MATH4512 – Fundamentals of Mathematical Finance

Topic Three — Capital asset pricing model and factor models

3.1 Capital asset pricing model and beta values

3.2 Interpretation and uses of the capital asset pricing model

3.3 Arbitrage pricing theory and factor models

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3.1 Capital asset pricing model and beta values

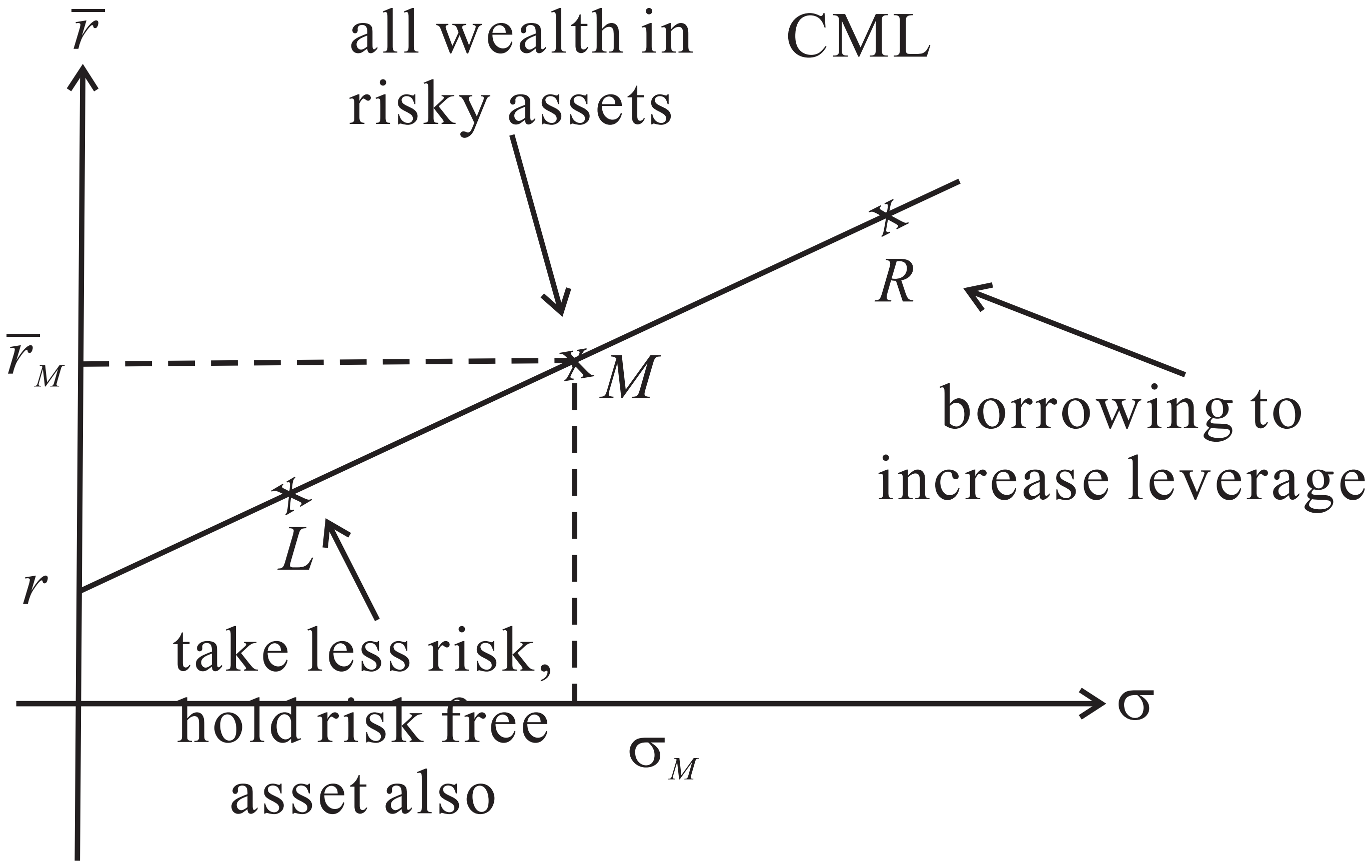
Capital market line (CML)

The CML is the tangent line drawn from the risk free point to the feasible region for risky assets. This line shows the relation between $\overline{r}_P$ and σ_P for efficient portfolios (risky assets plus the risk free asset). The tangency point M represents the *market portfolio*.

