MATH 4512 — Fundamentals of Mathematical Finance

Topic One — Bond portfolio management and immunization

1.1 Duration measures and convexity

1.2 Horizon rate of return: return from the bond investment over a time horizon

1.3 Immunization of bond investment

1.4 Optimal management and dynamic programming

1.1 Duration measures and convexity

Fixed coupon bond

Let i be the interest rate applicable to the cash flows arising from a fixed coupon bond, giving constant coupon c paid at times $1, 2, \ldots, T$ and par amount B_T paid at maturity T.

The fair bond value B is the sum of the coupons and par in present value, where

$$
\begin{aligned}
B &= \frac{c}{1+i} + \cdots + \frac{c}{(1+i)^T} + \frac{B_T}{(1+i)^T} \\
&= c\left[\frac{1}{1+i}\frac{1-\frac{1}{(1+i)^T}}{1-\frac{1}{1+i}}\right] + \frac{B_T}{(1+i)^T} = \frac{c}{i}\left[1-\frac{1}{(1+i)^T}\right] + \frac{B_T}{(1+i)^T}.
\end{aligned}
$$

Annuity factor and present value factor

The discrete coupons paid at times $1, 2, \ldots, T$ is called an annuity stream over the time period $[0, T]$. We define the annuity factor (i, T) to be

$$\text{annuity factor } (i, T) = \frac{1}{i}\left[1 - \frac{1}{(1+i)^T}\right].$$

The present value factor over $[0, T]$ at interest rate i is defined by

$$\text{PV factor } (i, T) = \frac{1}{(1+i)^T}.$$

- When $T \to \infty$, the annuity factor becomes $\frac{1}{i}$. For example, when $i = 5\%$, one needs to put \$20 upfront in order to generate a perpetual stream of annuity of \$1 paid annually.

- For an annuity of finite time horizon T, it can be visualized as the difference of two perpetual annuities starting on today and time T. The present value of the perpetual starting at time T is $\frac{1}{i}\frac{1}{(1+i)^T}$.

Par bond

Suppose the coupon rate is set to be the interest rate so that the coupon amount $c = iB_T$, then

$$B = \frac{iB_T}{i}\left[1 - \frac{1}{(1+i)^T}\right] + \frac{B_T}{(1+i)^T} = B_T.$$

This is called a *par bond*, so named since the bond value is equal to the par of the bond. Obviously, if the coupon rate is above (below) the discount rate, then the bond value B is above (below) the par value B_T.

Yield to maturity

Yield to maturity (YTM) of a bond is the rate of return anticipated on a bond if held until maturity. The yield to maturity is determined by finding the rate of return of the bond such that the sum of cash flows discounted at the rate of return is equal to the *observed bond price*. Here, the value of B is given, we find YTM i.

Duration

The duration of a bond is the weighted average of the times of payment of all the cash flows generated by the bond, the weights being the proportional shares of the bond's cash flows in the bond's present value.

Macauley's duration:

Let i denote the yield to maturity (YTM) of the bond. Bond duration is

$$D = 1\frac{c/B}{1+i} + 2\frac{c/B}{(1+i)^2} + \cdots + T\frac{(c+B_T)/B}{(1+i)^T}$$

$$= \frac{1}{B}\sum_{t=1}^{T} \frac{tc_t}{(1+i)^t}. \qquad (D1)$$

where c_t is the cash flow at time t. Note that $c_T = c + B_T$.

Measure of a bond's sensitivity to change in interest rate

Starting from

$$B = \sum_{t=1}^{T} c_t (1+i)^{-t},$$

$$\frac{dB}{di} = \sum_{t=1}^{T} (-t) c_t (1+i)^{-t-1} = -\frac{1}{1+i} \sum_{t=1}^{T} t c_t (1+i)^{-t};$$

$$\frac{1}{B}\frac{dB}{di} = -\frac{1}{1+i} \sum_{t=1}^{T} \frac{t c_t (1+i)^{-t}}{B} = -\frac{D}{1+i}.$$

Modified duration $= D_m = \frac{\text{duration}}{1+i}$

$$\frac{\Delta B}{B} \approx \frac{dB}{B} = -D_m \, di \quad \text{and} \quad \text{var}\left(\frac{dB}{B}\right) = D_m^2 \text{var}(di).$$

The standard deviation of the relative change in the bond price is a linear function of the standard deviation of the changes in interest rates, the coefficient of proportionality is the modified duration.

Suppose a bond is at par, its coupon is 9%, so YTM = 9%. The duration is found to be 6.99. Suppose that the interest rate (yield) increases by 1%, then the relative change in bond value is

$$\frac{\Delta B}{B} \approx \frac{dB}{B} = -\frac{D}{1+i}\,di = -\frac{6.99}{1.09} \times 1\% = -6.4\%.$$

How good is the linear approximation? For exact calculation, we have

$$\frac{\Delta B}{B} = \frac{B(10\%) - B(9\%)}{B(9\%)} = \frac{93.855 - 100}{100} = -6.145\%.$$

Later, we show how to obtain the quadratic approximation (an improvement over the linear approximation) with the inclusion of convexity (related to $\frac{d^2B}{di^2}$).

Example

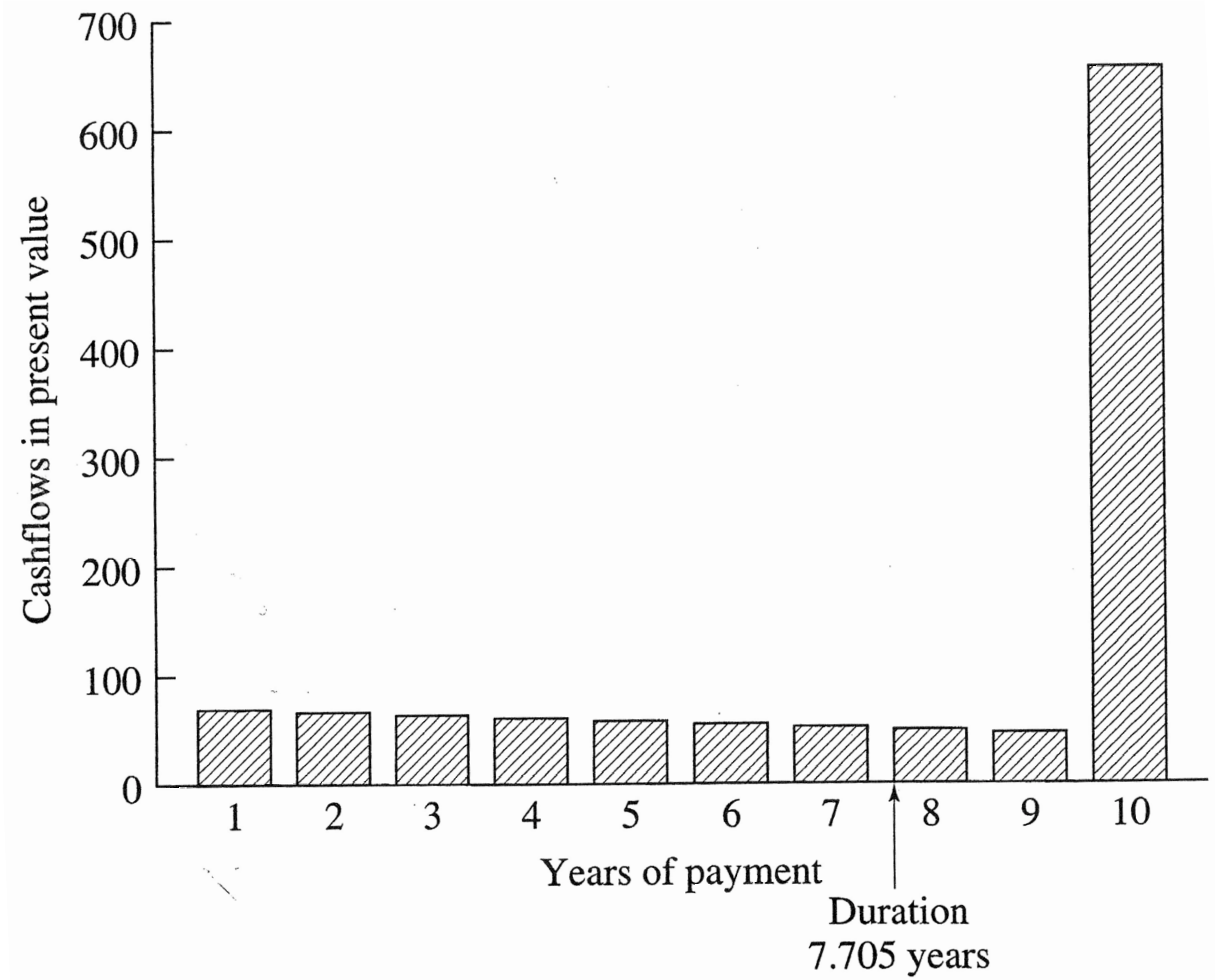
A bond with annual coupon 70, par 1000, and interest rate 5%; duration is found to be 7.7 years, modified duration $= \frac{7.7}{1.05} = 7.33$ yr.

A change in yield from 5% to 6% or 4% entails a relative change in the bond price approximately -7.33% or $+7.33\%$, respectively. The modified duration is seen to be the more appropriate proportional factor.

Calculation of the duration of a bond with a 7% coupon rate for $i = 5\%$

(1) Time of payment t	(2) Cash flow in current value	(3) Cash flows in present value (i = 5%)	(4) Share of cash flows in present value in bond's price	(5) Weighted time of payment (col. 1 X col. 4)
1	70	66.67	0.0577	0.0577
2	70	63.49	0.0550	0.1100
3	70	60.47	0.0524	0.1571
4	70	57.59	0.0499	0.1995
5	70	54.85	0.0475	0.2375
6	70	52.24	0.0452	0.2715
7	70	49.75	0.0431	0.3016
8	70	47.38	0.0410	0.3283
9	70	45.12	0.0391	0.3517
10	1070	656.89	0.5690	5.6901
Total	1700	1154.44	1	7.705 = duration

Bond value = 1154.44 = sum of cash flows in present value.



Duration of a bond as the center of gravity of its cash flows in present value (coupon: 7%; interest rate: 5%).

Duration in terms of coupon rate, maturity and interest rate

Recall $B = \frac{c}{i}\left[1 - \frac{1}{(1+i)^T}\right] + \frac{B_T}{(1+i)^T}$. We express $\dfrac{B}{B_T}$ in terms of $\dfrac{c}{B_T}$, i and T as

$$\frac{B}{B_T} = \frac{1}{i}\left\{\frac{c}{B_T}\left[1 - \frac{1}{(1+i)^T}\right] + \frac{i}{(1+i)^T}\right\}.$$

$$\begin{aligned}\frac{d\ln\left(\frac{B}{B_T}\right)}{di} &= \frac{d\ln B}{di} = \frac{1}{B}\frac{dB}{di} \\ &= -\frac{1}{i} + \frac{(c/B_T)T(1+i)^{-T-1} + (1+i)^{-T} + i(1+i)^{-T-1}(-T)}{(c/B_T)[1-(1+i)^{-T}] + i(1+i)^{-T}}\end{aligned}$$

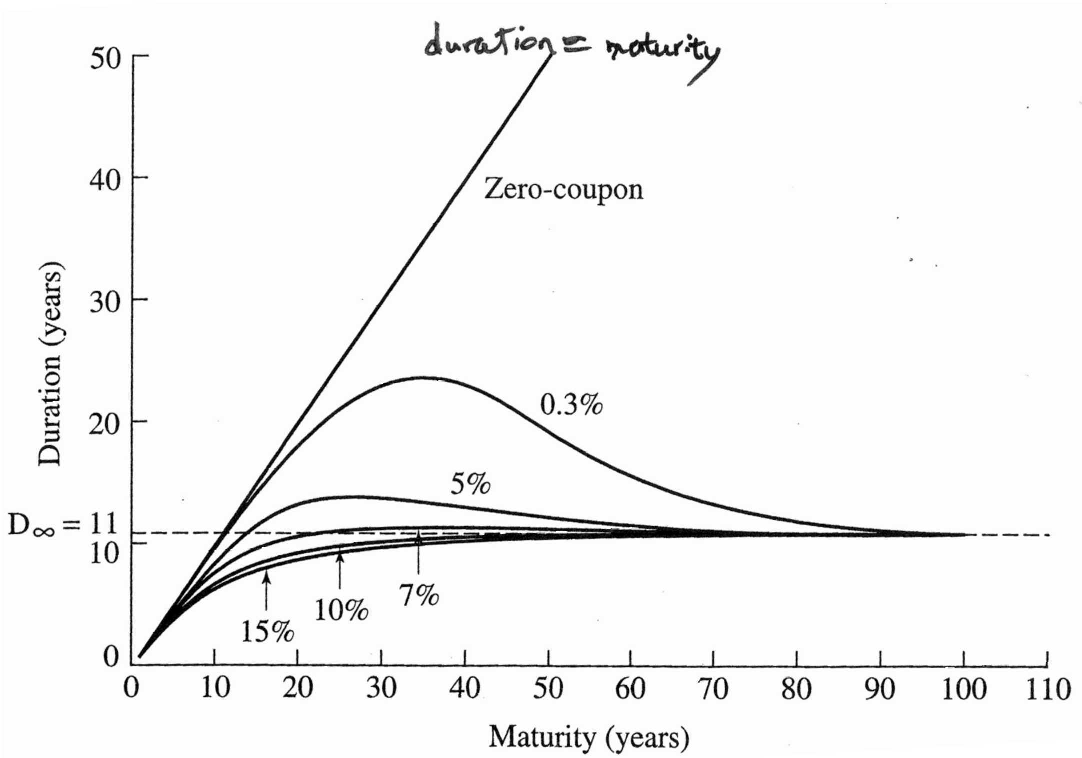
so that the duration D is related to c/B_T, i and T as

$$D = -\frac{1+i}{B}\frac{dB}{di} = 1 + \frac{1}{i} + \frac{T\left(i - \frac{c}{B_T}\right) - (1+i)}{\frac{c}{B_T}[(1+i)^T - 1] + i}. \qquad (D2)$$

- The impact of the coupon rate c/B_T and maturity T for a fixed value of i on duration D can be deduced from the last term.

