MATH 4512 – Fundamentals of Mathematical Finance

Solution to Homework Four

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1. If outcome j occurs, then the corresponding gain is given by

$$G_j = \sum_{i=1}^{m} g_{ij}\alpha_i,$$

where $\alpha_i = \dfrac{\dfrac{1}{1+d_i}}{1 - \displaystyle\sum_{i=1}^{m}\frac{1}{1+d_i}}$ and $g_{ij} = \begin{cases} d_i & \text{if } j = i \\ -1 & \text{if } j \neq i \end{cases}$.

We then have

$$\begin{aligned}
G_j &= g_{jj}\alpha_j - \sum_{\substack{i=1 \\ i\neq j}}^{m} \alpha_i \\
&= (1+g_{jj})\alpha_j - \sum_{i=1}^{m}\alpha_i \\
&= (1+d_j)\alpha_j - \sum_{i=1}^{m}\alpha_i \\
&= \frac{1}{1-\displaystyle\sum_{i=1}^{m}\frac{1}{1+d_i}} - \frac{\displaystyle\sum_{i=1}^{m}\frac{1}{1+d_i}}{1-\displaystyle\sum_{i=1}^{m}\frac{1}{1+d_i}} \\
&= 1, \qquad \text{for } j = 1, 2, \cdots, m.
\end{aligned}$$

Therefore, the betting game will always yield a gain of exactly 1.

2. Suppose we hold $(\alpha_1, \alpha_2, \alpha_3)$ units of the three securities, with $\displaystyle\sum_{i=1}^{3}\alpha_i \leq 1, \alpha_i \geq 0$. In this problem, we can set $\displaystyle\sum_{i=1}^{3}\alpha_i = 1$ since the random returns are greater than one under all states of world. Using the log-utility criterion, the growth factor is

$$\begin{aligned}
m = E[\ln R] &= \frac{1}{2}\ln(4\alpha_1 + 2\alpha_2 + 3(1-\alpha_1-\alpha_2)) + \frac{1}{2}\ln(2\alpha_1 + 4\alpha_2 + 3(1-\alpha_1-\alpha_2)) \\
&= \frac{1}{2}\ln(3+\alpha_1-\alpha_2) + \frac{1}{2}\ln(3-\alpha_1+\alpha_2).
\end{aligned}$$

Applying the first order condition, we obtain

$$\begin{aligned}
\frac{\partial m}{\partial \alpha_1} &= \frac{1}{2}\frac{1}{3+\alpha_1-\alpha_2} - \frac{1}{2}\frac{1}{3-\alpha_1+\alpha_2} = 0 \quad &\Leftrightarrow \quad & 3-\alpha_1+\alpha_2 = 3+\alpha_1-\alpha_2 \\
& &\Leftrightarrow \quad & \alpha_1 = \alpha_2; \\
\frac{\partial m}{\partial \alpha_2} &= \frac{1}{2}\frac{(-1)}{3+\alpha_1-\alpha_2} + \frac{1}{2}\frac{1}{3-\alpha_1+\alpha_2} = 0 \quad &\Leftrightarrow \quad & 3+\alpha_1-\alpha_2 = 3-\alpha_1+\alpha_2 \\
& &\Leftrightarrow \quad & \alpha_1 = \alpha_2.
\end{aligned}$$

Two possible optimal strategies are $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$.

In fact, for any portfolio choice with $\alpha_2 = \alpha_1, \alpha_3 = 1 - 2\alpha_1, \alpha_1 \geq 0$, the portfolio's return is either

$$4\alpha_1 + 2\alpha_2 + 3\alpha_3 = 4\alpha_1 + 2\alpha_1 + 3(1-2\alpha_1) = 3 \text{ if the first state occurs}$$

or

$$2\alpha_1 + 4\alpha_2 + 3\alpha_3 = 2\alpha_1 + 4\alpha_2 + 3(1-2\alpha_1) = 3 \text{ if the second state occurs.}$$

The optimal strategies always yield a return of 3 for all values of α_1.

3. Recall that the class of the power utility functions includes the logarithm utility since

$$\lim_{\gamma \to 0^+} \left[\frac{1}{\gamma} x^\gamma - \frac{1}{\gamma} \right] = \ln x.$$

This class of functions has the same recursive property as the log utility; that is, the structure is preserved from period to period. This is seen from

$$\begin{aligned}
E[U(X_k)] &= \frac{1}{\gamma} E[(R_k R_{k-1} \cdots R_1 X_0)^\gamma] = \frac{1}{\gamma} E[R_k^\gamma R_{k-1}^\gamma \cdots R_1^\gamma] X_0^\gamma \\
&= \frac{1}{\gamma} E[R_k^\gamma] E[R_{k-1}^\gamma] \cdots E[R_1^\gamma] X_0^\gamma
\end{aligned}$$

where the last equality follows from the fact that the expected value of a product of independent random variables is equal to the product of their expected values. To maximize $E[U(X_k)]$ with a fixed-proportions strategy it is only necessary to maximize $E[(R_1 X_0)^\gamma]$. Therefore, again if one wants to maximize $E[U(X_k)]$, one needs only to maximize $E[U(X_1)]$.

4. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in B$.

Reflexivity: $x_1 = x_1$ and $y_1 \geq y_1$ so that $(x_1, y_1) \succeq (x_1, y_1)$

Comparability: If $x_1 > x_2$, then $(x_1, y_1) \succeq (x_2, y_2)$.

If $x_2 > x_1$, then $(x_2, y_2) \succeq (x_1, y_1)$.

If $x_1 = x_2$, then

if $y_1 \geq y_2$, then $(x_1, y_1) \succeq (x_2, y_2)$

if $y_2 \geq y_1$, then $(x_2, y_2) \succeq (x_1, y_1)$.

Transitivity: Given $(x_1, y_1) \succeq (x_2, y_2), (x_2, y_2) \succeq (x_3, y_3)$. If $x_1 = x_2 > x_3$, with $y_1 \geq y_2$, then $x_1 > x_3$ so that $(x_1, y_1) \succeq (x_3, y_3)$.

If $x_1 > x_2 = x_3$, with $y_2 \geq y_3$, then $x_1 > x_3$ so that $(x_1, y_1) \succeq (x_3, y_3)$.

If $x_1 = x_2 = x_3$, with $y_1 \geq y_2 \geq y_3$, then $x_1 = x_3, y_1 \geq y_3$, so that $(x_1, y_1) \succeq (x_3, y_3)$.

If $x_1 > x_2 > x_3$, then $x_1 > x_3$, so that $(x_1, y_1) \succeq (x_3, y_3)$.

5. Suppose $(x_1, y_1) \succeq (x_2, y_2)$.

   Case I: $x_1 > x_2$

   1: $\alpha(x_1, y_1) + (1-\alpha)(x_2, y_2) = (1 + \alpha(x_1 - x_2), 1 + \alpha(y_1 - y_2))$
   2: $\beta(x_1, y_1) + (1-\beta)(x_2, y_2) = (1 + \beta(x_1 - x_2), 1 + \beta(y_1 - y_2))$.
   $\alpha > \beta \Leftrightarrow 1 + \alpha(x_1 - x_2) > 1 + \beta(x_1 - x_2)$ and since $x_1 - x_2 > 0$, so

$$\alpha > \beta \Leftrightarrow \alpha(x_1, y_1) + (1-\alpha)(x_2, y_2) \succeq \beta(x_1, y_1) + (1-\beta)(x_2, y_2).$$

   Case II: $x_1 = x_2, y_1 \geq y_2$

   $\alpha > \beta \Leftrightarrow 1 + \alpha(x_1 - x_2) = 1 = 1 + \beta(x_1 - x_2)$. However, we have $1 + \alpha(y_1 - y_2) \geq 1 + \beta(y_1 - y_2)$ and since $y_1 - y_2 \geq 0$, so

$$\alpha > \beta \Leftrightarrow \alpha(x_1, y_1) + (1-\alpha)(x_2, y_2) \succeq \beta(x_1, y_1) + (1-\beta)(x_2, y_2).$$

6. Let $u(x, y) = \ln(x + y)$, and consider $(1, 0)$ and $(0, 1) \in B$, we have $(1, 0) \succ (0, 1)$. But $u(1, 0) = \ln 1 = u(0, 1)$. Hence, u cannot be a utility function representing the Dictionary Order.

7. Consider the HARA class of utility functions

$$\begin{aligned} U(x) &= \frac{1-\gamma}{\gamma}\left(\frac{ax}{1-\gamma} + b\right)^{\gamma} \\ &= \left(\left(\frac{1-\gamma}{\gamma}\right)^{\frac{1}{\gamma}} \frac{a}{1-\gamma} x + \left(\frac{1-\gamma}{\gamma}\right)^{\frac{1}{\gamma}} b\right)^{\gamma} \end{aligned} \tag{1}$$

   (a) Let $a = (1-\gamma)\left(\dfrac{\gamma}{1-\gamma}\right)^{\frac{1}{\gamma}}, b \to 0^+$. Then

$$U(x) = x^{\gamma} \to x \quad \text{as} \quad \gamma \to 1.$$

   (b) Let $\gamma = 2$. Then
   $U(x) = -\frac{1}{2}(-ax + b)^2 = -\frac{b^2}{2} + abx - \frac{a^2}{2}x^2$ which is equivalent with $V(x) = abx - \frac{a^2}{2}x^2$ since they differ by only a constant. Now let $a^2 = c, b = 1/a$,

$$U(x) = x - \frac{1}{2}cx^2.$$

   (c) Note that $\left(1 + \frac{\alpha x}{n}\right)^n \to e^{\alpha x}$ as $n \to \infty$. Let $b = \left(\frac{\gamma}{1-\gamma}\right)^{\frac{1}{\gamma}}.a = \frac{1-\gamma}{\gamma}\left(\frac{\gamma}{1-\gamma}\right)^{\frac{1}{\gamma}}(-\alpha)$. Then

$$U(x) = \left(\frac{-\alpha x}{\gamma} + 1\right)^{\gamma} \to e^{-\alpha x} \quad \text{as} \quad \gamma \to \infty.$$

(d) Let $b \to 0^+, a = c^{1/\gamma}(1-\gamma)\left(\frac{r}{1-\gamma}\right)^{1/\gamma}$. Then

$$U(x) = (c^{1/\gamma}c)^\gamma = cx^\gamma.$$

(e) Take $c = \frac{1}{\gamma}$ from part (d). $U(x)$ is equivalent to $\frac{x^\gamma - 1}{\gamma} \to \ln x$ as $\gamma \to 0^+$.