- Every investor is a mean-variance investor and they all have homogeneous expectations on means and variances, then everyone buys the same portfolio. The proportional weights of this portfolio would be the same as those of the market portfolio. Prices adjust to drive the market to efficiency when market equilibrium prevails.

All portfolios on the CML are efficient, and they are composed of various mixes of the market portfolio and the risk free asset.

Based on the risk level that an investor can take, she combines the market portfolio of risky assets with the risk free asset.



Equation of the CML:

$$\overline{r} = r + \frac{\overline{r}_M - r}{\sigma_M}\sigma,$$

where $\overline{r}$ and σ are the mean and standard deviation of the rate of return of an efficient portfolio.

Slope of the CML $= \dfrac{\overline{r}_M - r}{\sigma_M} =$ price of risk of an efficient portfolio.

The market price of risk (Sharpe ratio) indicates how much the expected rate of return above the riskless interest rate should be demanded when the standard deviation increases by one unit.

The CML does not apply to an individual asset or portfolios that are inefficient.

Example Consider an oil drilling venture; current share price of the venture = \$875, expected to yield \$1,000 in one year. The standard deviation of return, $\sigma = 40\%$; and $r = 10\%$. Also, $\overline{r}_M = 17\%$ and $\sigma_M = 12\%$ for the market portfolio.

Question How does this venture compare with the investment on efficient portfolios on the CML?

Given this level of σ, the expected rate of return as predicted by the CML is

$$\overline{r} = 0.10 + \frac{0.17 - 0.10}{0.12} \times 0.40 = 33\frac{1}{3}\%.$$

The actual expected rate of return $= \dfrac{1,000}{875} - 1 = 14\%$, which is well below $33\frac{1}{3}\%$. This venture does not constitute an efficient portfolio. It bears certain type of risk that does not contribute to the expected rate of return.

Sharpe ratio

One index that is commonly used in performance measure is the Sharpe ratio, defined as

$$\frac{\overline{r}_i - r}{\sigma_i} = \frac{\text{excess expected rate of return above riskfree rate}}{\text{standard deviation}}.$$

We expect

$$\text{Sharpe ratio} \leq \text{slope of CML}.$$

When the Sharpe ratio is closer to the slope of CML, the better the performance of the fund in terms of return against risk.

In the previous example,

$$\begin{aligned} \text{Slope of CML} &= \frac{17\% - 10\%}{12\%} = \frac{7}{12} = 0.583 \\ \text{Sharpe ratio} &= \frac{14\% - 10\%}{40\%} = 0.1 < \text{ Slope of CML}. \end{aligned}$$

Capital Asset Pricing Model

Let M be the market portfolio M, then the expected rate of return $\overline{r}_i$ of any asset i satisfies

$$\overline{r}_i - r = \beta_i(\overline{r}_M - r)$$

where

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}.$$

Here, $\sigma_{iM} = \text{cov}(r_i, r_M)$ is the covariance between the rate of return of risky asset i and the rate of return of the market portfolio M.

Remark

Expected excess rate of return of a risky asset above r is related to the correlation of r_i with r_M.

Assumptions underlying the standard CAPM

1. No transaction costs.

2. Assets are infinitely divisible.

3. Absence of capital gain tax.

4. An individual cannot affect the price of a stock by his buying or selling action. All investors are *price takers*.

5. Unlimited short sales are allowed.

6. Unlimited long and short holding of the riskfree asset.

7. Investors are assumed to be concerned with the mean and variance of returns, and all investors set the same investment horizon.

8. All investors are assumed to have identical parameter estimation of the covariance matrix and expected rate of return vector in the mean-variance portfolio choice model.