Term structure of duration

- Obviously, $D = T$ when $c = 0$ since there is only one cash flow of par paid at maturity T.

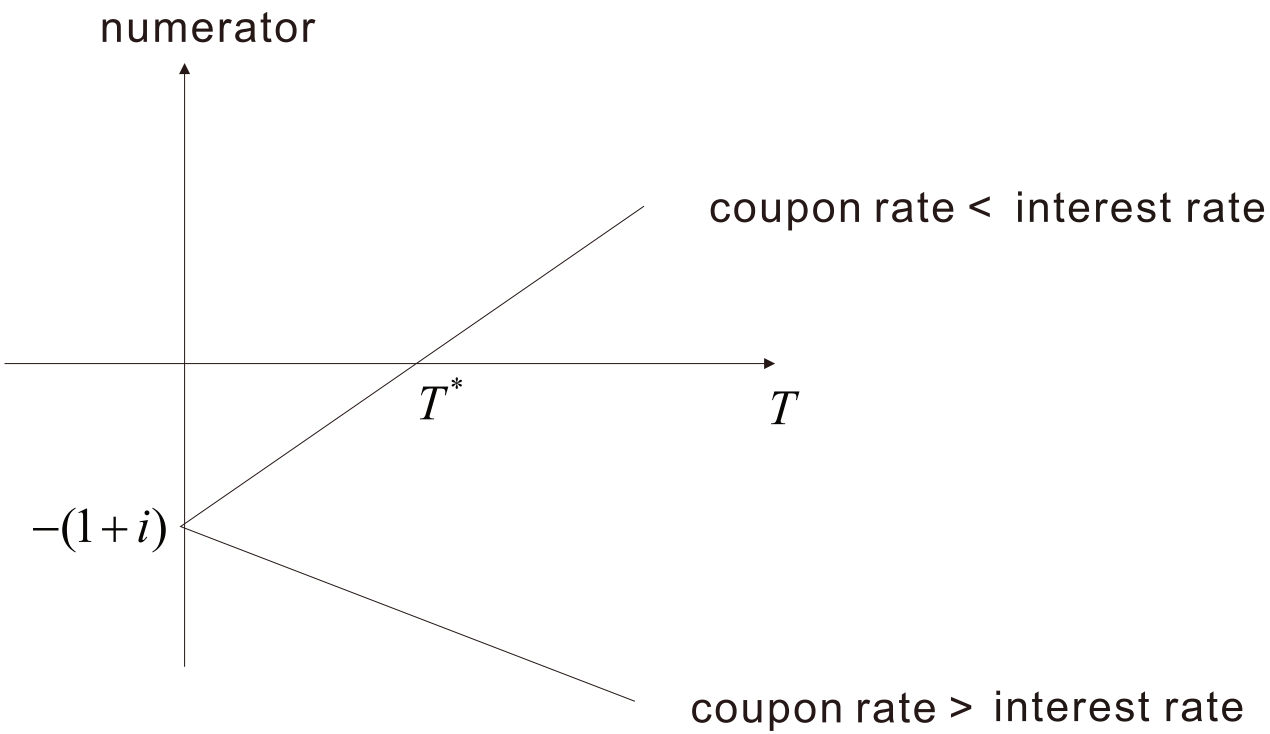


Duration of a bond as a function of its maturity for various coupon rates ($i = 10\%$).

Note that the numerator $= T\left(i - \frac{c}{B_T}\right) - (1+i)$ is a linear function in T. When the coupon rate $\frac{c}{B_T}$ is less than i, the numerator may change sign at T^* where

$$T^*\left(i - \frac{c}{B_T}\right) = 1 + i.$$

That is, D may assume value above $1 + \frac{1}{i}$ when $T > T^*$. When $\frac{c}{B_T} > i$, the numerator is always negative so D always stays below $1 + \frac{1}{i}$.



Impact of coupon rate on duration

With an increase in the coupon rate c/B_T, should there always be a decrease in duration for sure?

- From eq. (D2), it is seen that the numerator (denominator) in the last term decreases (increases) with increasing c/B_T. Hence, D always decreases with increasing c/B_T.

- Intuitively, when the coupon rate increases, the weights will be tilted towards the left, and the center of gravity will move to the left.

Perpetual bond – infinite maturity ($T \to \infty$)

- For a perpetual bond, $B = c/i$. The modified duration D^* is $-\frac{1}{B}\frac{dB}{di} = \frac{c}{i^2}\Big/\frac{c}{i} = \frac{1}{i}$. This gives $D = \frac{1+i}{i} = 1 + \frac{1}{i}$.
- Alternatively, we observe from eq.(D2) on P.11 that

$$\lim_{T\to\infty} \frac{T\left(i - \frac{c}{B_T}\right) - (1+i)}{\frac{c}{B_T}[(1+i)^T - 1] + i} = 0.$$

  This leads to

$$D \to 1 + \frac{1}{i} \quad \text{as} \quad T \to \infty. \qquad (D3)$$

  When $i = 10\%$, we have $D \to 11$ as $T \to \infty$.

In Qn 1 of HW 1, when there are m compounding periods in one year, we have $D \to \frac{1}{m} + \frac{1}{i}$ as $T \to \infty$. When $m \to \infty$, which corresponds to continuous compounding, we obtain

$$D \to \frac{1}{i} \text{ as } T \to \infty.$$

Relationship between duration and maturity

1. For zero-coupon bonds, duration is always equal to maturity.

   For all coupon-bearing bonds, we observe

   $$\text{duration} \to 1 + \frac{1}{i} \text{ when maturity increases infinitely.}$$

   The limit is independent of the coupon rate.

2. Coupon rate $\geq$ interest rate (bonds above par)

   An increase in maturity entails an increase in duration towards the limit $1 + \frac{1}{i}$.

3. Coupon rate $<$ interest rate (bonds below par)

   When maturity increases, duration first increases, pass through a maximum and decreases toward the limit $1 + \frac{1}{i}$.

Change of duration with respect to change in interest rate

Intuitively, since the discount factor for the cash flow at time t is $(1+i)^{-t}$, an increase in i will move the center of gravity to the left, and the duration is reduced. Actually

$$\frac{dD}{di} = -\frac{S}{1+i},$$

where S is the dispersion or weighted variance of the payment times of the bond. The respective weight is the present value of the cash flow at the corresponding payment time.

Proof

Starting from

$$D = \frac{1}{B}\sum_{t=1}^{T} tc_t(1+i)^{-t}$$

$$\begin{aligned}
\frac{dD}{di} &= -\frac{1}{B^2}\left[\sum_{t=1}^{T} t^2 c_t(1+i)^{-t-1}B(i) + \sum_{t=1}^{T} tc_t(1+i)^{-t}B'(i)\right] \\
&= -\frac{1}{1+i}\left[\sum_{t=1}^{T}\frac{t^2c_t(1+i)^{-t}}{B(i)} + \underbrace{(1+i)\frac{B'(i)}{B(i)}}_{-D}\underbrace{\frac{\sum_{t=1}^{T} tc_t(1+i)^{-t}}{B(i)}}_{D}\right] \\
&= -\frac{1}{1+i}\left[\frac{\sum_{t=1}^{T} t^2c_t(1+i)^{-t}}{B(i)} - D^2\right].
\end{aligned}$$

If we write

$$w_t = \frac{c_t(1+i)^{-t}}{B(i)}$$

so that

$$\sum_{t=1}^{T} w_t = 1 \quad \text{and} \quad D = \sum_{t=1}^{T} t w_t.$$

Here, w_t is the share of the bond's cash flow c_t (in the present value) in the bond's value. The bracket term becomes

$$\sum_{t=1}^{T} t^2 w_t - D^2 = \sum_{t=1}^{T} w_t (t-D)^2,$$

which is equal to the weighted average of the squares of the difference between the times t and their average D.

We obtain

$$\frac{dD}{di} = -\frac{1}{1+i} \sum_{t=1}^{T} w_t (t-D)^2 = -\frac{S}{1+i},$$

which is always negative.

Fair holding-period return

To illustrate built-in capital gains or losses, suppose a bond was issued several years ago when the interest rate was 7%. Suppose the bond was sold at par, then the bond's annual coupon rate was thus set at 7%. Let the par value be $\$1,000$.

We will suppose for simplicity that the bond pays its coupon annually. Now, with 3 years left in the bond's life, the interest rate is 8% per year. The bond's market price is the present value of the remaining annual coupons plus payment of par value. That present value is

$$\$70 \times \text{Annuity factor}(8\%, 3) + \$1,000 \times \text{PV factor}(8\%, 3) = \$974.23$$

which is less than par value.

One year later, after the next coupon is paid, the bond would sell at

$$\$70 \times \text{Annuity factor}(8\%, 2) + \$1,000 \times \text{PV factor}(8\%, 2) = \$982.17$$

thereby yielding a capital gain over the year of $\$982.17 - \$974.23 = \$7.94$.

An increase in bond value after one year is expected since the disadvantage of receiving coupon at the rate of 7% while the interest rate is 8% is diminished with the passage of one year.

If an investor had purchased the bond at \$974.23, the total return over the year would equal the coupon payment plus capital gain, or $\$70 + \$7.94 = \$77.94$. This represents a rate of return of $\$77.94/\$974.23$, or 8%, exactly the current rate of return available elsewhere in the market.

In an efficient financial market, the rate of return of holding bonds with different coupon rates would be the same. We have neglected default risk and liquidity risk that are typical in bonds investment. Mathematically, we have

$$i = \frac{c}{B} + \frac{\Delta B}{B}.$$

Here, c/B is called the direct rate of return of the bond (not to be confused with the coupon rate c/B_T).

Mystery behind duration

Recall that suppose the constant interest rate r is compounded continuously over $[0, T]$, then the growth factor is e^{rT}. This arises from the solution to the differential equation for the money market account M, where

$$dM = rM dt, \quad M(0) = 1.$$

Observe that

$$\begin{aligned} \int_0^T \frac{dM}{M} &= \int_0^T r \ dt \\ \ln \frac{M(T)}{M(0)} &= rT \end{aligned}$$

so that

$$M(T) = e^{rT}.$$

Here, e^{rT} is visualized as the growth factor of a fund over $[0, T]$. The reciprocal of the growth factor, namely e^{-rT}, is called the discount factor.

Under the continuous framework, the bond value $B(\vec{i})$ is given by

$$B(\vec{i}) = \int_0^T c(t)e^{-i(0,t)t}\,dt,$$

where $c(t)$ is the cash flow rate received at time t and $i(0,t)$ is the t-year spot rate compounded continuously over the interval $(0,t)$. For a given value t, $i(0,t)$ is the corresponding spot rate applicable for the cash flow amount $c(t)dt$ over $(t, t+dt)$ known at time zero. Therefore, the discount factor is $e^{-i(0,t)t}$. This is considered as a *functional* since this is a relation between a function $\vec{i}$ (term structure of the spot rates) and a number $B(\vec{i})$.

Naturally, duration of the bond with the initial term structure of the spot rate as characterized by $i(0,t)$ is given by

$$D(\vec{i}) = \frac{1}{B(\vec{i})}\int_0^T tc(t)e^{-i(0,t)t}\,dt,$$

where $\frac{c(t)e^{-i(0,t)t}}{B(\vec{i})}\,dt$ represents the weighted present value of the cash flow within $(t, t+dt)$. We would like to understand the financial intuition why duration is the multiplier that relates relative change in bond value and interest rate.

Suppose the whole term structure of spot rates move up in parallel shift by $\Delta\alpha$, then

$$B(\vec{i} + \Delta\alpha) = \int_0^T c(t)e^{-i(0,t)t}e^{-t\Delta\alpha}\,dt.$$

Note that when $\Delta\alpha$ is infinitesimally small, we have

$$e^{-t\Delta\alpha} \approx 1 - t\Delta\alpha$$

so that the discounted cash flow $e^{-i(0,t)t}c(t)\,dt$ within $(t, t+dt)$ decreases in proportional amount $t\Delta\alpha$. The corresponding contribution to the relative change in bond value as normalized by $B(\vec{i})$ is

$$-\frac{tc(t)e^{-i(0,t)t}\,dt}{B(\vec{i})}\Delta\alpha.$$

Note the role of the term $-t\Delta\alpha$, involving $(-t)$, which contributes to the relative change of the bond value.

This is the payment time t weighted by the discounted cash flow $\frac{c(t)dt}{B(i)} e^{-i(0,t)t}$ within $(t, t+dt)$ multiplied by the change in interest rate $\Delta\alpha$. Therefore, we have

$$\frac{B(\vec{i} + \Delta\alpha) - B(\vec{i})}{B(\vec{i})} \approx -\Delta\alpha \int_0^T \frac{tc(t)e^{-i(0,t)t}}{B(\vec{i})}\, dt.$$

The relative change in bond value

$$\begin{aligned} \frac{\Delta B}{B} &\approx -\Delta\alpha \left[\begin{array}{c} \text{weighted average of payment times that are} \\ \text{weighted according to present value of cash flow} \end{array} \right] \\ &= -D\Delta\alpha. \end{aligned}$$

In the differential limit, $\frac{B(\vec{i} + \Delta\alpha) - B(\vec{i})}{B(\vec{i})}$ becomes $\frac{dB}{B}$ and $\Delta\alpha$ is identified as the differential $d\alpha$, so we obtain

$$\frac{dB}{B} = -D(\vec{i})\, d\alpha.$$

Special case: discount bond (zero coupon bearing)

Since there is only one par payment P in a discount bond that is paid at T years, then

$$\frac{dB_{dis}}{di} = \frac{d}{di}\left[\frac{P}{(1+i)^T}\right] = -\frac{T}{1+i}B_{dis} = -\frac{D}{1+i}B_{dis},$$

where i is the interest rate per annum and $D = T$.

