MAFS 5250

Computational Methods for Pricing Structured Products

Final Examination – Spring 2019

Time allowed: 2 hours **Course Instructor:** *Prof. Y. K. Kwok*

[points]

1. The two-level four-point explicit scheme for pricing the Black-Scholes option model is given by

$$V_j^{n+1} = b_1 V_{j+1}^n + b_0 V_j^n + b_{-1} V_{j-1}^n,\ j = 1, 2, \ldots, N,\ n = 0, 1, 2, \ldots.$$

The coefficients are given by

$$\begin{aligned} b_1 &= \left[\frac{\sigma^2}{2}\frac{\Delta\tau}{\Delta x^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\Delta\tau}{2\Delta x}\right] e^{-r\Delta\tau}, \\ b_0 &= \left(1 - \sigma^2 \frac{\Delta\tau}{\Delta x^2}\right) e^{-r\Delta\tau}, \\ b_{-1} &= \left[\frac{\sigma^2}{2}\frac{\Delta\tau}{\Delta x^2} - \left(r - \frac{\sigma^2}{2}\right)\frac{\Delta\tau}{2\Delta x}\right] e^{-r\Delta\tau}. \end{aligned}$$

(a) Find the restrictions on the time step $\Delta\tau$ and stepwidth Δx. **[1]**

(b) An implicit scheme requires the solution of a tridiagonal system of equations at every time step. Explain in two reasons why most practitioners prefer to use an implicit scheme even with this computational complexity. **[1]**

(c) Explain why the usual dynamic programming procedure cannot be applied to implicit schemes for numerical calculations of an American option model. Outline briefly the numerical procedure in an iterative scheme to resolve the difficulty. **[4]**

2. Consider the down-and-out proportional step call option whose terminal payoff is defined by

$$\exp(-\rho\tau_B^-)\max(S_T - X, 0),$$

where ρ is the killing rate (assuming to be constant), X is the strike price, S_T is the terminal asset price, and τ_B^- is the occupation time in the knock-out region that is defined by

$$\tau_B^- = \int_{t_0}^{T} \mathbf{1}_{\{S_t \le B\}}\, dt.$$

Here, B is the constant down barrier and $\mathbf{1}_{\{\ \ \}}$ is the indicator function, and t_0 is the inception time of the option. We assume the usual Black-Scholes model where the dynamics of the asset price S_t is governed by

$$\frac{dS_t}{S_t} = r\, dt + \sigma\, dZ_t,$$

where r is the riskfree interest rate and Z_t is the standard Brownian motion. The usual trinomial tree algorithm takes the form:

$$V_j^{n+1} = \left[\frac{\alpha + \mu}{2} V_{j+1}^n + (1 - \alpha) V_j^n + \frac{\alpha - \mu}{2} V_{j-1}^n\right] e^{-r\Delta\tau},$$

where $e^{-r\Delta\tau}$ is the discount factor over the time interval $\Delta\tau$, $\mu = \left(r - \frac{\sigma^2}{2}\right)\frac{\Delta\tau}{\Delta x}$ and $\alpha = \frac{\sigma^2 \Delta\tau}{\Delta x^2}$. How do we modify the trinomial scheme so as to incorporate the "proportional step" feature? Give an explanation to your answer. **[3]**

Hint Consider the damping factor being applied over one time step $\Delta\tau$ when the asset price lies below B (down region).

3. In the participating policy model, there are two state variables: $A(t)$ and $P(t)$. The dynamics of the asset $A(t)$ under a risk neutral measure Q is governed by the Geometric Brownian motion:

$$dA(t) = rA(t)\ dt + \sigma A(t)\ dW^Q(t).$$

The updating of the policy account $P(t)$ is based on

$$P(t^+) = P(t^-) + \max(r_G P(t^-), \alpha\{[A(t) - P(t^-)] - \gamma P(t^-)\})$$

across a fixing date t, where

r_G	:	guaranteed rate of return
γ	:	target buffet ratio
α	:	distribution ratio.

Explain in details the key procedures (interpolation and extrapolation) in the incorporation of the jump condition (crediting mechanism) across a fixing date in the finite difference scheme. **[4]**

Hint Write $P(t^+) = \widetilde{j}\Delta P$ and $P(t^-) = j\Delta P$. Let $A = i\Delta A$, where ΔA is the stepwidth for A. Explain why

$$\widetilde{j} = j + \max\left\{r_G j, \alpha\left[\left(i\frac{\Delta A}{\Delta P} - j\right) - \gamma j\right]\right\}.$$

4. In a convertible bond, the issuer can call back the bond at call price K and the holder can convert prematurely with payoff denoted by *conv*. Let *cont* denote the continuation value of holding the convertible bond.

   (a) Give the financial intuition in the design of the dynamic programming procedure as given by

   $$\min(K, \max(cont, conv)).$$ **[1]**

   (b) Suppose the three values K, *cont* and *conv* at the lattice tree node has the following relative order of magnitude: $K < cont < conv$. The procedure in part (a) produces the outcome K while another dynamic programming procedure:

   $$\min(\max(cont, conv), \max(K, conv))$$

   produces *conv* as the outcome. Explain the occurrence of such inconsistency. Which dynamic programming procedure is correct, or indeed such inconsistency does not occur under the context of pricing algorithm of convertible bond? Give justification to your answer. **[3]**

5. The governing equation for pricing a defaultable coupon-paying convertible bond is given by

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - q + h) S \frac{\partial V}{\partial S} - [r + (1 - R)h]V + c(t) = 0.$$