Both (7) and (8) are called the "homogeneity of expectations".

The CAPM relies on the mean-variance approach, homogeneity of expectation of investors, and no market frictions. In equilibrium, every investor must invest in the same fund of risky assets (market portfolio) and the risk free asset.

Proof

Consider the portfolio with α portion invested in asset i and $1-\alpha$ portion invested in the market portfolio M. The expected rate of return of this portfolio is

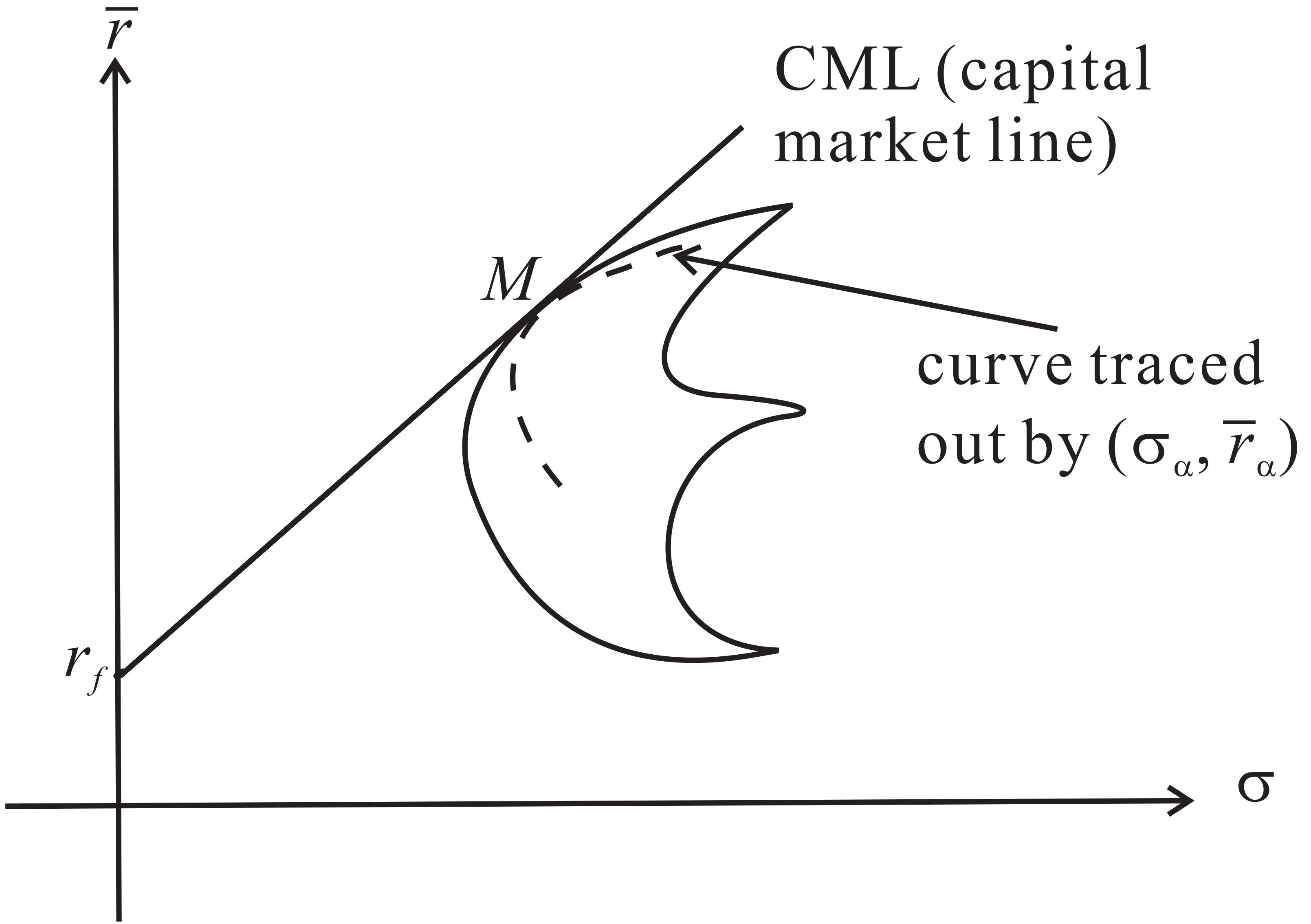
$$\overline{r}_\alpha = \alpha\overline{r}_i + (1-\alpha)\overline{r}_M$$

and its variance is

$$\sigma_\alpha^2 = \alpha^2\sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2\sigma_M^2.$$

