MATH 4512 – Fundamentals of Mathematical Finance

Solution to Homework Two

Course Instructor: Prof. Y.K. Kwok

---

1. (a) The portfolio variance σ_P^2 is given by

$$\sigma_P^2 = \alpha^2\sigma_A^2 + (1-\alpha)^2\sigma_B^2 + 2\alpha(1-\alpha)\rho\sigma_A\sigma_B.$$

Differentiating σ_P^2 with respect to α, we have

$$\frac{d\sigma_P^2}{d\alpha} = 2\alpha\sigma_A^2 - 2(1-\alpha)\sigma_B^2 + (2-4\alpha)\rho\sigma_A\sigma_B.$$

Setting $\dfrac{d\sigma_P^2}{d\alpha} = 0$, we obtain

$$\alpha = \frac{\sigma_B^2 - \rho\sigma_A\sigma_B}{\sigma_A^2 + \sigma_B^2 - 2\rho\sigma_A\sigma_B} = 0.8261.$$

(b) Substituting $\alpha = 0.8261$ into σ_P^2, the portfolio variance of the optimal portfolio is

$$\sigma_P^2 = \alpha^2\sigma_A^2 + (1-\alpha)^2\sigma_B^2 + 2\alpha(1-\alpha)\rho\sigma_A\sigma_B = 0.01937$$

so that $\sigma_P = 0.1392$.

(c) The expected rate of return of the optimal portfolio:

$$\mu_P = \alpha\overline{r}_A + (1-\alpha)\overline{r}_B = 0.1139.$$

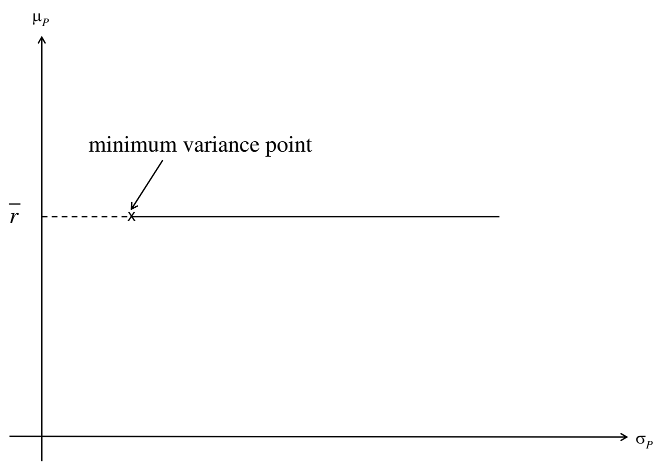
2. (a) The expected rate of return is given by

$$E[r] = \frac{0.5 \times 3 \times 10^6 + 0.5 \times u}{10^6 + 0.5u} - 1.$$

(b) It is seen that buying 3 million units of insurance eliminates all uncertainty regarding the return, resulting in zero variance. The corresponding expected rate of return is

$$E[r] = \frac{0.5 \times 3 \times 10^6 + 0.5 \times 3 \times 10^6}{10^6 + 0.5 \times 3 \times 10^6} - 1 = \frac{3}{2.5} - 1 = 0.2.$$

3. (a)



The expected portfolio rate of return always remains to be $\overline{r}$. The set of minimum variance portfolio (also called the efficient set) reduces to one portfolio, which is represented by the minimum variance point in the above σ_P-μ_P diagram.

(b) The minimum variance point is the global minimum variance portfolio. Recall

$$\boldsymbol{w}_g = \frac{\Omega^{-1}\mathbf{1}}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}, \quad \text{where} \quad \Omega = \begin{pmatrix} \sigma_1^2 & \cdots & 0 \\ \vdots & \sigma_2^2 & \vdots \\ 0 & \cdots & \sigma_n^2 \end{pmatrix}.$$

Note that

$$\Omega^{-1}\mathbf{1} = \begin{pmatrix} 1/\sigma_1^2 \\ 1/\sigma_2^2 \\ \vdots \\ 1/\sigma_n^2 \end{pmatrix} \quad \text{and} \quad \mathbf{1}^T\Omega^{-1}\mathbf{1} = \sum_{i=1}^n \frac{1}{\sigma_i^2}.$$

The minimum variance is given by

$$\sigma_P^2 = \frac{1}{\mathbf{1}^T\Omega^{-1}\mathbf{1}} = \frac{1}{\sum_{i=1}^n \frac{1}{\sigma_i^2}} = \overline{\sigma}^2.$$

Note that $\overline{\sigma}^2$ is the harmonic mean of $\sigma_i^2, i = 1, 2, \cdots, n$. Hence, the minimum variance point is $(\overline{\sigma}, \overline{r})$.