If the interest rate is compounded m times per year, then

$$\frac{dB_{dis}}{di} = \frac{d}{di}\left[\frac{P}{(1+\frac{i}{m})^{mT}}\right] = -\frac{T}{1+\frac{i}{m}}B_{dis} = -\frac{D}{1+\frac{i}{m}}B_{dis},$$

As $m \to \infty$, which corresponds to continuous compounding, we obtain

$$\frac{dB_{dis}}{B_{dis}} = -D\ di.$$

The factor $\frac{1}{1+i}$ disappears in continuous compounding.

Summary of formulas for continuous bond models

- Value of a bond in continuous time, with $\vec{i} = i(0,t)$ being the term structure of spot rates:

$$B(\vec{i}) = \int_0^T c(t)e^{-i(0,t)t}\,dt$$

- Duration of the bond:

$$D(\vec{i}) = \frac{1}{B(\vec{i})}\int_0^T tc(t)e^{-i(0,t)t}\,dt$$

- Duration of the bond when $\vec{i}$ receives a (constant) drift α:

$$D(\vec{i}+\alpha) = \frac{1}{B(\vec{i}+\alpha)}\int_0^T tc(t)e^{-[i(0,t)+\alpha]t}\,dt$$

- Fundamental property of duration:

$$-\frac{1}{B(\vec{i})}\frac{dB(\vec{i})}{d\alpha} = \frac{1}{B(\vec{i})}\int_0^T tc(t)e^{-i(0,t)t}\,dt = D$$

1.2 Horizon rate of return: return from the bond investment over a time horizon

Horizon rate of return, r_H – bond is kept for a time horizon H

Suppose a bond investor bought a bond valued at $B(i_0)$ when the interest rate common to all maturities was i_0 (flat rate). On the following day, the interest rate moves up to i (parallel shift). The new future value at H given the bond price $B(i)$ at the new interest rate level i is given by $B(i)(1+i)^H$ since the future cash flows from the bond are assumed to be compounded annually at the new interest rate i (though H may not be an integer). To the investor, by paying B_0 as the initial investment, the new horizon rate of return based on the new future value is given by $B_0(1+r_H)^H = B(i)(1+i)^H$ and so

$$r_H = \left[\frac{B(i)}{B_0}\right]^{1/H}(1+i) - 1.$$

The impact on the bond value on changing interest rate is spread out in H years.

Example – Calculation of r_H

A 10-year bond with coupon rate of 7% was bought when the interest rates were at 5%. We have $B(i_0) = \$1154.44$.

Suppose on the next day, the interest rates move up to 6%. The bond drops in value to \$1073.60. If he holds his bond for 5 years (horizon is chosen to be 5 years), and if interest rates stay at 6%, then

$$r_H = \left(\frac{1073.60}{1154.44}\right)^{1/5}(1.06) - 1 = 4.47\%.$$

Observation

Though the rate of interest at which the investor can reinvest his coupons (which is now 6%) is higher, his overall performance will be lower than 5% (r_H is only 4.47%).

- As a function of i, the horizon rate of return r_H is a product of a decreasing function $B(i)$ and an increasing function $(1+i)$. This represents a counterbalance between an immediate capital gain/loss and rate of return based on new i on the cashflows from now till H.

- Whatever the horizon, the horizon rate of return will always be i_0 if i does not move away from this value. In this case,

$$F_H = B_0(1+i_0)^H = B_0(1+r_H)^H$$

so that $r_H = i_0$ for any H (see the column in the table on the next page under $i = 5\%$).

- If $H \to \infty$, then $r_H = \left[\frac{B(i)}{B_0}\right]^{1/H}(1+i) - 1 \to i$. With infinite time of horizon, the immediate change of bond price is immaterial. The horizon rate of return is simply the new prevailing interest rate i.

The table shows the horizon rate of return (in percentage per year) on the investment in a 7% coupon, 10-year maturity bond bought at 1154.44 when interest rates were at 5%, should interest rates move immediately either to 6% or 4%. At $H = 7.7$, which is the duration of the bond, r_H increases when i either increases or decreases.

		Interest rates		
Horizon (years)		4%	5%	6%
1		12.01	5	-1.42
2	increasing	7.93	5	2.22
3	r_H ↑	6.60	5	3.47
4		5.95	5	4.09
5		5.55	5	4.47
6		5.29	5	4.73
7		5.11	5	4.92
7.7		**5.006**	**5**	**5.006**
8		4.97	5	5.04
9		4.86	5	5.15
10		4.77	5	5.23
↓		↓		↓
∞		4.00		6.00

- Comparing the one-year horizon and four-year horizon, if interest rates rise (see the last column in the previous table under $i = 6\%$), the four-year horizon return is higher than the one-year horizon return. This is because the longer the horizon and the longer the reinvestment of the coupons at a higher rate, the greater the chance that the investor will outperform the initial yield of 5%.

- The duration of the 10-year 7%-coupon rate bond is found to be 7.7 years. When the horizon is chosen to be 7.7 years, then the horizon rate of return will be slightly above 5% (5.006%) whether the interest rate falls to 4% or increases to 6%.

- For the extreme case of $H \to \infty, r_H = i$. The immediate capital gain/loss is immaterial since all cash flows from the bonds remain the same while they can be reinvested at the rate of return i.

Example

An insurance company issues a guaranteed investment contract (GIC) for $\$10,000$. Essentially, GICs are zero-coupon bonds issued by the insurance company to its customers. They are popular products for individuals' retirement-savings accounts. If the GIC has a 5-year maturity and a guaranteed interest rate of 8%, the insurance company promises to pay $\$10,000 \times (1.08)^5 = \$14,693.28$ in 5 years.

Suppose that the insurance company chooses to fund its obligation with $\$10,000$ of 8% *annual* coupon bonds, selling at par value, with 6 years to maturity. It happens that this 8%-coupon bond with 6 years to maturity has a duration that matches with the time horizon of 5 years. As long as the market interest rate stays at 8%, the company has fully funded the obligation, as the present value of the obligation exactly equals the value of the bonds.

The following table shows that if interest rates remain at 8%, the accumulated funds from the bond will grow to exactly the $14,693.28 obligation.

Payment Number	Years Remaining until Obligation	Accumulated Value of Invested Payment		
A. Rates remain at 8%				
1	4	$800 \times (1.08)^4$	=	1,088.39
2	3	$800 \times (1.08)^3$	=	1,007.77
3	2	$800 \times (1.08)^2$	=	933.12
4	1	$800 \times (1.08)^1$	=	864.00
5	0	$800 \times (1.08)^0$	=	800.00
Sale of bond	0	10,800/1.08	=	10,000.00
				14,693.28

Over the 5-year period, year-end coupon income of $800 is reinvested at the prevailing 8% market interest rate. At the end of the period, the bonds can be sold for $10,000; they still will sell at par value because the coupon rate still equals the market interest rate. Total income after 5 years from reinvested coupons and the sale of the bond is precisely $14,693.28.

Price risk and reinvestment risk are offsetting

If interest rates change, two offsetting influences will affect the ability of the fund to grow to the targeted value of $\$14,693.28$. If interest rate rise, the fund will suffer a capital loss, impairing its ability to satisfy the obligation. The bonds will be worth less in 5 years than if interest rates had remained at 8%. However, at a higher interest rate, reinvested coupons will grow at a faster rate, offsetting the capital loss.

In other words, fixed-income investors face two offsetting types of interest rate risk: *price risk* and *reinvestment rate risk*. Increases in interest rates cause capital losses but at the same time increase the rate at which reinvested income will grow. If the portfolio duration is chosen appropriately, these two effects will cancel out exactly.

When the portfolio duration is set equal to the investor's horizon date, the accumulated value of the investment fund at the horizon date will be unaffected by interest rate fluctuations. *For a horizon equal to the portfolio's duration, price risk and reinvestment risk exactly cancel out*.

In this example, the duration of the 6-year maturity bonds used to fund the GIC is 5 years. Since the fully funded plan has equal duration for its assets and liabilities, the insurance company should be immunized against interest rate fluctuations. To confirm this, we know that bond can generate enough income to pay off the obligation in 5 years regardless of interest rate movements.

Rates fall to 7%				
1	4	$800 \times (1.07)^4$	=	1,048.64
2	3	$800 \times (1.07)^3$	=	980.03
3	2	$800 \times (1.07)^2$	=	915.92
4	1	$800 \times (1.07)^1$	=	856.00
5	0	$800 \times (1.07)^0$	=	800.00
Sale of bond	0	10,800/1.07	=	10,093.46
				14,694.05

Rates increase to 9%				
1	4	$800 \times (1.09)^4$	=	1,129.27
2	3	$800 \times (1.09)^3$	=	1,036.02
3	2	$800 \times (1.09)^2$	=	950.48
4	1	$800 \times (1.09)^1$	=	872.00
5	0	$800 \times (1.09)^0$	=	800.00
Sale of bond	0	10,800/1.09	=	9,908.26
				14,696.02

- The horizon rate of return r_H is a decreasing function of i when the horizon H is short and an increasing one for long horizons. For a horizon equal to the duration of the bond, the horizon rate of return first decreases, goes through a minimum for $i = i_0$ then increases by i.

- There is a critical value for H such that r_H changes from a decreasing function of i to an increasing function of i. This critical value is the bond duration. Why? Recall

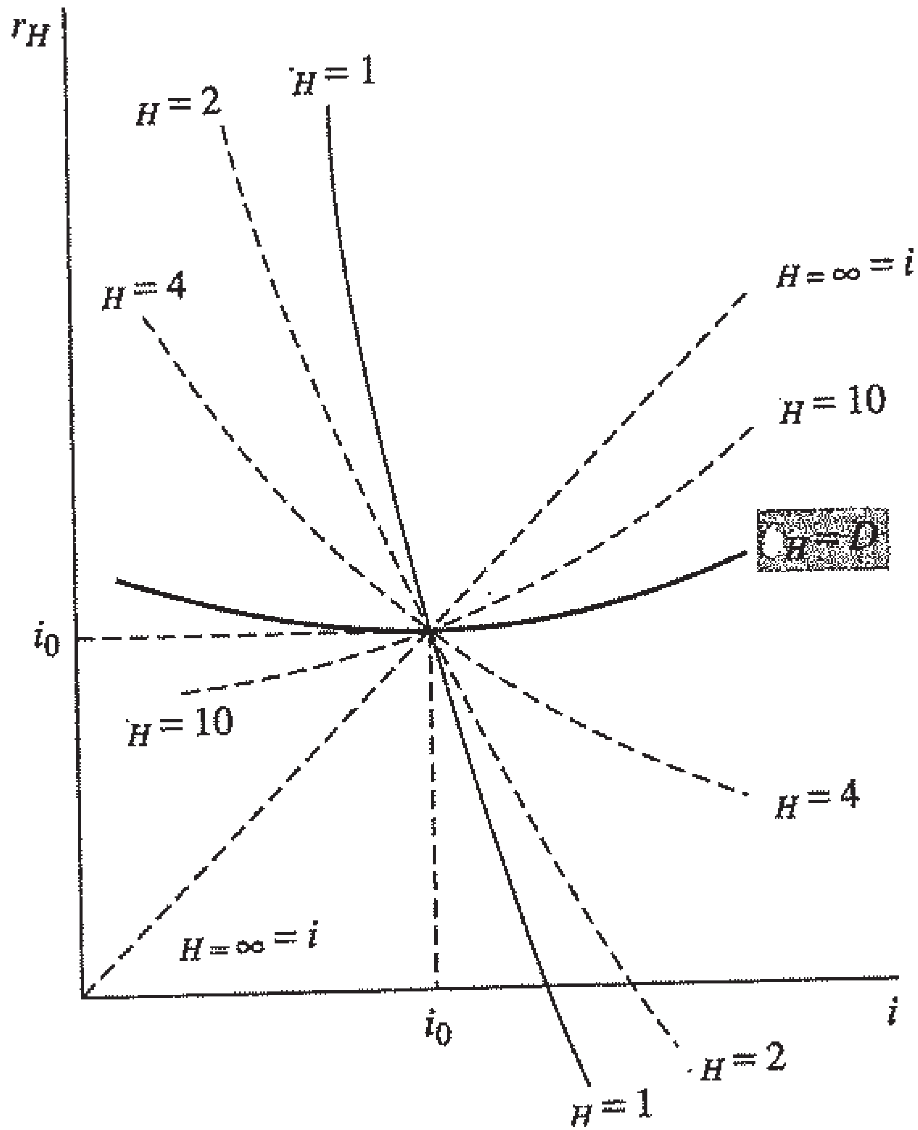
$$\frac{\Delta B}{B} \approx -D \times \Delta i$$

- The immediate capital loss of amount $D\Delta i$ is spread over H years.

- The gain in a higher rate of return of the future cash flows is $H\Delta i$ over H years of horizon of investment.

- These two effects are counterbalanced if $H = D$.

Dependence of r_H on i with varying H



Remark

Suppose an investor is targeting at a time horizon of investment H, he should choose a bond whose duration equals H so that the rate of return at the target horizon is immunized from any change in the interest rate.

Stronger mathematical result

There exists a horizon H such that r_H always increases when the interest rate moves up or down from the initial value i_0. The more precise statement is stated in the following theorem.

Theorem

There exists a horizon H such that the rate of return for such a horizon goes through a minimum at point i_0.

Proof

Minimizing r_H is equivalent to minimizing any positive function of it, and so it is equivalent to minimizing $B_0(1+r_H)^H = F_H = B(i)(1+i)^H$. Consider

$$\frac{dF_H}{di} = \frac{d}{di}[B(i)(1+i)^H] = B'(i)(1+i)^H + HB(i)(1+i)^{H-1},$$

we would like to find H such that the first order condition: $\frac{dF_H}{di} = 0$ at $i = i_0$ is satisfied. This gives

$$B'(i_0)(1+i_0) + HB(i_0) = 0$$

and

$$H = -\frac{1+i_0}{B(i_0)}B'(i_0) = \text{duration}.$$

The horizon H must be chosen to be equal to the duration at the initial rate of return i_0 for F_H to run through a minimum. If otherwise, then $\frac{dF_H}{di} = 0$ at $i = i_0$ cannot be satisfied. This is revealed by the other curves (see P.39) that pass through $i = i_0$, where they are either monotonic increasing or decreasing in i.