8. By setting $U(c) = E[U(x)]$, where c is the certainty equivalent, we obtain

$$U'(x)(c - \overline{x}) \approx \frac{U''(\overline{x})}{2}\text{var}(x)$$

so that

$$c \approx \overline{x} + \frac{U''(\overline{x})}{2U'(\overline{x})}\text{var}(x).$$

9. (a)

Outcomes (%)	F_A	$\int F_A$	F_B	$\int F_B$	F_C	$\int F_C$
4	0.7	0.7	0	0	0	0
5	0.7	1.4	0.1	0.1	0.1	0.1
6	0.8	2.2	0.3	0.4	0.1	0.2
7	0.9	3.1	0.3	0.7	0.2	0.4
8	0.9	4.0	0.4	1.1	0.4	0.8
9	1.0	5.0	0.6	1.7	0.6	1.4
10	1.0	6.0	1.0	2.7	0.6	2.0
11	1.0	7.0	1.0	3.7	1.0	3.0

$F_A > F_B > F_C$, for all outcomes; $\int F_A \geq \int F_B \geq \int F_C$, for all outcomes.

(b) Consider the geometric average of the 3 investments:

$$\begin{aligned}
\overline{X}_{geo}(A) &= 3^{0.4}4^{0.3}6^{0.1}7^{0.1}9^{0.1} = 4.2581 \\
\overline{X}_{geo}(B) &= 5^{0.1}6^{0.2}8^{0.1}9^{0.2}10^{0.4} = 8.0665 \\
\overline{X}_{geo}(C) &= 5^{0.1}7^{0.1}8^{0.2}9^{0.2}11^{0.4} = 8.7585.
\end{aligned}$$

Hence, C is preferred to B and A, and B is preferred to A, according to the geometric mean criterion.

10. We would like to show that $F(x)$ dominates $G(x)$ by the 3^{th} order stochastic dominance (TSD) if

$$\text{(i)} \quad \int_a^x \int_a^t F(y)\,dy\,dt \leq \int_a^x \int_a^t G(y)\,dy\,dt \qquad \text{for all } x \in [a,b]$$

$$\text{and} \quad \text{(ii)} \quad \int_a^b F(t)\,dt \leq \int_a^b G(t)\,dt.$$

According to the above definition, $F(x)$ dominates $G(x)$ in TSD if and only if

$$\int_c u(x)\,dF(x) \geq \int_c u(x)\,dG(x) \qquad (*)$$

for all utility functions with $u'(x) > 0, u''(x) < 0$ and $u'''(x) > 0$ for all $x \in C$, where C is the set of all possible outcomes.

Let a and b be the smallest and largest values F and G can take on, where $F(a) = G(a) = 0$, $F(b) = G(b) = 1$. Consider

$$\int_a^b u(x)d(F(x) - G(x)) = u(x)[F(x) - G(x)]\Big|_a^b - \int_a^b u'(x)[F(x) - G(x)]\,dx$$
$$= -\int_a^b u'(x)[F(x) - G(x)]\,dx$$
$$= -u'(x)\int_a^x [F(y) - G(y)]\,dy\Big|_a^b + \int_a^b u''(x)\int_a^x [F(y) - G(y)]\,dy\,dx$$
$$= -u'(b)\int_a^b [F(y) - G(y)]\,dy + \int_a^b u''(x)\int_a^x [F(y) - G(y)]\,dy\,dx.$$

By parts integration, we obtain

$$\int_a^b u''(x)\int_a^x [F(y) - G(y)]\,dy\,dx = u''(x)\int_a^x\int_a^t [F(y) - G(y)]\,dy\,dt\Big|_a^b$$
$$- \int_a^b u'''(x)\int_a^y\int_a^t [F(y) - G(y)]\,dy\,dt\,dx.$$

By property (i), we have

$$\int_a^b u''(x)\int_a^y [F(y) - G(y)]\,dy\,dx \geq 0.$$

Here, we assume that both

$$\int_a^x\int_a^t F(y)\,dy\,dt \quad \text{and} \quad \int_a^x\int_a^t G(y)\,dy\,dt$$

are continuous function of x, otherwise (*) holds only at discontinuous point, since $u''(x) < 0$ and $u'''(x) > 0$. Also, by (ii) and $u'(x) > 0$, we obtain

$$-u'(b)\int_a^b [F(y) - G(y)]\,dy \geq 0.$$

Hence, the combination of properties (i) and (ii) implies TSD.