4. (a) Solve for $\mathbf{v}_g$ such that

$$\Omega\mathbf{v}_g = \mathbf{1} \quad \text{or} \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} v_g^1 \\ v_g^2 \\ v_g^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

We obtain

$$\mathbf{v}_g = \begin{pmatrix} 0.5 \\ 0 \\ 0.5 \end{pmatrix}.$$

It happens that the sum of components in $\mathbf{v}_g$ is already equal to 1. So, the optimal weight vector corresponding to the global minimum variance portfolio is $\boldsymbol{w}_g = (0.5 \quad 0 \quad 0.5)^T$.

(b) The other efficient portfolio is obtained by first solving for

$$\Omega\mathbf{v}_d = \overline{\boldsymbol{r}}$$

and normalize the components so that the sum of components equals 1. Consider

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} v_d^1 \\ v_d^2 \\ v_d^3 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 0.8 \\ 0.8 \end{pmatrix},$$

we obtain

$$\mathbf{v}_d = (0.1 \quad 0.2 \quad 0.3)^T.$$

Upon normalization, we obtain the weight vector of another efficient portfolio to be

$$\boldsymbol{w}_d = \left(\frac{1}{6} \quad \frac{1}{3} \quad \frac{1}{2}\right)^T.$$

(c) With the inclusion of the riskfree asset, we solve for

$$\Omega \mathbf{v} = \overline{\boldsymbol{r}} - r_f \mathbf{1}$$

and normalize the components so that the condition on target expected rate of return of the portfolio is met. It is seen that

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_g - r_f \mathbf{v}_d \\ &= (0.1 \quad 0.2 \quad 0.3)^T - 0.2(0.5 \quad 0 \quad 0.5)^T \\ &= (0 \quad 0.2 \quad 0.2)^T. \end{aligned}$$

The optimal weight vector $\boldsymbol{w}^* = \lambda \mathbf{v}$, where λ is determined by enforcing

$$\lambda \sum_{j=1}^{3} (\overline{r}_j - r_f) v_j = \mu_P - r_f, \quad \text{where} \quad \mu_P = 0.4.$$

We then obtain

$$\lambda(0.6 \times 0.2 + 0.6 \times 0.2) = 0.4 - 0.2 = 0.2$$

so that $\lambda = \frac{1}{1.2}$. The weights of the risky assets are

$$w_1 = 0, \quad w_2 = \frac{0.2}{1.2} = \frac{1}{6} \quad \text{and} \quad w_3 = \frac{0.2}{1.2} = \frac{1}{6}.$$

The weight of the risk free asset is $1 - \frac{1}{6} - \frac{1}{6} = \frac{2}{3}$.

5. Consider the betting wheel which has n segments. Let Y be the random variable of the outcome, where $Y = i$ if the outcome of the wheel is i. The payoff of a \$1 bet on the segment i is given by $A_i\, I_{\{Y=i\}}$, where the indicator function $I_{\{Y=i\}} = \begin{cases} 1, \text{if } Y = i \\ 0, \text{otherwise} \end{cases}$.