Checking the second order condition

Recall $\dfrac{\mathrm{d}^2 \ln f}{\mathrm{d}x^2} = \dfrac{f''f - f'^2}{f^2}$ so $\dfrac{\mathrm{d}^2 \ln f}{\mathrm{d}x^2} > 0 \Rightarrow f'' > \dfrac{f'^2}{f} > 0$ for $f > 0$. It suffices to show that $\ln F_H(i) = \ln B(i) + H \ln(1+i)$ has a positive second order derivative.

$$\begin{aligned}\frac{d \ln F_H}{di} &= \frac{d}{di} \ln B(i) + \frac{H}{1+i} = \frac{1}{B(i)}\frac{dB(i)}{di} + \frac{H}{1+i} = \frac{-D+H}{1+i} \\ \frac{d^2}{di^2} \ln F_H &= \frac{1}{(1+i)^2}\left[-\frac{dD}{di}(1+i) + D - H\right].\end{aligned}$$

Setting $H = D$, we obtain

$$\frac{d^2}{di^2} \ln F_H = -\frac{1}{1+i}\frac{dD}{di} = \frac{S}{(1+i)^2} > 0.$$

Therefore, F_H and r_H go through a global minimum at point $i = i_0$ whenever $H = D$.

1.3 Immunization of bond investment

- In the case of either a drop or a rise in interest rates, when the horizon was properly chosen, the *horizon rate of return* for the bond's owner was about the same as if interest rates had not moved. **This horizon is the duration of the bond.**

- Immunization is the set of *bond management procedures* that aim at protecting the investor against changes in interest rates.

- It is dynamic since the passage of time and changes in interest rates will modify the portfolio's duration by an amount that will not necessarily correspond to the steady and natural decline of the investor's horizon.

- Even if interest rates do not change, the simple passage of one year will reduce duration of the portfolio by less than one year. The money manager will have to change the composition of the portfolio so that the duration is reduced by a whole year. The new bond portfolio's duration is adjusted and targeted at the updated horizon.

- Changes in interest rates will also modify the portfolio's duration.

- *Immunization* may be defined as the process by which an investor can protect himself against interest rate changes by suitably choosing a bond or a portfolio of bonds such that *its duration is kept equal to his horizon dynamically*.

Duration matching and rebalancing

An insurance company must make a payment of $\$19,487$ in seven years. The market interest rate is 10%, so the present value of the obligation is $\$10,000$. The company's portfolio manager wishes to fund the obligation using three-year zero-coupon bonds and perpetuities paying annual coupons.

How can the manager immunize the obligation?

Immunization requires that the duration of the portfolio of assets equal the duration of the liability. We can proceed in four steps:

Step 1. Calculate the duration of the liability. It is a single-payment obligation with duration of seven years.

Step 2. Calculate the duration of the asset portfolio. The portfolio duration is the weighted average of duration of each component asset, with weights proportional to the funds placed in each asset.

- The duration of the zero-coupon bond is simply its maturity, three years.

- The duration of the perpetuity is $\frac{1}{0.1} + 1 = 11$ years.

If the fraction of the portfolio invested in the zero is called w, and the fraction invested in the perpetuity is $(1 - w)$, then

$$\text{asset duration } = w \times 3 \text{ years} + (1 - w) \times 11 \text{ years}$$

Step 3. Find the asset mix that sets the duration of assets equal to the seven-year duration of liabilities. This requires us to solve for w in the following equation

$$w \times 3 \text{ years} + (1 - w) \times 11 \text{ years} = 7 \text{ years}.$$

This gives $w = 1/2$.

Step 4. Fully fund the obligation. Since the obligation has a present value of $\$10,000$, and the fund will be invested equally in the zero and the perpetuity, the manager must purchase $\$5,000$ of the zero-coupon bond and $\$5,000$ of the perpetuity. Note that the face value of the zero-coupon bond will be $\$5,000 \times (1.10)^3 = \$6,655$.

Rebalancing

Suppose that one year has passed, and the interest rate remains at 10%. The portfolio manager needs to reexamine her position.

Funding

The present value of the obligation will have grown to $\$11,000$, as it is one year closer to maturity. The manager's funds also have grown to $\$11,000$: The zero-coupon bonds have increased in value from $\$5,000$ to $\$5,500$ with the passage of time, while the perpetuity has paid its annual $\$500$ coupons and remains worth $\$5,000$. Therefore, the obligation is still fully funded since $\$11,000 = \$5,500 + (\$5,000 + \$500)$.

The portfolio weights must be changed, however. The zero-coupon bond now will have a duration of two years, while the perpetuity duration remains at 11 years. The obligation is now due in six years. The weights must now satisfy the equation

$$w \times 2 + (1 - w) \times 11 = 6$$

which implies that $w = 5/9$.

To rebalance the portfolio and maintain the duration match, the manager now must invest a total of $\$11,000 \times 5/9 = \$6,111.11$ in the zero-coupon bond. This indicates an increase of amount equals $\$6,111.11 - \$5,500 = \$611.11$ in holding the zero-coupon bond. This requires that the entire \$500 coupon payment be invested in the zero, with an additional \$111.11 of the perpetuity sold and invested in the zero-coupon bond.

Numerical example – matching duration

A company has an obligation to pay \$1 million in 10 years. That is, the future value at the time of horizon of 10 years is \$1 million. It wishes to invest money now that will be sufficient to meet this obligation. The purchase of a single zero-coupon bond would provide one solution, but such discount bonds are not always available in the required maturities.

	coupon rate	maturity	price	yield	duration
bond 1	6%	30 yr	69.04	9%	11.44
bond 2	11%	10 yr	113.01	9%	6.54
bond 3	9%	20 yr	100.00	9%	9.61

- The above 3 bonds all have the yield of 9%. Present value of the obligation of \$1 million in 10 years at 9% yield is $\$414,643$.

- Since bond 2 and bond 3 have their duration shorter than 10 years, it is not possible to attain a portfolio with duration 10 years using these two bonds. A bond with a longer maturity is required (say, bond 1) to be included in the portfolio. The coupons received are reinvested earning rate of return at the prevailing yield.

Suppose we use bond 1 and bond 2 of notional amount V_1 and V_2 in the portfolio, by matching the present value and duration, we obtain

$$\begin{aligned} V_1 + V_2 &= PV = \$414,643 \\ \frac{D_1 V_1 + D_2 V_2}{PV} &= 10 \end{aligned}$$

giving

$$V_1 = \$292,788.64 \quad \text{and} \quad V_2 = \$121,854.78.$$

Number of shares of bond 1 to be held is $\$292,798.64/\$69.04 = 4,241$, which will be held fixed. Similarly, the number of bond 2 to be held is $\$121,824.78/\$113.01 = 1,078$.

What would happen when we have a sudden change in the prevailing yield?

	Yield 9.0	8.0	10.0
Bond 1			
Price	69.04	77.38	62.14
shares	4,241	4,241	4,241
value	292,798.64	328,168.58	263,535.74
Bond 2			
Price	113.01	120.39	106.23
shares	1,078	1,078	1,078
value	121,824.78	129,780.42	114,515.94
Obligation value	414,642.86	456,386.95	376,889.48
Surplus	-19.44	1,562.05	1,162.20

- Obligation value at 8% yield $= 1,000,000/(1.08)^{10} = 456,387$.
- Surplus at 8% yield $= 328,168.58 + 129,780.42 - 456,386.95$
$= 1,562.05$.

Observation

At different yields (8% and 10%), the value of the portfolio almost agrees with that of obligation (at the new yield) with a small amount of surplus.

Difficulties in the implementation of immunization

- It is quite unrealistic to assume that both the long- and short-duration bonds can be found with identical yields. Usually, the longer-maturity bonds have higher yields.
- When interest rates change, it is unlikely that the yields on all bonds will change by the same amount.

Bankruptcy of Orange County, California *(see Qn 9 in HW 1)*

"A prime example of the interest rate risk incurred when the duration of asset investments is not equal to the duration of fund needs."

Orange County (like most municipal governments) maintained an operating account of cash from which operating expenses were paid. During the 1980s and early 1990s, interest rates in US had been falling.

Seeing the larger returns being earned on long-term securities, the treasurer of Orange County decided to invest in long-term fixed income securities.

- The County has \$7.5 billion and borrowed \$12.5 billion from Wall Street brokerages. We illustrate how leverage triggered default in the numerical calculations in the Homework Problem.

- Between 1991 and 1993, the County enjoyed more than a 8.5% return on investments.

- Started in February 1994, the Federal Reserve Board raised the interest rate in order to cool an expanding economy. All through the year, paper losses on the fund led to margin calls from Wall Street brokers that had provided short-term financing.

- In December 1994, as news of the loss spread, brokers tried to pull out their money. Finally, as the fund defaulted on payments of additional collateral, brokers started to liquidate their collateral.

- Bankruptcy caused the County to have difficulties to meet payrolls, 40% cut in health and welfare benefits and school employees were laid off.

- County officials blamed the county treasurer, Bob Citron, for undertaking risky investments. He claimed that there was no risk in the portfolio since he was holding the bond portfolio to maturity.

- Since the government accounting standards do not require municipal investment pools to report "paper" gains or losses, Citron did not report the market value of the portfolio. The immediate loss in value in the long-term bonds due to an increased interest rate can be compensated by the higher interest rate earned in the remaining life of the long-term bonds. Indeed, if the targeted horizon of investment is sufficiently long, the horizon rate of return may increase with increasing interest rate.

Convexity of a bond (second order effect to changing interest rate)

We define convexity C to be $\frac{1}{B}\frac{d^2B}{di^2}$. To relate C to S and D, we consider the derivative of D and equate the result to $-\frac{S}{1+i}$. Now

$$D = -\frac{1+i}{B(i)}B'(i) \quad \text{so} \quad \frac{dD}{di} = -\left[\frac{B-(1+i)B'}{B^2}\right]B'(i) - \frac{1+i}{B}B''.$$

Recall $\frac{dD}{di} = -\frac{S}{1+i}$ so that

$$\frac{1}{B}(1+D)B'(i) + \frac{1+i}{B}B'' = \frac{S}{1+i}.$$

Writing $\frac{B'}{B} = -\frac{D}{1+i}$ and $\frac{B''}{B} = C$ so that

$$-D(D+1) + (1+i)^2C = S.$$

Finally, we obtain

$$C = \frac{S + D(D+1)}{(1+i)^2}.$$

The convexity of a bond is affected by the dispersion of the payment times of the cash flows.

Convexity and its uses in bond portfolio management

	Coupon rate	maturity	price	yield to maturity	duration
Bond A	9%	10 years	$1,000	9%	6.99 years
Bond B	3.1%	8 years	$673		

Bond B is found such that it has the same duration and yield to maturity (YTM) as Bond A. Bond B has coupon rate 3.1% and maturity equals 8 years. Its price is $673.

- Portfolio α consists of 673 units of Bond A ($\$673,000$)
- Portfolio β consists of $1,000$ units of Bond B ($\$673,000$)

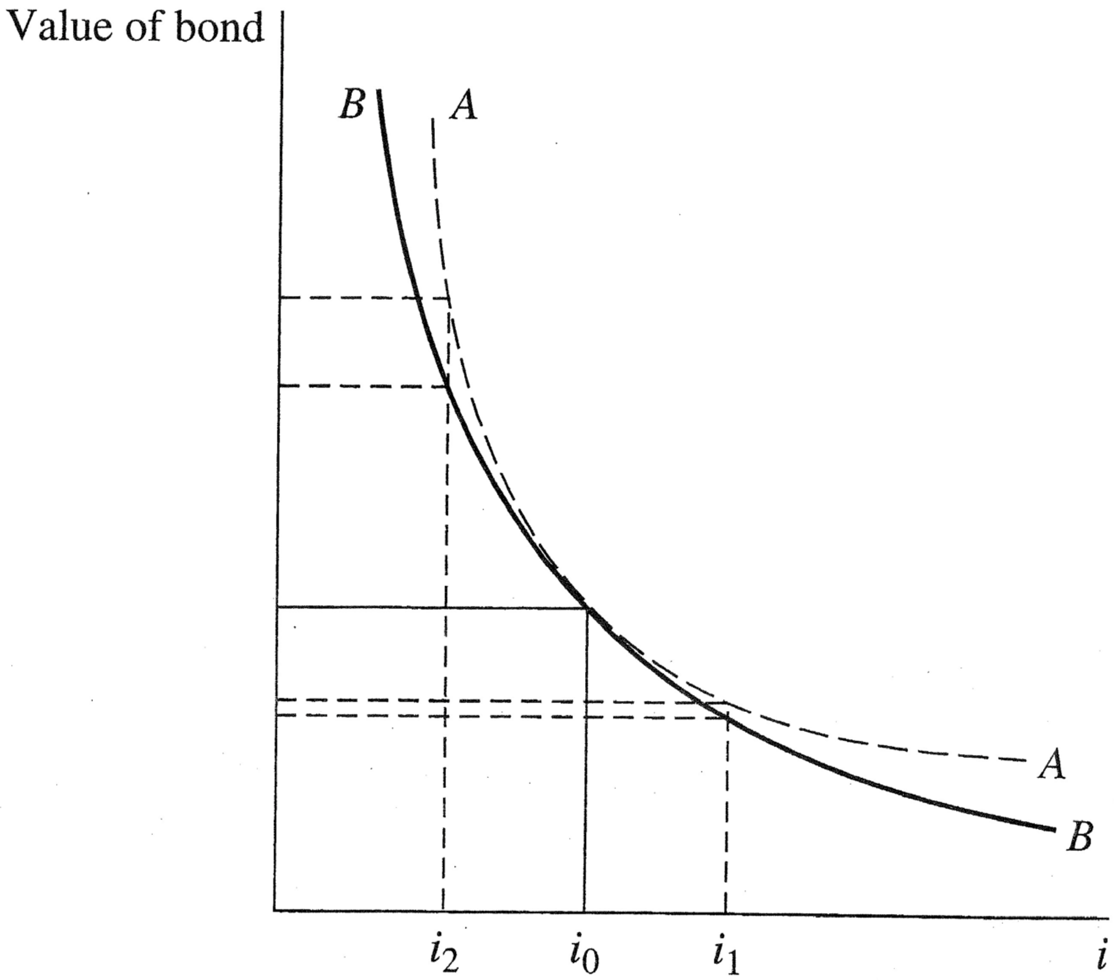
Would an investor be indifferent to these two portfolios since they are worth exactly the same, offer the same YTM and have the same duration (apparently faced with the same interest rate risk)?