(a) Using financial interpretation (not from the above equation directly), explain why the hazard rate h can be visualized as negative dividend yield. **[2]**

(b) The discrete coupons are modeled by the coupon rate

$$c(t) = \sum_{i=1}^{n} c_i \delta(t - t_i),$$

where c_i is the discrete coupon amount paid on the coupon date t_i, $i = 1, 2, \cdots, n$. Explain how the Dirac terms in $c(t)$ arise? **[3]**

6. In the pricing model of variable annuities with guaranteed minimum withdrawal benefit, we define the surrogate unrestricted fund value process by

$$\begin{aligned} \mathrm{d}\widetilde{W}_t &= (r - \alpha)\widetilde{W}_t \,\mathrm{d}t - G \,\mathrm{d}t + \widetilde{W}_t \sigma \,\mathrm{d}B_t, \ t > 0, \\ \widetilde{W}_0 &= w_0. \end{aligned}$$

Here, α is the participating fee rate, G is the constant withdrawal rate, σ is the volatility and B_t is a standard Brownian motion.

(a) Give the financial justification of the solution:

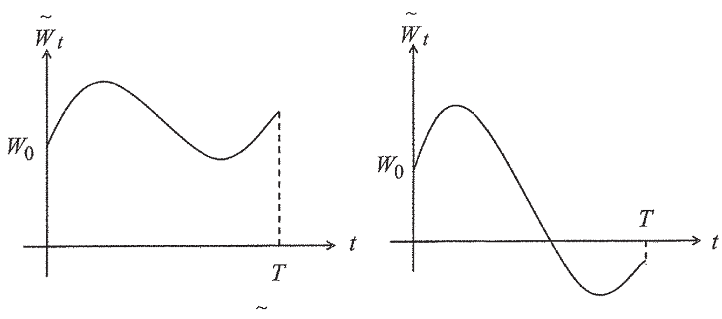
$$\widetilde{W}_t = X_t \left(w_0 - G \int_0^t \frac{1}{X_u} \,\mathrm{d}u \right),$$

where

$$X_t = e^{\left(r - \alpha + \frac{\sigma^2}{2}\right)t + \sigma B_t}.$$ **[3]**

Hint Note that $X_0 = 1$. The number of fund units is depleted continuously due to withdrawal.

(b) We observe that either $\widetilde{W}_t > 0$ for $t \leq T$ or $\widetilde{W}_T$ remains negative once W_t reaches the negative region at some earlier time prior to T (see the figures below).



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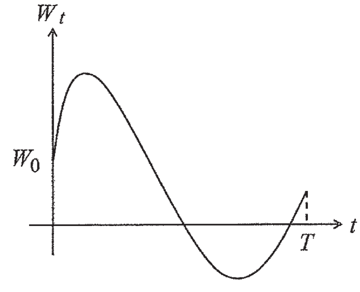

Show mathematically that if $\widetilde{W}_T > 0$, then $\widetilde{W}_t > 0$ for any $t < T$. **[3]**

Hint If some units of fund remains at T, then the fund would not be completely depleted before T. Give the mathematical statement of this observation.

7. Consider the scenario where we require n correlated samples from standardized (zero mean and unit variance) normal distribution with the correlation coefficient between sample i and sample j being ρ_{ij}. We first sample 3 independent random variables x_1, x_2 and x_3 from univariate standardized normal distribution. The required correlated samples, ϵ_1, ϵ_2 and ϵ_3, are then computed by

$$\begin{aligned} \epsilon_1 &= \alpha_{11}x_1 \\ \epsilon_2 &= \alpha_{21}x_1 + \alpha_{22}x_2 \\ \epsilon_3 &= \alpha_{31}x_1 + \alpha_{32}x_2 + \alpha_{33}x_3. \end{aligned}$$

Write down the governing equations for the determination of the parameters $\alpha_{11}, \alpha_{21}, \alpha_{22}, \alpha_{31}, \alpha_{32}$ and α_{33}. DO NOT SOLVE THEM! **[3]**

Hint Use the following 6 relations:

$$\begin{aligned} &\text{var}(\epsilon_1) = 1, \text{var}(\epsilon_2) = 1, \text{var}(\epsilon_3) = 1, \\ &\text{cov}(\epsilon_1, \epsilon_2) = \rho_{12}, \text{cov}(\epsilon_1, \epsilon_3) = \rho_{13}, \text{cov}(\epsilon_2, \epsilon_3) = \rho_{23}. \end{aligned}$$

Alternatively, we may apply the Cholesky decomposition:

$$MM^T = \Sigma,$$

where Σ is the correlation matrix and

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = M \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

8. (a) Let c_i and $\widetilde{c}_i$ denote the simulated call value in the i^{th} simulation run in the antithetic variates method in pricing a European call option to achieve variance reduction in Monte Carlo simulation. Explain why

$$\text{var}\left(\frac{c_i + \widetilde{c}_i}{2}\right) = \frac{1}{2}[\text{var}(c_i) + \text{cov}(c_i, \widetilde{c}_i)].$$ **[1]**

(b) Show that the antithetic variates method improves *computational efficiency* provided that

$$\text{cov}(c_i, \widetilde{c}_i) < 0.$$

Explain why the above negative correlation property is in general valid. **[2]**

9. (a) Discuss the intrinsic difficulty of applying the Monte Carlo simulation method to pricing of American style options with the early exercise right. **[2]**

(b) Consider the Grant-Vora-Weeks method with regard to the numerical valuation of American options using the Monte Carlo simulation approach. Discuss how the method determines the optimal stopping rules at discrete time points? **[4]**

— *End* —