By using the strategy stated in the question, the payoff is

$$\sum_{i=1}^{n} \frac{1}{A_i} A_i I_{\{Y=i\}} = 1,$$

which is independent of the outcome of the wheel. Following this strategy, the initial total amount betted $= \sum_{i=1}^{n} \frac{1}{A_i}$ and the final payoff is always 1 (risk free). Therefore, the corresponding deterministic rate of return is given by

$$\frac{1}{\sum_{i=1}^{n} \frac{1}{A_i}} - 1.$$

For example, suppose the wheel has 4 segments with $A_1 = 3, A_2 = 4, A_3 = 5, A_6 = 6$. The betting strategy is to bet $\frac{1}{3}$ on segment 1, $\frac{1}{4}$ on segment 2, $\frac{1}{5}$ on segment 3, and $\frac{1}{6}$ on segment 4. The riskfree return is

$$\frac{1}{\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}} - 1 = \frac{1}{\frac{57}{60}} - 1 = \frac{3}{57}.$$

6. (a) Consider the variance of the difference of $r - r_M$

$$\begin{aligned} &\text{var}(r - r_M) \\ &= \text{var}(r) + \text{var}(r_M) - 2\text{cov}(r, r_M) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\alpha_j\sigma_{ij} + \sigma_M^2 - 2\sum_{i=1}^{n}\alpha_i\sigma_{iM}, \text{ where } \sigma_{iM} = \text{cov}(r_i, r_M). \end{aligned}$$

To minimize $\text{var}(r - r_M)$ subject to $\sum_{i=1}^{n}\alpha_i = 1$, we set up the Lagrangian

$$L = \frac{1}{2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\sigma_{ij} + \sigma_M^2 - 2\sum_{i=1}^{n}\alpha_i\sigma_{iM}\right] - \lambda\left(\sum_{i=1}^{n}\alpha_i - 1\right).$$

Differentiating L with respect to α_i and λ, we obtain

$$\sum_{j=1}^{n}\alpha_j\sigma_{ij} - \sigma_{iM} - \lambda = 0, \quad i = 1, 2, \cdots, n,$$

$$\sum_{i=1}^{n}\alpha_i = 1.$$

In matrix form:

$$\begin{aligned} \Omega\boldsymbol{\alpha} - \boldsymbol{\sigma}_M - \lambda\mathbf{1} &= 0 \\ \mathbf{1}^T\boldsymbol{\alpha} &= 1, \end{aligned}$$

where $\boldsymbol{\sigma}_M = (\sigma_{1M} \quad \sigma_{2M} \quad \cdots \quad \sigma_{nM})^T$. Assuming Ω^{-1} exists, we have

$$\boldsymbol{\alpha} = \Omega^{-1}\boldsymbol{\sigma}_M + \lambda\Omega^{-1}\mathbf{1}.$$

Applying the constraint: $\mathbf{1}^T\boldsymbol{\alpha} = 1$, we obtain

$$\mathbf{1}^T\Omega^{-1}\boldsymbol{\sigma}_M + \lambda\mathbf{1}^T\Omega^{-1}\mathbf{1} = 1$$

so that

$$\lambda = \frac{1 - \mathbf{1}^T\Omega^{-1}\boldsymbol{\sigma}_M}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}.$$

Finally, we obtain

$$\boldsymbol{\alpha} = \Omega^{-1}\boldsymbol{\sigma}_M + \frac{1 - \mathbf{1}^T\Omega^{-1}\boldsymbol{\sigma}_M}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}\Omega^{-1}\mathbf{1}.$$

(b) The modified Lagrangian is given by

$$L = \frac{1}{2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\sigma_{ij} - 2\sum_{i=1}^{n}\alpha_i\sigma_{iM} + \sigma_M^2\right] - \lambda_1\left(\sum_{i=1}^{n}\alpha_i - 1\right) - \lambda_2\left(\sum_{i=1}^{n}\alpha_i\overline{r}_i - m\right),$$

where m is the target mean. Differentiating L with respect to $\alpha_i, \lambda_1, \lambda_2$, we obtain

$$\sum_{j=1}^{n}\alpha_j\sigma_{ij} - \sigma_{iM} - \lambda_1 - \lambda_2\overline{r}_i = 0, \quad i = 1, 2, \cdots, n,$$

$$\sum_{i=1}^{n}\alpha_i = 1$$

$$\sum_{i=1}^{n}\alpha_i\overline{r}_i = m.$$

In matrix form:

$$\Omega\boldsymbol{\alpha} - \boldsymbol{\sigma}_M - \lambda_1\mathbf{1} - \lambda_2\overline{\boldsymbol{r}} = 0 \qquad (i)$$
$$\mathbf{1}^T\boldsymbol{\alpha} = 1 \qquad (ii)$$
$$\overline{\boldsymbol{r}}^T\boldsymbol{\alpha} = m. \qquad (iii)$$