What makes a bond more convex than the other one if they have the same duration? The key is the dispersion of payment times. Recall the formula:

$$\text{Convexity} = \frac{\text{dispersion} + \text{duration (duration} + 1)}{(1+i)^2}.$$

- The convexity has the second order effect on bond portfolio management.

- Higher convexity is resulted with higher dispersion and duration of payment times, properties that are exhibited by bonds with longer maturities.



The effect of a greater convexity for bond A (longer maturity) than for bond B enhances an investment in A compared to an investment in B in the event of change in interest rates. Investment in A will gain more value than investment in B if interest rates drop and it will lose less value if interest rates rise (too good to be true, but it is true).

$$C = \frac{1}{B}\frac{d^2B}{di^2} = \frac{1}{B(1+i)^2}\sum_{t=1}^{T} t(t+1)c_t(1+i)^{-t}$$

Calculation of the convexity of bond *A* (coupon: 9%; yield to maturity: 10 years)

(1) Time of payment t	(2) $t(t+1)$	(3) Cash flow in nominal value c_t	(4) Share of the discounted cash flows in bond's value $c_t(1+i)^{-t}/B$	(5) $t(t+1)$ times share of discounted cash flows = (2) × (4) $t(t+1)c_t(1+i)^{-t}/B$
1	2	9	0.0826	0.165
2	6	9	0.0758	0.456
3	12	9	0.0695	0.834
4	20	9	0.0638	1.275
5	30	9	0.0585	1.755
6	42	9	0.0537	2.254
7	56	9	0.0492	2.757
8	72	9	0.0452	3.252
9	90	9	0.0414	3.729
10	110	109	0.4604	50.647

Convexity: total of (5) $\times \frac{1}{(1.09)^2} = 56.5$ (years^2)

By the Taylor series approximation of ΔB with respect to di, the relative increase in the bond's value is given in quadratic approximation by

$$\frac{\Delta B}{B} \approx \frac{1}{B}\frac{dB}{di}\Delta i + \frac{1}{2}\frac{1}{B}\frac{d^2B}{di^2}(\Delta i)^2.$$

Taking $\Delta i = 1\%$, duration $= 6.99$, convexity $= 56.5$, we have

$$\begin{aligned}\frac{\Delta B}{B} &\approx \left(-\frac{6.99}{1.09} + \frac{0.01}{2} \times 56.5\right)\% \\ &= (-6.4176 + 0.2825)\% = -6.135\%.\end{aligned}$$

On the other hand, suppose i decreases by 1%. With di equal to -1%, we obtain

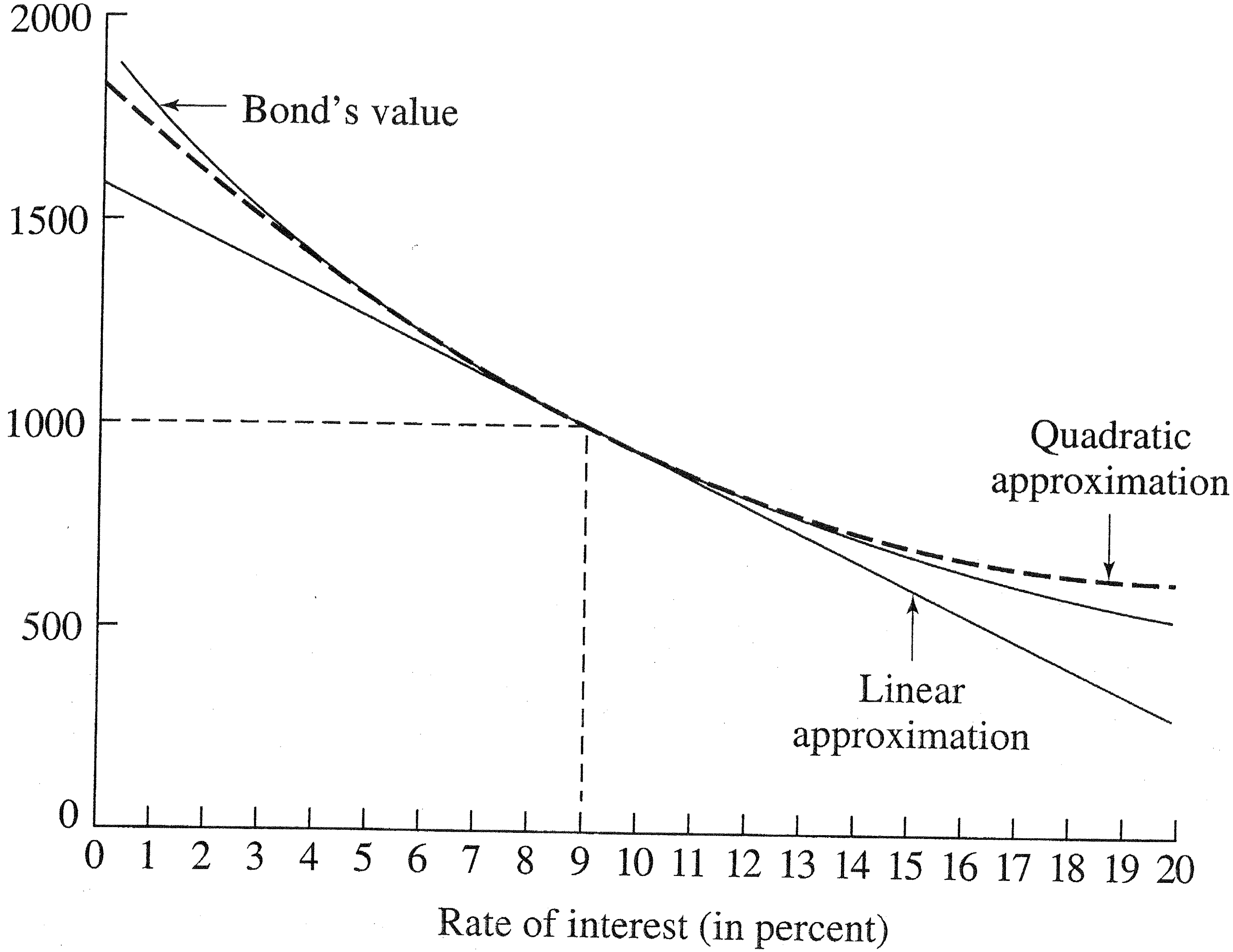
$$\frac{\Delta B}{B} \approx (+6.4176 + 0.2825)\% = 6.700\%.$$

Note that $\frac{1}{B}\frac{d^2B}{di^2}\frac{1}{200} = 0.2825$ as percentage point is added to the modified duration with change of interest rate of $\pm 1\%$ to give 6.700% and -6.135%, respectively, on $\dfrac{\Delta B}{B}$ (see the table and figure on the next two pages).

Improvement in the measurement of a bond's price change by using convexity

	Change in the bond's price		
Change in the rate of interest	in linear approximation (using duration)	in quadratic approximation (using both duration and convexity)	in exact value
+1%	-6.4176%	-6.135%	-6.14%
-1%	+6.4176%	+6.700%	+6.71%

The quadratic approximation undershoots (overshoots) when change in interest rate is negative (positive). For details, see HW 1, Qn 6.



Linear and quadratic approximations of a bond's value

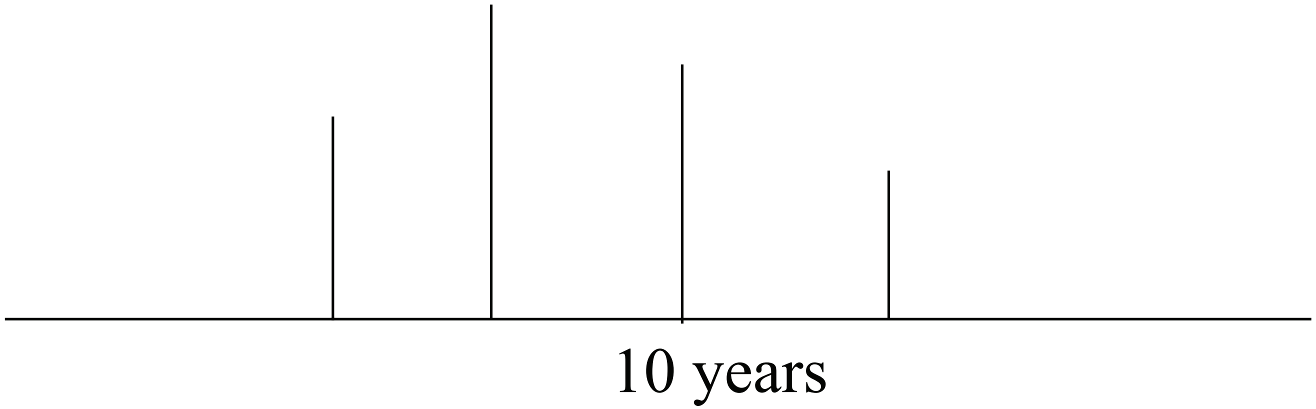
Yield curve strategies

- Seek to capitalize on investors' market expectations based on the short-term movements in yields.
- Source of return depends on the maturity of the securities in the portfolio since different parts of the yield curve respond differently to the same economic stock.

In most circumstances, yield curve is upward sloping with maturity and eventually level off at sufficiently high value of maturity. How does an investor choose the spread of the maturity of bonds in the portfolio to increase portfolio return or achieve higher convexity (both are offsetting) under the same duration in the sense that portfolio with higher convexity would have lower yield?

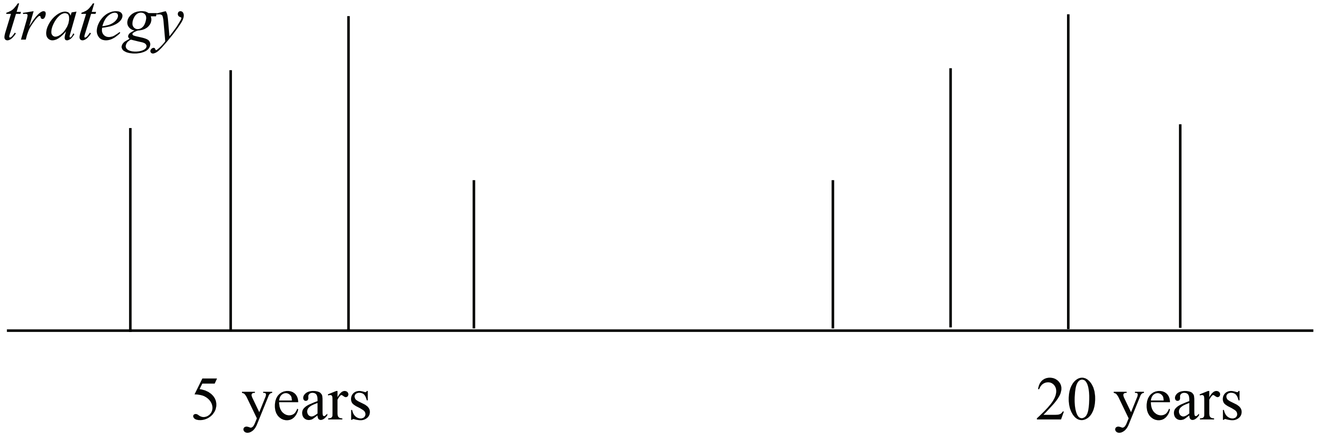
Spread of maturity of bonds put in a portfolio

Bullet strategy



Maturity of bonds are highly concentrated around one maturity date.

Barbell strategy



Maturity of bonds are concentrated at two extreme maturities.

Bond	Coupon	Maturity	Price	YTM	Duration	Convexity
A	8.5%	5	100	8.50	4.005	19.8164
B	9.5%	20	100	9.50	8.882	124.1702
C	9.25%	10	100	9.25	6.434	55.4506

In general, yield increases with maturity while the increase in convexity is more significant with increasing maturity. There is a 75 bps increase from 5-year maturity to 10-year maturity but only a 25 bps increase from 10-year maturity to 20-year maturity. However, bond B's convexity is more than double that of bond C.

- Bullet portfolio: 100% bond C
- Barbell portfolio: 50.2% bond A and 49.8% bond B

$$\begin{aligned}\text{duration of barbell portfolio} &= 0.502 \times 4.005 + 0.498 \times 8.882 \\ &= 6.434\end{aligned}$$

$$\begin{aligned}\text{convexity of barbell portfolio} &= 0.502 \times 19.8164 + 0.498 \times 124.1702 \\ &= 71.7846.\end{aligned}$$

Yield

Since the bond prices are equal to par, so YTM = coupon rate. We have

$$\begin{aligned} & \text{portfolio yield for the barbell portfolio} \\ = \ & 0.502 \times 8.5\% + 0.498 \times 9.5\% = 8.998\% < 9.25\% = \text{yield of bond C} \end{aligned}$$

Duration

For the purpose of examining the impact of convexities on bond investment strategies, we choose the portfolio weights in the barbell so that the two portfolios have the same duration.

Convexity

convexity of barbell $= 71.7846 >$ convexity of bullet $= 55.4506$

Tradeoff between yield and convexity

The lower value of yield for the barbell portfolio is a reflection of the level off effect of yield at higher maturity. When both bullet and barbell portfolios have the same duration, the barbell strategy gives up yield in order to achieve a higher convexity.

