scheme has been well discussed in the lecture note. The key steps are summarized below:

1. Start at time T and apply the terminal condition on a suitable grid in (A, P)-space, where $V_T = P(T^+)$. The crediting mechanism remains to be applicable at T^+ since the policyholder is eager to receive the last credit bonus.

2. For every value of P, solve the Black-Scholes partial differential equation, via a finite difference scheme applied to the corresponding vector of constract values. This first step will determine $V_{(T-1)^+}$ everywhere in the grid.

3. Apply the no-jump condition to obtain $V_{(T-1)^-}$ everywhere in the grid.

4. Repeat steps 2 and 3 to obtain V_{t^-} from $V_{(t+1)^-}$ everywhere in the grid working backwards from $t = T-1$ to $t = 0$.

An important aspect in the finite difference scheme is the no-jump (continuity) condition on the value function across a sampling date t, where

$$V_{t^-} = V_{t^+} \Leftrightarrow V(t^-, A(t), P(t^-)) = V(t^+, A(t), P(t^+)).$$

Note that $A(t)$ is continuous across t while $P(\cdot)$ has a jump at t. A brief overview of the procedure is presented below:

1. For each i and j in the grid, we compute

$$\begin{aligned}\widetilde{j} &\equiv \frac{j\Delta P + \max\{(j\Delta P)r_G, \alpha((i\Delta A - j\Delta P) - (j\Delta P)\gamma)\}}{\Delta P} \\ &= j + \max\left\{jr_G, \alpha\left[\left(i\frac{\Delta A}{\Delta P} - j\right) - j\gamma\right]\right\}.\end{aligned}$$

   Denote the integer part of $\widetilde{j}$ as $\underline{j}$.

2. If $\underline{j} + 1 \leq J$, we compute $V_{t-1,0}^{i,j}$ by using the linear interpolation

$$V_{t-1,0}^{i,j} = [1 - (\widetilde{j} - \underline{j})]V_{t,K}^{i,\underline{j}} + (\widetilde{j} - \underline{j})V_{t,K}^{i,\underline{j}+1}.$$

3. If $\underline{j} + 1 > J$ and hence lies outside the computational domain, then the above equation cannot be used. Instead, since for large values of P, the contract value V is approximately linear in P, we can apply the linear extrapolation

$$V_{t-1,0}^{i,j} = V_{t,K}^{i,J} + (\widetilde{j} - J)\left(V_{t,K}^{i,J} - V_{t,K}^{i,J-1}\right).$$

Work elements

Jensen *et al.* (2001) report various studies on the pricing properties of the participating policies. In this computer assignment, you are asked to verify some of their numerical results and explore several new phenomena in pricing behavior. By using the fully implicit scheme as presented in the lecture note (or paper), we compute the contract value with and without the surrender right under various sets of parameter values.

1. As a warm up for checking accuracy of your program, reproduce the tables of values on the contract value as documented on P.27 of Topic 4. Use the same set of parameter values and try the three pairs (I, J) : $(100, 100)$, $(200, 200)$, and $(400, 400)$. Also, report the CPU required.

2. We would like to explore the impact of the distribution ratio α and target budget ratio γ on the option value of the surrender right with $r_G = 2\%$ and $r = 6\%$. The surrender option value is the difference between the values of the contract with and without the surrender right. You are asked to plot the surrender option value against time to maturity (up to 20 years) using the set of parameter values on P.27 of Topic 4 (other than the variation of the parameter values for r_G, r, α and γ). Try the following pairs of (α, γ):

$$\text{(i)}\ \ (\alpha, \gamma) = (0.3, 0.1), \quad \text{(ii)}\ (\alpha, \gamma) = (0.4, 0.1), \quad \text{(iii)}\ (\alpha, \gamma) = (0.3, 0.15).$$

   In the figure, show the 3 curves of the surrender option value corresponding to the 3 pairs of (α, γ) listed above.

(i) Explain the implementation of the surrender right at $t = 1, 2, \ldots, T-1$. Note that the surrender right is immaterial at $t = T$ (maturity date). Similar to an American put on a discrete dividend paying asset, it may not be optimal to surrender at instants that are right before a sampling date.

(ii) Give the financial interpretation of your numerical plots.

3. By choosing $(\alpha, \gamma) = (0, 3, 0.1)$ and $r_G = 2\%$, show the plots of the contract value against maturity (up to 20 years) at $r = 4\%, 7\%$ and 10%. Explore the interplay between the increasing property of the surrender value and decreasing property of the bond component when maturity lengthens.

Reference

B. Jensen, P.L. J∅rgensen and A. Grosen, "A finite difference approach to the valuation of path dependent life insurance liabilities," Geneva Papers on Risk and Insurance Theory, vol.26, p.57-84 (2001).