MAFS 5250 – Computational Methods for Pricing Structured Products

Mid-term Test, 2015

Time allowed: 90 minutes *Instructor: Prof. Y. K. Kwok*

[points]

1. Consider the dynamic programming procedure applied to incorporate the game option between the issuer and holder of a callable American call option in a binomial tree calculation. Let X be the strike price and K be the call price.

   (a) Give the financial interpretation of the following dynamic programming procedure:

   $$C_j^n = \min\left(\max\left(\frac{pC_{j+1}^{n+1} + (1-p)C_j^{n+1}}{R}, S_j^n - X\right), \max\left(K, S_j^n - X\right)\right),$$

   where p is the risk neutral probability of an upward move and R is the growth factor over one time step. **[3]**

   (b) Devise an alternative form of the above dynamic programming procedure and provide the corresponding financial interpretation of this alternative scheme. **[5]**

2. (a) Recall that the risk neutral probability of an upward move is given by

   $$p_{i+1}^n = \frac{F_i^n - S_i^{n+1}}{S_{i+1}^{n+1} - S_i^{n+1}},$$

   where $F_i^n = e^{r\Delta t} S_i^n$. Explain why we require

   $$F_i^n < S_i^{n+1} < F_{i+1}^n$$

   in order that the risk neutral probability stays within $(0, 1)$. **[2]**

   (b) Recall that the risk neutral probability of an upward move in the Cox-Ross-Rubinstein binomial tree is

   $$p = \frac{e^{r\Delta t} - d}{u - d}, \quad \text{where } u = e^{\sigma\sqrt{\Delta t}} = \frac{1}{d}.$$

   Show that when the time step Δt becomes too large, where

   $$\left|(r-q)\sqrt{\Delta t}\right| > \sigma,$$

   then p may become negative. **[4]**

3. The key step in applying the forward shooting grid technique in pricing an Asian style option is the derivation of the evolution grid function for the path dependent averaging state variable.

   (a) For the arithmetic averaging state variable A_t, where

   $$A_t = \frac{1}{t}\int_0^t S_u\, du.$$

   Suppose we discretize S_t and A_t as

   $$S_j^n = S_0 e^{j\Delta W} \quad \text{and} \quad A_k^n = S_0 e^{k\rho\Delta W}.$$

   Show that the grid function is found to be

   $$g(n,k,j+1) = \frac{\ln \frac{(n+1)e^{k\rho\Delta W} + e^{(j+1)\Delta W}}{n+2}}{\rho\Delta W}.$$ **[5]**

   (b) (i) Note that $g(n,k,j+1)$ assumes non-integer value in general. How do we modify the forward shooting grid algorithm so that we can limit the calculations of the Asian option values at integer value of the index k at every time step? **[3]**

   (ii) How do we estimate the range of possible values of the integer k at each time step in the forward shooting grid algorithm? **[2]**

4. In the Derman-Kani algorithm for constructing the implied binomial tree, it is necessary to relate the Arrow Debreu price λ_i^n with the risk neutral probabilities and the observed call option prices with various implied binomial tree parameters.

   (a) Recall that λ_i^n is defined by

   $$\lambda_i^n = e^{-rn\Delta t} E_Q\left[\mathbf{1}_{\{S(n\Delta t)=S_i^n\}} \middle| S(0) = S_0\right].$$

   Explain why

   $$\begin{aligned}
   \lambda_0^{n+1} &= e^{-r\Delta t}\left[\lambda_0^n(1-p_1^n)\right] \\
   \lambda_{i+1}^{n+1} &= e^{-r\Delta t}\left[\lambda_i^n p_{i+1}^n + \lambda_{i+1}^n(1-p_{i+2}^n)\right],\ i = 1,2,\ldots,n-1, \\
   \lambda_{n+1}^{n+1} &= e^{-r\Delta t}\lambda_n^n p_{n+1}^n.
   \end{aligned}$$ **[2]**

   (b) The observed call option price with strike S_i^n and maturity date $(n+1)\Delta t$ is given by

   $$c(S_i^n,(n+1)\Delta t) = e^{-r\Delta t}\left[\lambda_i^n p_{i+1}^n(S_{i+1}^{n+1} - S_i^n) + \sum_{j=i+1}^{n} \lambda_j^n(F_j^n - S_i^n)\right],$$

   where

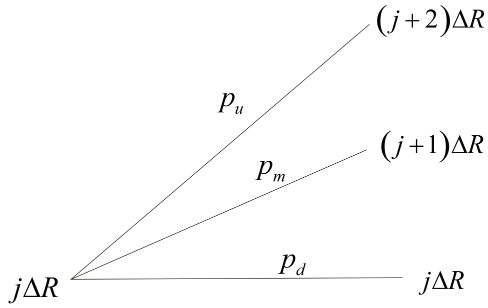
   $$F_j^n = e^{r\Delta t}S_j^n \quad \text{and} \quad p_{j+1}^n = \frac{F_j^n - S_j^{n+1}}{S_{j+1}^{n+1} - S_j^{n+1}}.$$

   Give a financial interpretation of the above formula. **[4]**

5. Consider the Hull-White implied interest rate tree, where the trinomial tree may adopt upward branching, normal branching and downward branching.

   (a) Explain why upward branching or downward branching in the implied interest rate tree has to be adopted in view of the mean reversion feature of the short rate process. **[2]**

   (b) Consider the upward branching for the tree of at node $R^* = j\Delta r$,



   the mean change in R^* in time Δt is $-aR^*\Delta t$ and the variance of the change is $\sigma^2\Delta t$. Find the governing equations for the determination of p_u, p_m and p_d. **[3]**

   (c) The key step in the Hull-White interest rate tree construction is the determination of the parameter α_m using the following formula:

$$P_{m+1} = \sum_{j=-n_m}^{n_m} Q_{m,j} \exp(-(\alpha_m + j\Delta R)\Delta t),$$

   where the short rate R at node (m, j) is $\alpha_m + j\Delta R$, $Q_{m,j}$ is the Arrow-Debreu price of the state at node (m, j), P_{m+1} is the price of a zero-coupon bond maturing at $(m+1)\Delta t$, and n_m is the number of nodes on each side of the central node at time $m\Delta t$. Give the theoretical justification of the above formula. **[5]**

   Hint Recall

$$P_{m+1} = E\left[D(t_0, t_{m+1}) \middle| \mathcal{F}_{t_0}\right],$$

   where $D(t_0, t_{m+1})$ is the discount factor from t_0 to t_{m+1}. Apply the rule on nested expectation.

— *End* —