Assume a 6-month investment horizon

1. Yield curve shifts in a parallel fashion

   When the change in yield $\Delta\lambda < 100$ basis points, the bullet portfolio outperforms the barbell portfolio in return; vice versa if otherwise.

   If λ shifts parallel in a small amount, the bullet portfolio with less convexity remains to provide a better total return. The change in yield has to be more significant in order that the high convexity portfolio can outperform. Recall that portfolio with higher convexity increases more (decreases less) in value when the interest rate drops (rises).

2. Non-parallel shift (flattening of the yield curve)

$$\Delta\lambda \text{ of bond } A = \Delta\lambda \text{ of bond } C + 45 \text{ bps}$$
$$\Delta\lambda \text{ of bond } B = \Delta\lambda \text{ of bond } C - 15 \text{ bps}$$

   The barbell strategy always outperforms the bullet strategy. This is due to the yield pickup (45 bps) for shorter-maturity bonds.

3. Non-parallel shift (steepening of the yield curve)

$$\Delta\lambda \text{ of bond } A = \Delta\lambda \text{ of bond } C - 25 \text{ bps}$$
$$\Delta\lambda \text{ of bond } B = \Delta\lambda \text{ of bond } C + 25 \text{ bps}$$

The bullet portfolio remains to provide higher yield than that of the barbell portfolio.

Conclusion

The barbell portfolio with higher convexity may outperform only when the yield change is significant and/or yield curve flattens (loss of yield with higher convexity is less significant). The performance depends on the magnitude of the change in yields and how the yield curve shifts.

Barbell strategy (higher convexity + lower yield) versus
bullet strategy (lower convexity + higher yield).

Comparing two coupon-bearing bonds with differing convexities

- Note that increasing the coupon rate decreases both the duration and convexity. This is because higher coupon rate leads to lower percentage weight on the present value of the par.
- The interest rate is 9%.

Characteristics of bonds A and B

	Bond A	Bond B
Maturity	10 years	20 years
Coupon	1	13.5
Duration	9.31 years	9.31 years
Convexity	84.34 years^2	115.97 years^2

Bond A is way below par (48.66) and Bond B is above par (141.08). They have the same duration but differing convexities.

Duration and convexity for 10-year bond and 20-year bond with varying coupon rates

Coupon (c)	Type I bond Maturity: 10 years Duration (years) $D = -\frac{1+i}{B}\frac{dB}{di}$	Type I bond Maturity: 10 years Convexity (years^2) $CONV = \frac{1}{B}\frac{d^2B}{di^2}$	Type II bond Maturity: 20 years Duration (years) $D = -\frac{1+i}{B}\frac{dB}{di}$	Type II bond Maturity: 20 years Convexity (years^2) $CONV = \frac{1}{B}\frac{d^2B}{di^2}$
0	10.00	92.58	20.00	353.50
1	**9.31**	**84.34**	15.90	251.50
2	8.79	78.02	13.81	215.98
3	8.37	73.02	12.59	188.85
4	8.03	68.96	11.78	170.80
5	7.75	65.60	11.21	158.00
6	7.52	62.79	10.77	148.40
7	7.32	60.38	10.44	140.94
8	7.15	58.31	10.17	134.98
9	6.99	56.50	9.95	130.10
10	6.86	54.90	9.76	126.04
11	6.76	53.49	9.61	122.61
12	6.64	52.23	9.48	119.70
13	6.55	51.10	9.36	117.12
13.5	6.50	50.58	**9.31**	**115.97**
14	6.46	50.08	9.29	114.89
15	6.38	49.16	9.18	112.93

The 20-year bond (B) can be made to have the same duration as that of the 10-year bond (A) by setting a very high coupon rate. Bond B still has a higher convexity.

Case 1: $H = D$

Set the horizon H to be the common duration of 9.31 years.

The horizon rates of return for bonds A and B move up even i increases or decreases from $i_0 = 9\%$ (see the tables on the next page). Comparing the future value at $H = 9.31$ for the same initial value of $\$1,000,000$, a difference of $\$14,023$ is resulted under different convexities.

- Suppose that the initial rates are 9% and that they quickly move up by 1 or 2% or drop by the same amount. The horizon rate of return in excess of 9% for the more convex bond is tenfold that of the bond with lower convexity.

Horizon rates of return for A and B with $H = 9.31$ years when the rates move quickly from 9% to another value and stay there

	Scenario				
	$i = 7\%$	$i = 8\%$	$i = 9\%$	$i = 10\%$	$i = 11\%$
Bond A	9.008%	9.002%	9%	9.002%	9.008%
Bond B	9.085%	9.021%	9%	9.020%	9.079%

Suppose the same current value of 1 million and interest rate decreases from 9% to 7%. We observe

- investment in A: $1,000,000(1 + 0.9008)^{9.31} = 2,232,222$
- investment in B: $1,000,000(1 + 0.9085)^{9.31} = 2,246,245$

This implies a difference of $\$2,246,245 - \$2,232,222 = \$14,023$ in the future value for no trouble at all, except looking up the value of convexity.

Case 2: $H < D$

- Two short horizons have been chosen in the two bonds: $H = 1$ and $H = 2$.

  The gain in bond value when the interest rate decreases is more substantial for the bond with higher convexity.

- Shorter horizon, the gain of horizon rate of return of Bond B is more significant.

- At $H = 1$, with an increase in interest rate from 9% to 11%, the convexity of B will cushion the loss from 6.2% to 5.7%.

Horizon rates of return when i takes a new value immediately after the purchase of bond A and bond B (in annualized percentage)

Horizon (in years) and rates of return for A and B	Scenario				
	$i = 7\%$	$i = 8\%$	$i = 9\%$	$i = 10\%$	$i = 11\%$
$H = 1$					
R^A	27.2	17.1	9.0	1.0	-6.2
R^B	28.1	17.9	9.0	1.2	-5.7
$H = 2$					
R^A	16.7	12.7	9.0	5.4	2.0
R^B	17.1	12.8	9.0	5.5	2.3

Duration and Convexity for a zero-coupon bond

	Bond A (zero-coupon)	Bond B
Coupon	0	9
Maturity	10.58 years	25 years
Duration	10.58 years	10.58 years
Convexity	103.12 years^2	159.17 years^2

Rates of return of A and B when i moves from $i = 9\%$ to another value after the bond has been bought (horizon is set equal the duration)

	Scenario				
	$i = 7\%$	$i = 8\%$	$i = 9\%$	$i = 10\%$	$i = 11\%$
Bond A (zero-coupon)	9	9	9	9	9
Bond B	9.14	9.04	9	9.02	9.08

Recall $C = \frac{S+D(D+1)}{(1+i)^2}$. Though $S = 0$ for a zero coupon bond, its convexity remains positive since $C = \frac{D(D+1)}{(1+i)^2}$ when $S = 0$.

For the zero-coupon bond, when the horizon H is set to be the bond's maturity date T (so $H = D = T$), the future value F_H remains to be equal to par under an increase or decrease of the interest rate since there is no coupon within the time horizon. We then have $F_H = \text{par} = B_0(1 + r_H)^H$, so r_H does not change.

Looking for convexity in building a bond portfolio

Suppose we are unable to find a bond with the same duration and higher convexity as the one we are considering buying. We may build a barbell portfolio that have the same duration but higher convexity.

	Price	duration (years)	Convexity (years)
Bond 1	\$105.96	5	28.62
Bond 2	\$102.80	1	1.75
Bond 3	\$97.91	9	95.72
Portfolio	\$105.96	5	48.73

Portfolio consists of $N_2 = 0.5153$ units of Bond 2 and $N_3 = 0.5411$ units of Bond 3. The weights N_2 and N_3 are obtained by equating the bond value and duration, which give

$$N_2 B_2 + N_3 B_3 = B_1 \quad \text{and} \quad \frac{N_2 B_2}{B_1} D_2 + \frac{N_3 B_3}{B_1} D_3 = D_1.$$

The barbell portfolio is seen to have higher convexity (48.73 for the portfolio versus 28.62 for Bond 1). However, it is likely that the portfolio has lower yield than that of Bond 1 due to convexity of the yield curve.

1. Comparison of bond values under changes in interest rate

Values of bond 1 and the barbell portfolio P for various values of interest rate

i	$B_1(i)$	$B_P(i)$
4%	122.28	123.48
5%	116.50	116.99
6%	111.06	111.18
7%	105.96	105.96
8%	101.16	101.25
9%	96.64	97.00
10%	92.38	93.15

$B_P(i)$ always achieves higher value than $B_1(i)$ under varying values of i.

The numerical results on the bond values are consistent with the plots of bond values shown on P.59.

2. Comparison of horizon rate of return under changes in interest rate

5-year horizon rates of return for bond 1 and the barbell portfolio

i	$r^1_{H=5}$	$r^P_{H=5}$
4%	7.023%	7.230%
5%	7.010%	7.100%
6%	7.003%	7.025%
7%	7%	7%
8%	7.003%	7.024%
9%	7.010%	7.092%
10%	7.023%	7.203%

$r^1_{H=5} < r^P_{H=5}$ for all varying values of i.

Asset and liabilities management

- How should a pension fund, or an insurance company, set up its asset portfolio in such a way as to be practically certain that it will be able to meet its payment obligations in the future?

Redington conditions

Assume that the liabilities flow L_t, $t = 1, \ldots, T$ and assets flow A_t, $t = 1, \ldots, T$, are known.

Interest rate term structure is flat, equal to i. The present value of the liabilities and assets are

$$L = \sum_{t=1}^{T} \frac{L_t}{(1+i)^t} \text{ and } A = \sum_{t=1}^{T} \frac{A_t}{(1+i)^t}.$$

We assume that the net value $N = A - L = 0$ initially.

How should one choose the structure of the assets such that this net value does not change in the event of a change in interest rate?

First order condition (first Redington condition):

$N = A - L$ to be insensitive to i.

$$\text{Set} \qquad \frac{dN}{di} = \frac{1}{1+i}\sum_{t=1}^{T} t(L_t - A_t)(1+i)^{-t} = \frac{1}{1+i}(D_L L - D_A A)$$
$$= \frac{A}{1+i}(D_L - D_A) = 0, \ \text{(since } L = A\text{)},$$

where

$$D_L = \sum_{t=1}^{T} \frac{tL_t}{L}\frac{1}{(1+i)^t} \quad \text{and} \quad D_A = \sum_{t=1}^{T} \frac{tA_t}{L}\frac{1}{(1+i)^t}.$$

To satisfy the first Redington condition, we need to observe equality of the two durations, D_L and D_A.

Recall that

$$\Delta N = \Delta(A - L) \approx \left(\frac{\mathrm{d}^2 A}{\mathrm{d}i^2} - \frac{\mathrm{d}^2 L}{\mathrm{d}i^2}\right) \Delta i^2$$

when A and L have the same duration.

In order that N remains positive, a sufficient condition is given by $N(i)$ being a convex function of i within that interval. This is captured by

Second Redington condition: $\dfrac{d^2 A}{di^2} > \dfrac{d^2 L}{di^2}$.

Once the duration is set to be the same for both A and L, convexity depends positively on the dispersion S of the cash flows. Therefore, a sufficient condition for the second Redington condition is that the dispersion of the inflows from the assets is larger than that of the outflows to the liabilities.

Example (Savings and Loan Associations in US in early 1980s)

They had deposits with short maturities (duration) while their loans to mortgage developers had very long durations, since they financed mainly housing projects. Their assets are loans to housing projects while their liabilities are deposits.

- When the interest rates climbed sharply, the net worth of the Savings and Loans Associations fall drastically.

  In this case, even the first Redington condition was not met. This spelled disaster.

Numerical example (Net initial position of the financial firm is zero)

- Asset: Investing \$1 million in a 20-year, 8.5% coupon bond.
- Liability: Financed with a 9-year loan carrying an 8% interest rate.

We set the initial value of the asset and liability to be the same. Recall the generalized duration formula:

$$D = \frac{1}{i} + \theta + \frac{\frac{N}{m}(i - \frac{c}{B_T}) - (1 + \frac{i}{m})}{\frac{c}{B_T}[(1 + \frac{i}{m})^N - 1] + i},$$

where

$$\theta = \text{time to wait for the next coupon to be paid } (0 \leq \theta \leq \frac{1}{m})$$
$$m = \text{number of times a payment is made within one year}$$
$$N = \text{total number of coupons remaining to be paid.}$$

Here, $\theta = \frac{1}{2}$ when $m = 2$ (semi-annual payments); $N = 40$ for the 20-year bond and $N = 18$ for the 9-year loan.

We obtain the respective duration of the asset and liability as

$$D_A = 9.944 \text{ years} \quad \text{and} \quad D_L = 6.583 \text{ years}.$$

To secure profits in operating loans and savings, the housing loans interest rate should be higher than the deposits interest rates. Therefore, we must have $i_A > i_L$. For $i_A = 8.5\%$ and $i_L = 8\%$, the modified durations are

$$D_{mA} = \frac{D_A}{1 + i_A} = 9.165 \text{ years} \quad \text{and} \quad D_{mL} = \frac{D_L}{1 + i_L} = 6.095 \text{ years}.$$

Note that

$$\frac{\Delta V_A}{V_A} \approx \frac{dV_A}{V_A} = -D_{mA}\ di_A \quad \text{and} \quad \frac{\Delta V_L}{V_L} \approx \frac{dV_L}{V_L} = -D_{mL}\ di_L$$

so that

$$\Delta V_P = \Delta V_A - \Delta V_L \approx -(D_{mA}V_A\ di_A - D_{mL}V_L\ di_L).$$

Suppose i_A and i_L receive the same increment and $V_A = V_L$, we have

$$\Delta V_P \approx -(D_{mA} - D_{mL})V_A\ di = -3.070V_A\ di.$$

Based on the linear approximation, if the interest rate increases by 1%, the net value of the project diminishes by 3.070% of the asset. Its risk exposure presents a net modified duration of $D_{mA} - D_{mL} = 3.070$.