As α varies, $(\sigma_\alpha, \overline{r}_\alpha)$ traces out a curve in the $\sigma - \overline{r}$ diagram. The market portfolio M corresponds to $\alpha = 0$.

The curve must lie within the feasible region consisting of risky assets. As α passes through zero, the curve traced out by $(\sigma_\alpha, \overline{r}_\alpha)$ must be tangent to the CML at M since the CML intersects the dotted curve at one single point M.



Tangency condition Slope of the curve at M = slope of the CML.

First, we obtain $\frac{d\bar{r}_\alpha}{d\alpha} = \bar{r}_i - \bar{r}_M$, which has no dependence on α since $\bar{r}_\alpha$ is linear in α. Also,

$$\frac{d\sigma_\alpha}{d\alpha} = \frac{\alpha\sigma_i^2 + (1-2\alpha)\sigma_{iM} + (\alpha-1)\sigma_M^2}{\sigma_\alpha}$$

so that $\left.\frac{d\sigma_\alpha}{d\alpha}\right|_{\alpha=0} = \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}$.

Next, we apply the relation $\frac{d\bar{r}_\alpha}{d\sigma_\alpha} = \frac{\frac{d\bar{r}_\alpha}{d\alpha}}{\frac{d\sigma_\alpha}{d\alpha}}$ to obtain

$$\left.\frac{d\bar{r}_\alpha}{d\sigma_\alpha}\right|_{\alpha=0} = \frac{(\bar{r}_i - \bar{r}_M)\sigma_M}{\sigma_{iM} - \sigma_M^2}.$$

However, $\left.\frac{d\bar{r}_\alpha}{d\sigma_\alpha}\right|_{\alpha=0}$ should be equal to the slope of the CML, that is,

$$\frac{(\bar{r}_i - \bar{r}_M)\sigma_M}{\sigma_{iM} - \sigma_M^2} = \frac{\bar{r}_M - r}{\sigma_M}.$$

Solving for $\overline{r}_i$, we obtain

$$\overline{r}_i = r + \underbrace{\frac{\sigma_{iM}}{\sigma_M^2}}_{\beta_i}(\overline{r}_M - r) = r + \beta_i(\overline{r}_M - r).$$

Now, $\beta_i = \dfrac{\overline{r}_i - r}{\overline{r}_M - r}$

$$= \frac{\text{expected excess rate of return of asset } i \text{ over } r}{\text{expected excess rate of return of market portfolio over } r}.$$

Predictability of equilibrium return

The CAPM implies that in equilibrium the expected excess rate of return on any single risky asset is proportional to the expected excess rate of return on the market portfolio. The constant of proportionality is called the beta of the risky asset.

Alternative proof of CAPM

Recall $\boldsymbol{w}_M^* = \dfrac{\Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1})}{b - ar}$ so that for any portfolio P, we have

$$\sigma_{PM} = \boldsymbol{w}_P^T \Omega \boldsymbol{w}_M^* = \frac{\boldsymbol{w}_P^T(\boldsymbol{\mu} - r\mathbf{1})}{b - ar} = \frac{\mu_P - r}{b - ar}.$$