We write $a = \mathbf{1}^T\Omega^{-1}\mathbf{1}, b = \mathbf{1}^T\Omega^{-1}\overline{\boldsymbol{r}}, c = \overline{\boldsymbol{r}}^T\Omega^{-1}\overline{\boldsymbol{r}}, s_1 = \mathbf{1}^T\Omega^{-1}\boldsymbol{\sigma}_M, s_2 = \overline{\boldsymbol{r}}^T\Omega^{-1}\boldsymbol{\sigma}_M$. Assuming Ω^{-1} exists, eq. (i) can be expressed as

$$\boldsymbol{\alpha} = \Omega^{-1}\boldsymbol{\sigma}_M + \lambda_1\Omega^{-1}\mathbf{1} + \lambda_2\Omega^{-1}\overline{\boldsymbol{r}}. \qquad (iv)$$

Invoking conditions (ii) and (iii), we obtain the following pair of algebraic equations for λ_1 and λ_2:

$$\begin{aligned} 1 &= s_1 + \lambda_1 a + \lambda_2 b \\ m &= s_2 + \lambda_1 b + \lambda_2 c. \end{aligned}$$

Solving for λ_1 & λ_2:

$$\lambda_1 = \frac{\begin{vmatrix} 1-s_1 & b \\ m-s_2 & c \end{vmatrix}}{\begin{vmatrix} a & b \\ b & c \end{vmatrix}} = \frac{c(1-s_1) - b(m-s_2)}{ac-b^2},$$
$$\lambda_2 = \frac{\begin{vmatrix} a & 1-s_1 \\ b & m-s_2 \end{vmatrix}}{\begin{vmatrix} a & b \\ b & c \end{vmatrix}} = \frac{a(m-s_2) - b(1-s_1)}{ac-b^2}.$$

Both λ_1 and λ_2 are linear functions of m. We are able to express $\boldsymbol{\alpha}$ in terms of m [see eq. (iv)].

7. (a) Recall $\boldsymbol{w}_0 = \dfrac{\Omega^{-1}\mathbf{1}}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}, \sigma_0^2 = \boldsymbol{w}_0^T\Omega\boldsymbol{w}_0 = \dfrac{1}{\mathbf{1}^T\Omega^{-1}\mathbf{1}}$, so that

$$\text{cov}(r_0, r_1) = \boldsymbol{w}_0^T\Omega\boldsymbol{w}_1 = \frac{\mathbf{1}^T\Omega^{-1}\Omega\boldsymbol{w}_1}{\mathbf{1}^T\Omega^{-1}\mathbf{1}} = \frac{1}{\mathbf{1}^T\Omega^{-1}\mathbf{1}} = \sigma_0^2$$

giving $A = 1$. Consider the variance σ_α^2

$$\begin{aligned} \sigma_\alpha^2 &= \text{cov}((1-\alpha)r_0 + \alpha r_1, (1-\alpha)r_0 + \alpha r_1) \\ &= (1-\alpha)^2\sigma_0^2 + 2\alpha(1-\alpha)\text{cov}(r_0, r_1) + \alpha^2\sigma_1^2 \\ &= (1-\alpha)^2\sigma_0^2 + 2\alpha(1-\alpha)\sigma_0^2 + \alpha^2\sigma_1^2 \\ &= \sigma_0^2 + \alpha^2(\sigma_1^2 - \sigma_0^2). \end{aligned}$$

The result agrees with the intuition that variations of the variance of the given portfolio around $\alpha = 0$ should be second order in α.