Summary

1. Immunization is a short-term series of measures destined to match sensitivities of assets and liabilities. As time passes, these sensitivities continue to change since the duration does not generally decrease in the same amount as the planning horizon with the passage of time.

2. Whenever interest rates change, the duration also changes. Financial manager may also want to pay special attention to the convexity of his assets and liabilities as well.

3. So far we have considered flat term structures and parallel displacements of them. More refined duration measures and analysis are required if we do not face such flat structures. The next level of more refined analysis is the use of deterministic term structure of spot rates.

Measuring the riskiness of foreign currency-denominated bonds

Let B be the value of a foreign bond in foreign currency, e be the exchange rate (value of one unit of foreign currency in domestic currency), V be the value of the foreign bond in domestic currency. We have

$$V = eB$$

so that

$$\frac{dV}{V} = \frac{dB}{B} + \frac{de}{e}.$$

All the three relative changes are random variables. Observe that

$$\frac{dB}{B} = \frac{1}{B}\frac{dB}{di}di = -\frac{D}{1+i}di$$

so that

$$\frac{dV}{V} = -\frac{D}{1+i}di + \frac{de}{e}.$$

Recall that

$$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y) + 2\text{cov}(X,Y).$$

We then deduce that

$$\mathrm{var}\left(\frac{dV}{V}\right) = \left(\frac{D}{1+i}\right)^2 \mathrm{var}(di) + \mathrm{var}(\frac{de}{e}) - 2\frac{D}{1+i}\mathrm{cov}(di, \frac{de}{e}).$$

- The covariance between changes in interest rates and modifications in the exchange rates is usually very low. The bulk of the variance of changes in the foreign bond's value stems mainly from the variance of the exchange rate.

- Empirical studies show that the share of the exchange rate variance is easily two-thirds of the total variance.

For finite changes in e and B, the correct formula should be

$$\frac{\Delta V}{V} = \frac{(B+\Delta B)(e+\Delta e) - Be}{Be} = \frac{\Delta B}{B} + \frac{\Delta e}{e} + \frac{\Delta B}{B}\frac{\Delta e}{e}.$$

The second order term can be significant when $\frac{\Delta B}{B}$ and $\frac{\Delta e}{e}$ are large.

Numerical example

Suppose that the loss on the Jakarta stock market was 60% in a given period and that the rupee lost 60% of its value against the dollar in the same period.

- The rate of change in the investment's value in American dollars cannot be $(-60\%) + (-60\%) = (-120\%)$. It does not make sense to have a loss of more than 100%.

- It is more proper to use

$$
\begin{aligned}
\frac{\Delta V}{V} &= (-60\%) + (-60\%) + (-60\%)(-60\%) \\
&= -120\% + 36\% = -84\%.
\end{aligned}
$$

Cash matching problem – Linear programming with constraints

- A known sequence of future monetary obligations over n periods:

$$\mathbf{y} = (y_1 \ldots y_n).$$

- Purchase bonds of various maturities and use the coupon payments and redemption values to meet the obligations.

Suppose there are m bonds, and the cash stream on dates $1, 2, \ldots, n$ associated with one unit of bond j is $\mathbf{c}_j = (c_{1j} \ldots c_{nj})$, $j = 1, 2, \ldots, m$.

$$p_j = \text{price of bond } j$$

$$x_j = \text{amount of bond } j \text{ held in the portfolio}$$

$$\begin{array}{lll}
\text{Minimize} & \displaystyle\sum_{j=1}^{m} p_j x_j & \\
\text{subject to} & \displaystyle\sum_{j=1}^{m} c_{ij} x_j \geq y_i & i = 1, 2, \cdots, n \\
& x_j \geq 0 & j = 1, 2, \cdots, m.
\end{array}$$

Numerical example – Six-year match of cash obligations using 10 bonds

Cash matching example

	Bonds											
Yr	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Req'd**	**Actual**
1	10	7	8	6	7	5	10	8	7	100	100	171.74
2	10	7	8	6	7	5	10	8	107		200	200.00
3	10	7	8	6	7	5	110	108			800	800.00
4	10	7	8	6	7	105					100	119.34
5	10	7	8	106	107						800	800.00
6	110	107	108								1,200	1,200.00
p	109	94.8	99.5	93.1	97.2	92.9	110	104	102	95.2	2,381.14	
x	0	11.2	0	6.81	0	0	0	6.3	0.28	0	Cost	

In two of the 6 years (Year One and Year Four), some extra cash is generated beyond what is required. Note that in Year Four, we have

$$11.2 \times 7 + 6.81 \times 6 = 119.34 > 100.$$

Difficulties and weaknesses

- There are high liabilities outflows in some years so a larger number of bonds must be purchased that mature on those dates. These bonds generate coupon payments in earlier years and only a portion of these payments is needed to meet obligations in these early years. Such problem is alleviated with a smoother set of liabilities outflows.

- At the end of Year 6, the cash flow required is $\$1,200$, so either Bond 1, Bond 2 and / or Bond 3 must be purchased. Bond 2 is chosen since it has lower coupon payments in earlier years so that less extra cash is generated in those earlier years.

- Liabilities are being met by coupon payments or principals of maturing bonds. Under passive static bond portfolio management, there will be no sale or acquisition of bonds. The only risk is default risk. Adverse interest rate changes do not affect the ability to meet liabilities.

- For typical liabilities schedule and bonds available for cash flow matching, perfect matching is unlikely.

- We strike the tradeoff between (i) avoidance of the risk of not satisfying the liabilities stream (ii) lower cost in building the bond portfolio that meets the liabilities.

- How to combine immunization with cash matching? More precisely, how one strikes the balance between lower cost of constructing the bond portfolio against smaller difference in duration (more ideal to have higher convexity as well).

- The extra surpluses should be reinvested so this creates reinvestment risks. This requires the estimation of the future interest rate movements as well.

1.4 Optimal management and dynamic programming

Dynamic programming solves a problem step by step, starting at the terminal time and working back to the beginning. This is commonly called the "backward induction procedure". From the end of a problem or situation, we determine a sequence of optimal actions. It proceeds by first considering the last time an optimal decision might be made and choosing what to do in any situation at that time. Using this information, one can then determine what to do at the second-to-last time of decision.

We use the one-period discount factor d_k and evaluate the present value step by step. In running dynamic programming, we assign to each node a value equal to the best running present value that can be obtained from that node.

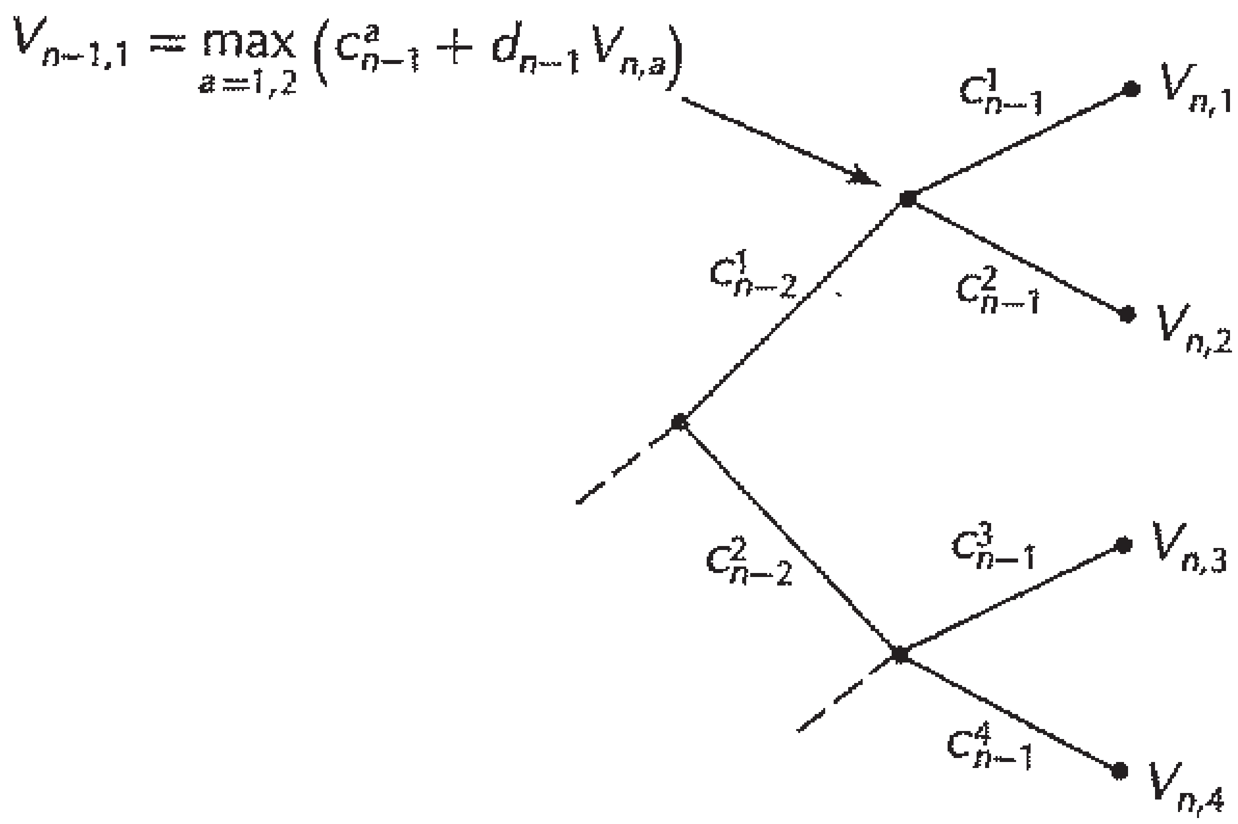
For the ith node at time k, denoted by (k, i), the best running value is called V_{ki}. We refer to these values as V-values.

The V-values at the final nodes are just the terminal values of the investment process. Usually, the V-values at the final nodes are known as part of the problem description.

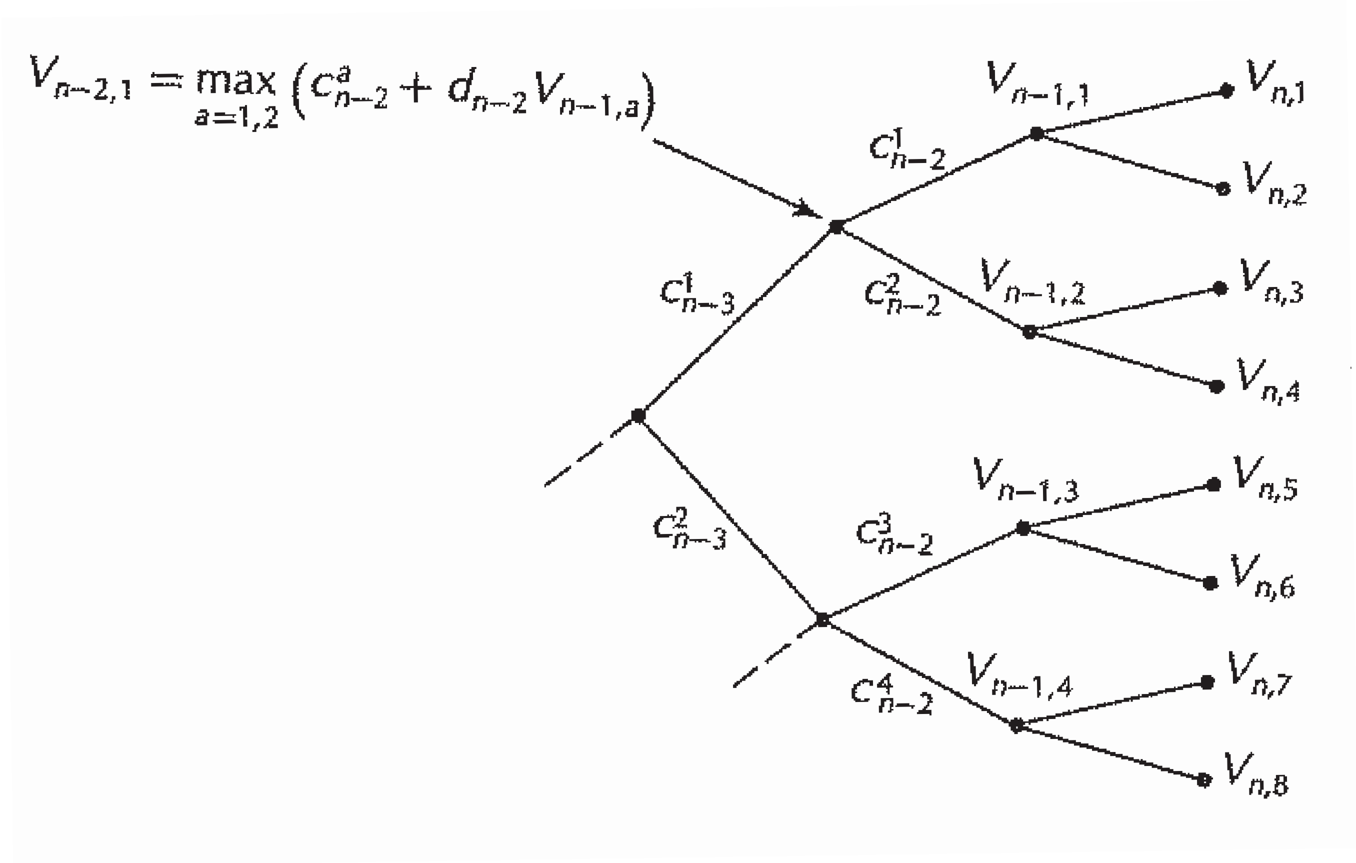
Recurrence relation

Define c_{ki}^a to be the cash flow generated by moving from node (k, i) to node $(k+1, a)$. The recursion procedure is

$$V_{ki} = \underset{a}{\text{maximize}}(c_{ki}^a + d_k V_{k+1,a}).$$



First recursive step of dynamic programming. For any node at time $n-1$, we find the maximum running present value from that node.



Second stage of dynamic programming. We evaluate the best running present value for the remaining two steps.

