# MATH 4512 – Fundamentals of Mathematical Finance

**Homework Two**

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1. (Two correlated assets) The correlation coefficient $\rho$ between the random rates of return of assets $A$ and $B$ is 0.1, and other data are given in the following table. [*Note*: $\rho = \sigma_{AB}/(\sigma_A \sigma_B)$.]

| **Asset** | $\overline{r}$ | $\sigma$ |
|---|---|---|
| A | 10.0% | 15% |
| B | 18.0% | 30% |

   (a) Find the proportions $\alpha$ of $A$ and $(1-\alpha)$ of $B$ that define a portfolio of $A$ and $B$ having minimum standard deviation of portfolio return.
   (b) What is the value of this minimum standard deviation?
   (c) What is the expected rate of return of this portfolio?

2. (Rain insurance) Gavin Jones's friend is planning to invest \$1 million in a rock concert to be held 1 year from now. The friend figures that he will obtain \$3 million revenue from his \$1 million investment – unless, my goodness, it rains. If it rains, he will lose his entire investment. There is a 50% chance that it will rain on the day of the concert. Gavin suggests that he buys rain insurance. He can buy one unit of insurance for \$0.50, and this unit pays \$1 if it rains and nothing if it does not. He may purchase as many units as he wishes, up to \$3 million.

   (a) What is the expected rate of return on his investment if he buys $u$ units of insurance? (The cost of insurance is in addition to his \$1 million investment.)
   (b) What number of units will minimize the variance of his return? What is this minimum value? And what is the corresponding expected rate of return? [*Hint*: Before calculating a general expression for variance, think about a simple answer.]

3. (Wild cats) Suppose there are $n$ assets which are uncorrelated. (They might be $n$ different "wild cat" oil well prospects.) You may invest in any one or in any combination of them. The mean rate of return $\overline{r}$ is the same for each asset but the variances are different. The return on asset $i$ has a variance of $\sigma_i^2$ for $i = 1, 2, \cdots, n$.

   (a) Show the situation on an $\sigma - \overline{r}$ diagram. Describe the efficient set.
   (b) Find the minimum-variance point. Express your result in terms of

$$\overline{\sigma}^2 = \left( \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \right)^{-1}.$$

4. (Markowitz fun) There are three assets with their rates of return $r_1, r_2$ and $r_3$, respectively. The covariance matrix and the expected rates of return vector are

$$\Omega = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad \boldsymbol{\overline{r}} = \begin{pmatrix} 0.4 \\ 0.8 \\ 0.8 \end{pmatrix}.$$

   (a) Find the global minimum-variance portfolio by setting $\lambda_2 = 0$. [*Hint*: By symmetry $w_1 = w_3$.]

(b) Find another efficient portfolio by setting $\lambda_1 = 0$.

(c) Let the risk-free rate $r_f$ of the risk free asset be 0.2, find the efficient portfolio consisting of the 3 risky assets and the risk free asset with target expected rate of return of the portfolio set at 0.4.

5. (Betting wheel) Consider a general betting wheel with $n$ segments. The payoff for a \$1 bet on a segment $i$ is $A_i, i = 1, 2, \cdots, n$. Suppose you bet an amount $B_i = 1/A_i$ on segment $i$ for each $i, i = 1, 2, \cdots, n$. Show that the amount you win is independent of the outcome of the wheel. What is the risk-free rate of return for the betting wheel?

6. Suppose that it is impractical to use all the assets that are incorporated into a specified portfolio (such as a given efficient portfolio). One alternative is to find the portfolio, made up a given set of $n$ stocks, that tracks the specified portfolio in the sense of minimizing the variance of the difference in returns. Specifically, suppose that the target portfolio has (random) rate of return $r_M$. Suppose that there are $n$ assets with (random) rate of return $r_1, r_2, \cdots, r_n$. We wish to find the portfolio rate of return

$$r = \alpha_1 r_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n$$

$\left(\text{with } \sum_{i=1}^{n} \alpha_i = 1\right)$ minimizing $\mathrm{var}(r - r_M)$.

(a) Find the set of equations for the $\alpha_i$'s. Solve for $\alpha_i$'s.

(b) Although this portfolio tracks the desired portfolio most closely in terms of variance, it may sacrifice the mean. Hence a logical approach is to minimize the variance of the tracking error subject to achieving a given mean return. As the mean is varied, this results in a family of portfolios that are efficient in a new sense, called tracking efficient. Find the set of equations for the $\alpha_i$'s that are tracking efficient. Solve for $\alpha_i$'s.

7. Let $\boldsymbol{w}_0$ be the portfolio (weights) of risky assets corresponding to the global minimum-variance point in the feasible region. Let $\boldsymbol{w}_1$ be any other portfolio on the efficient frontier. Define $r_0$ and $r_1$ to be the corresponding rate of returns.

(a) Suppose we set $\mathrm{cov}(r_0, r_1) = A\sigma_0^2$, then find $A$. Consider the set of portfolios $(1-\alpha)\boldsymbol{w}_0 + \alpha\boldsymbol{w}_1$ with varying values of $\alpha$. Let $\sigma_\alpha^2$ denote the variance of such portfolio with parameter $\alpha$. Show that

$$\sigma_\alpha^2 = \sigma_0^2 + \alpha^2(\sigma_1^2 - \sigma_0^2).$$

(b) Corresponding to the portfolio $\boldsymbol{w}_1$, there is a portfolio $\boldsymbol{w}_z$ on the minimum-variance set such that $\mathrm{cov}(r_1, r_z) = 0$, where $r_z$ is the rate of return of portfolio $\boldsymbol{w}_z$. This portfolio can be expressed as $\boldsymbol{w}_z = (1-\alpha)\boldsymbol{w}_0 + \alpha\boldsymbol{w}_1$. Find the proper value of $\alpha$.

(c) Show the relation of these three portfolios on a $\sigma - \bar{r}$ diagram that includes the feasible region.