(b) Writing $\boldsymbol{r} = (r_1 \cdots r_N)^T$, consider

$$\begin{aligned} \text{cov}(r_1, r_z) &= \text{cov}(\boldsymbol{w}_1^T\boldsymbol{r}, [(1-\alpha)\boldsymbol{w}_0 + \alpha\boldsymbol{w}_1]^T\boldsymbol{r}) \\ &= \text{cov}(\boldsymbol{w}_1^T\boldsymbol{r}, (1-\alpha)\boldsymbol{w}_0^T\boldsymbol{r} + \alpha\boldsymbol{w}_1^T\boldsymbol{r}) \\ &= \text{cov}(\boldsymbol{w}_1^T\boldsymbol{r}, (1-\alpha)\boldsymbol{w}_0^T\boldsymbol{r}) + \text{cov}(\boldsymbol{w}_1^T\boldsymbol{r}, \alpha\boldsymbol{w}_1^T\boldsymbol{r}) \\ &= (1-\alpha)\sigma_0^2 + \alpha\sigma_1^2. \end{aligned}$$

Setting $\text{cov}(r_1, r_z) = 0$, we obtain

$$0 = (1-\alpha)\sigma_0^2 + \alpha\sigma_1^2$$

giving

$$\alpha = -\frac{\sigma_0^2}{\sigma_1^2 - \sigma_0^2} < 0.$$

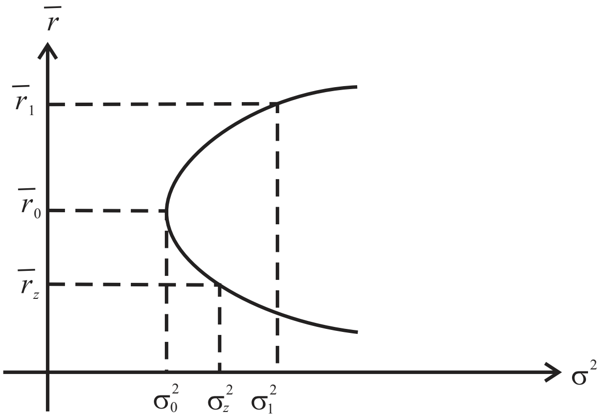
(c) We have

$$\overline{r}_z = (1-\alpha)\overline{r}_0 + \alpha\overline{r}_1 = \overline{r}_0 + \alpha(\overline{r}_1 - \overline{r}_0)$$

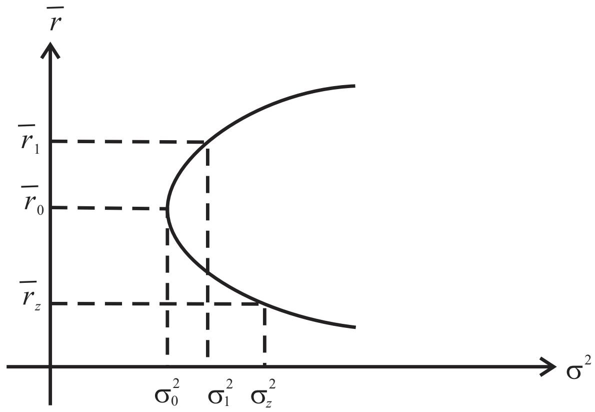
so that $\overline{r}_z < \overline{r}_0$ (since $\alpha < 0$ and $\overline{r}_1 - \overline{r}_0 > 0$). Now,

$$\begin{aligned}\sigma_z^2 = \text{var}(r_z) &= (1-\alpha)^2\sigma_0^2 + \alpha^2\sigma_1^2 + 2\alpha(1-\alpha)\text{cov}(r_0, r_1) \\ &= (1-\alpha)^2\sigma_0^2 + 2\alpha(1-\alpha)\sigma_0^2 + \alpha^2\sigma_1^2 = \frac{\sigma_0^2\sigma_1^2}{\sigma_1^2 - \sigma_0^2}.\end{aligned}$$

Note that Portfolio z is a minimum variance portfolio but it is not efficient.



(i) When $2\sigma_0^2 < \sigma_1^2$, then $\sigma_z^2 < \sigma_1^2$



(ii) When $2\sigma_0^2 > \sigma_1^2$, then $\sigma_z^2 > \sigma_1^2$