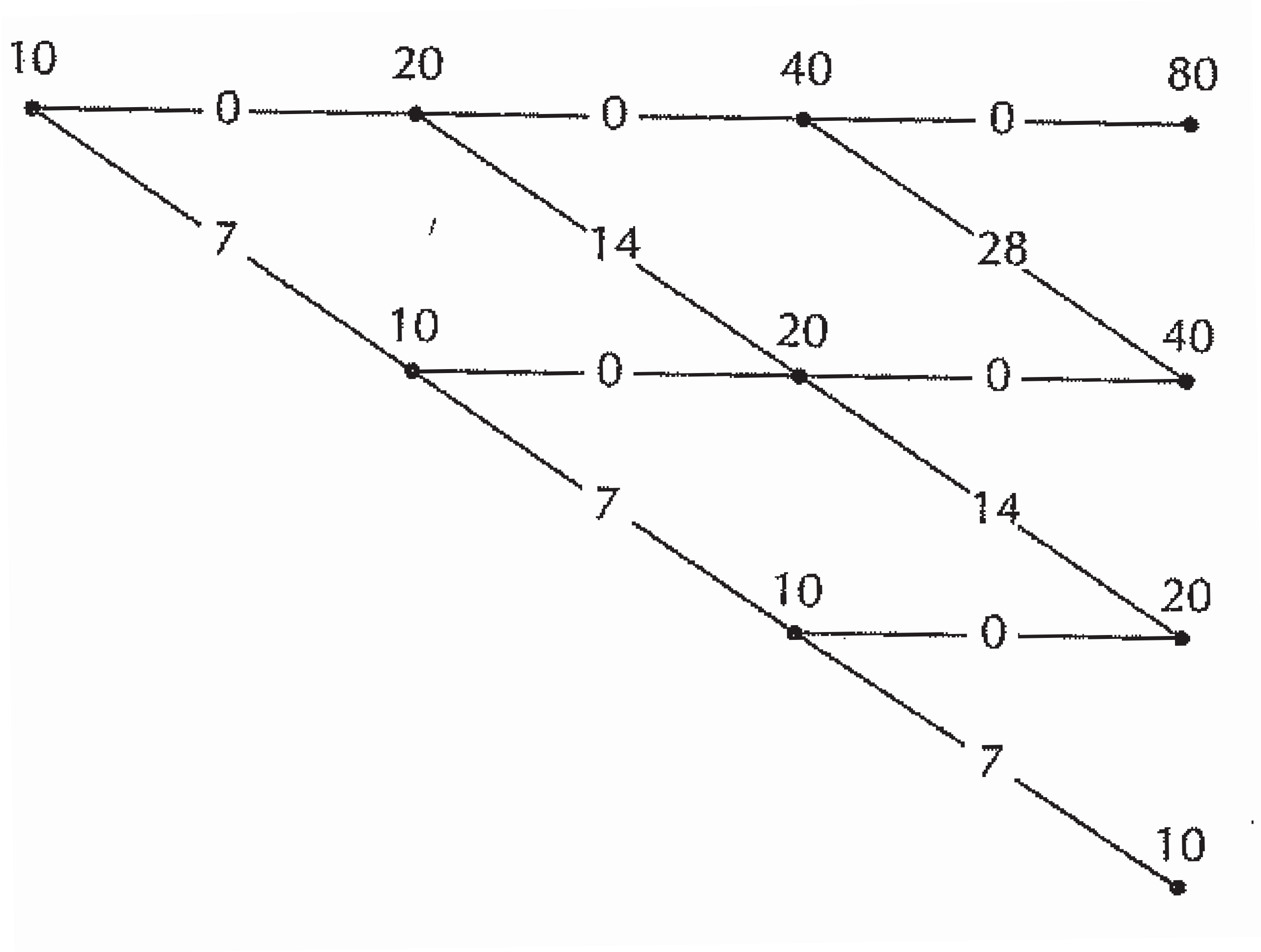
Example – Fishing problem

- If you do not fish, the fish population doubles in the next year.
- If you fish and extract 70% of the fish, then the fish population will grow to the same amount at the beginning of next fishing season.

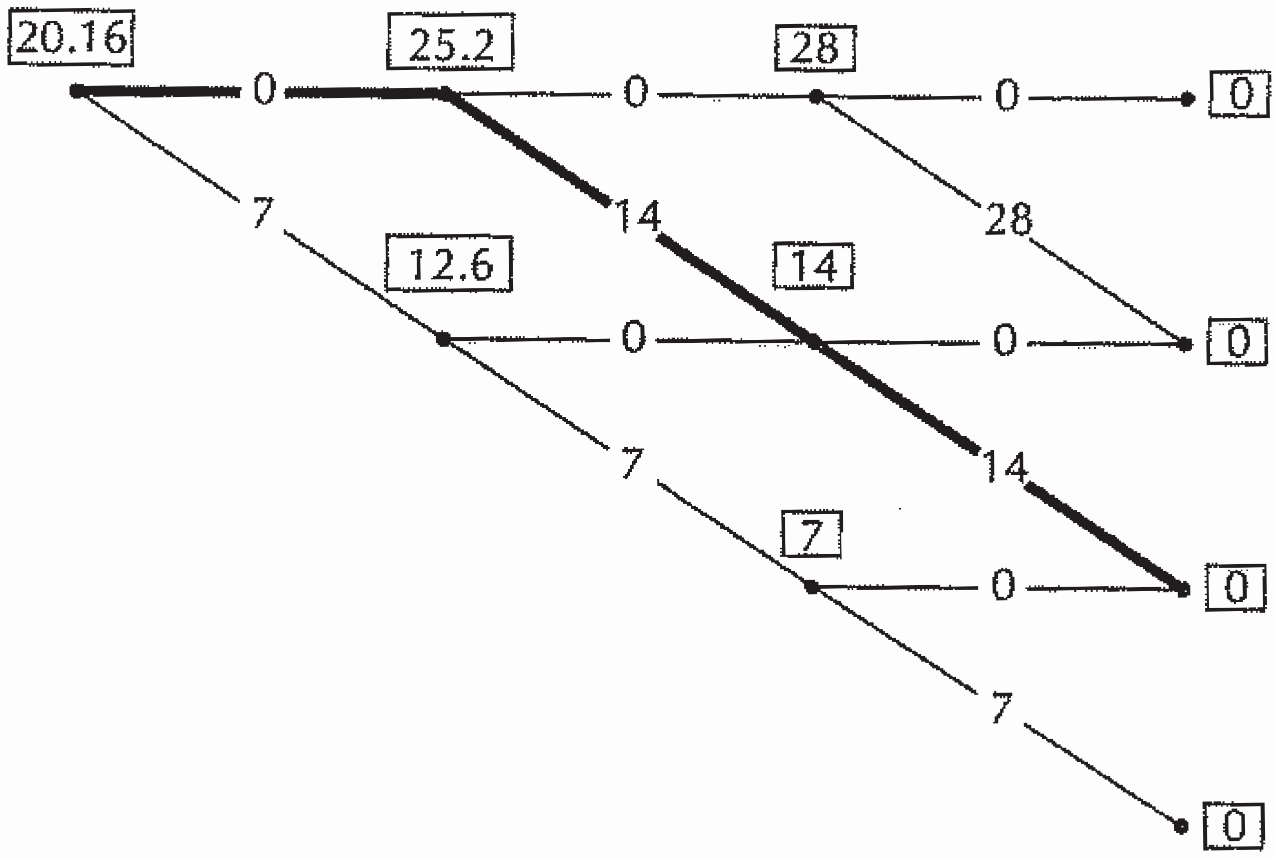
The interest rate is constant at 25%, which means that the discount factor is 0.8 per year. The initial fish population is 10 tons. The profit is \$1 per ton.

Remark

In this example, we fix the percentage of fish extracted to be 70%. In the next example, we relax this constraint of fixing the percentage of extraction.



The node values are the tonnage of fish in the lake; the branch values are cash flows.



The node values are now the optimal running present values, found by working backward from the terminal nodes. The branch values are cash flows.

Once we are in Year Three, we can no longer fish. Therefore, we assign the value of 0 to each of the final nodes. By backward induction, at each of the nodes one step from the end, we determine the maximum possible cash flow.

This determines the cash flow received in that season, and we assume that we obtain that cash of selling the fish at the beginning of the season. Hence we do *not* discount the profit. The value obtained is the (running) present value, as viewed from that time.

Next we back up one time period and calculate the maximum present values at that time. For example, for the node just to the right of the initial node, we have

$$V = \max(0.8 \times 28, 14 + 0.8 \times 14).$$

The maximum is attained by the second choice, corresponding to the downward branch, and hence $V = 14 + 0.8 \times 14 = 25.2$. The discount rate of $1/1.25 = 0.8$ is applicable at every stage.

For the node below, we compute

$$V = \max(0.8 \times 14, 7 + 0.8 \times 7) = 12.6.$$

For the initial node, we obtain

$$V = \max(0.8 \times 25.2, 7 + 0.8 \times 12.6) = 20.16.$$

The value at the initial node gives the maximum present value. The optimal path is the path determined by the optimal choices we discovered in the procedure.

The optimal path is indicated by the heavy line. In words, the solution is not to fish the first season (to let the fish population increase) and then fish the next two seasons (to harvest the population).

Example – Mining

The mine has been worked heavily and is approaching depletion. If x is the amount of gold remaining in the mine at the beginning of a year, the cost to extract $z < x$ ounces of gold in that year is \$$500z^2/x$. Note that as x decreases, it becomes more difficult to obtain gold. This particular form of the cost as a square function of z allows simple solution in the backward induction procedure.

It is estimated that the current amount of gold remaining in the mine is $x_0 = 50,000$ ounces. The price of gold is \$400/oz.

We are considering the purchase of a 10-year lease of the mine. The interest rate is 10%. How much is this lease worth?

Note that the extraction amount z in each year is the control variable whose value is to be determined in the dynamic programming procedure at each time step.

Backward induction procedure

We begin by determining the value of a lease on the mine at time 9, when the remaining deposit is x_9. Only 1 year remains on the lease, so the value is obtained by maximizing the profit for that year. If we extract z_9 ounces, the revenue from the sale of the gold will be gz_9, where g is the price of gold, and the cost of mining will be $500z_9^2/x_9$. Hence the optimal value of the mine at time 9 if x_9 is the remaining deposit level is

$$V_9(x_9) = \max_{z_9}(gz_9 - 500z_9^2/x_9).$$

We find the maximum by setting the derivative with respect to z_9 equal to zero. This yields

$$z_9 = gx_9/1,000.$$

We should check that $z_9 \leq x_9$, which does hold with the values we use.

We substitute this value in the formula for profit to find

$$V_9(x_9) = \frac{g^2 x_9}{1,000} - \frac{500 g^2 x_9}{1,000 \times 1,000} = \frac{g^2 x_9}{2,000}.$$

We write this as $V_9(x_9) = K_9 x_9$, where $K_9 = g^2/2,000$ is a constant. Hence the value of the lease is directly proportional to how much gold remains in the mine; the proportionality factor is K_9.

Next we back up and solve for $V_8(x_8)$. In this case we account for the profit generated during the ninth year and also for the value that the lease will have at the end of that year – a value that depends on how much gold we leave in the mine. Hence,

$$V_8(x_8) = \max_{z_8}[g z_8 - 500 z_8^2/x_8 + d V_9(x_8 - z_8)].$$

Note that we have discounted the value associated with the mine at the next year by a factor d. As in the previous example, the discount rate is constant because the spot rate curve is flat. In this case, $d = 1/1.1$.

Using the explicit form for the function V_9, we may write

$$V_8(x_8) = \max_{z_8}[gz_8 - 500z_8^2/x_8 + dK_9(x_8 - z_8)].$$

We again set the derivative with respect to z_8 equal to zero and obtain

$$z_8 = \frac{(g - dK_9)x_8}{1,000}.$$

This value can be substituted into the expression for V_8 to obtain

$$V_8(x_8) = \left[\frac{(g - dK_9)^2}{2,000} + dK_9\right] x_8.$$

This is proportional to x_8, and we may write it as $V_8(x_8) = K_8 x_8$.

We can continue backward in this way, determining the functions $V_7, V_6, \ldots,$ V_0. Each of these functions will be of the form $V_j(x_j) = K_j x_j$. It is seen that the same algebra applies at each step, and hence we have the recursive formula

$$K_j = \frac{(g - dK_{j+1})^2}{2,000} + dK_{j+1}.$$

If we use the specific values $g = 400$ and $d = 1/1.1$, we begin the backward recursion with $K_9 = g^2/2,000 = 80$. We can then easily solve for all the other values, as shown in the Table, working from the right to the left.

Year	0	1	2	3	4	5	6	7	8	9
K-value	213.81	211.45	208.17	203.58	197.13	187.96	174.79	155.47	126.28	80.00

K-values for the mine

It is K_0 that determines the value of the original lease, which is determined by finding the value of the lease when there is $50,000$ ounces of gold remaining. Hence $V_0(50,000) = 213.82 \times 50,000 = \$10,691,000$.

The optimal plan is determined as a by-product of the dynamic programming procedure. At any time j, the amount of gold to extract is the value z_j found in the optimization problem. Hence $z_9 = gx_9/1,000$ and $z_8 = (g - dK_9)x_8/1,000$. In general, we obtain

$$z_j = (g - dK_{j+1})x_j/1,000.$$

We start from $x_0 = 50,000$, then

$$z_0 = \frac{8 - dK_1}{1,000}50,000 = \frac{400 - \frac{211.45}{1.1}}{1,000}50,000 = 10,388.6.$$

Next, $x_1 = x_0 - z_0 = 39,611.4$. We then have

$$z_1 = \frac{400 - \frac{208.17}{1.1}}{1,000}39,611.4 = 7,716.$$

In Qn 10 of HW 1, we consider the perpetual ownership of the mine. In this case, the K-values are independent of the years of operation. The corresponding equation for the common K is given by

$$K = \frac{(g - dK)^2}{2000} + dK.$$