Taking P to be M, we obtain

$$\sigma_M^2 = \frac{\mu_M - r}{b - ar}.$$

By eliminating the common term $b - ar$ in σ_{PM} and σ_M^2, we then deduce a similar formula of the CAPM for any portfolio P, where

$$\mu_P - r = \frac{\sigma_{PM}}{\sigma_M^2}(\mu_M - r).$$

The CAPM remains valid if we take the portfolio P to consist of a single asset i only. This gives the same result as in the earlier proof:

$$\overline{r}_i - r = \frac{\sigma_{iM}}{\sigma_M^2}(\overline{r}_M - r).$$

Beta of a portfolio

Consider a portfolio containing n risky assets with weights $w_1, \cdots, w_n$. Since $r_P = \sum_{i=1}^{n} w_i r_i$, we have $\text{cov}(r_P, r_M) = \sum_{i=1}^{n} w_i \text{cov}(r_i, r_M)$ so that

$$\beta_P = \frac{\text{cov}(r_P, r_M)}{\sigma_M^2} = \frac{\sum_{i=1}^{n} w_i \text{cov}(r_i, r_M)}{\sigma_M^2} = \sum_{i=1}^{n} w_i \beta_i.$$

The portfolio beta is given by the weighted average of the beta values of the risky assets in the portfolio.

Since $\overline{r}_P = \sum_{i=1}^{n} w_i \overline{r}_i$ and $\beta_P = \sum_{i=1}^{n} w_i \beta_i$, and for each asset i, the CAPM gives: $\overline{r}_i - r = \beta_i(\overline{r}_M - r)$. Noting $\sum_{i=1}^{n} w_i = 1$, we then have

$$\overline{r}_P - r = \beta_P(\overline{r}_M - r).$$

Various interpretations of the CAPM

- If we write $\sigma_{iM} = \rho_{iM}\sigma_i\sigma_M$, then the CAPM can be rewritten as

$$\frac{\overline{r}_i - r}{\sigma_i} = \rho_{iM}\frac{\overline{r}_M - r}{\sigma_M}.$$

  The Sharpe ratio of asset i is given by the product of ρ_{iM} and the slope of CML. When ρ_{iM} is closer to one, the asset is closer to (but remains to stay below) the CML. For an efficient portfolio e that lies on the CML, we then deduce that $\rho_{eM} = 1$.

- For any two risky assets i and j, we have

$$\frac{\overline{r}_i - r}{\beta_i} = \frac{\overline{r}_j - r}{\beta_j} = \overline{r}_M - r.$$

  Under the CAPM, the expected excess rate of return above r normalized by the beta value is constant for all assets. On the other hand, the Sharpe ratios of two assets are related by

$$\frac{(\text{Sharpe ratio})_i}{\rho_{iM}} = \frac{\overline{r}_i - r}{\rho_{iM}\sigma_i} = \frac{\overline{r}_j - r}{\rho_{jM}\sigma_j} = \frac{(\text{Sharpe ratio})_j}{\rho_{jM}} = \frac{\overline{r}_M - r}{\sigma_M}.$$

Beta of an efficient portfolio

Let P be an efficient portfolio on the CML, then

$$r_P = \alpha r_M + (1-\alpha)r$$

where α is the proportional weight of the market portfolio M. Consider

$$\begin{aligned}
\text{cov}(r_P, r_M) &= \text{cov}(\alpha r_M + (1-\alpha)r, r_M) = \alpha \text{var}(r_M) = \alpha\sigma_M^2 \\
\text{var}(r_P) &= \alpha^2\sigma_M^2; \text{hence} \\
\rho_{PM} &= \frac{\text{cov}(r_P, r_M)}{\sigma_P\sigma_M} = \frac{\alpha\sigma_M^2}{\alpha\sigma_M\sigma_M} = 1,
\end{aligned}$$

thus verifying the earlier claim. Furthermore, it is seen that

$$\beta_P = \frac{\text{cov}(r_P, r_M)}{\text{var}(r_M)} = \alpha\frac{\text{var}(r_M)}{\text{var}(r_M)} = \alpha.$$

The beta value of an efficient portfolio is equal to the proportional weight α of the market portfolio in the efficient portfolio. This is obvious since $\overline{r}_P - r$ is contributed only by the α proportion of market portfolio in the efficient portfolio.

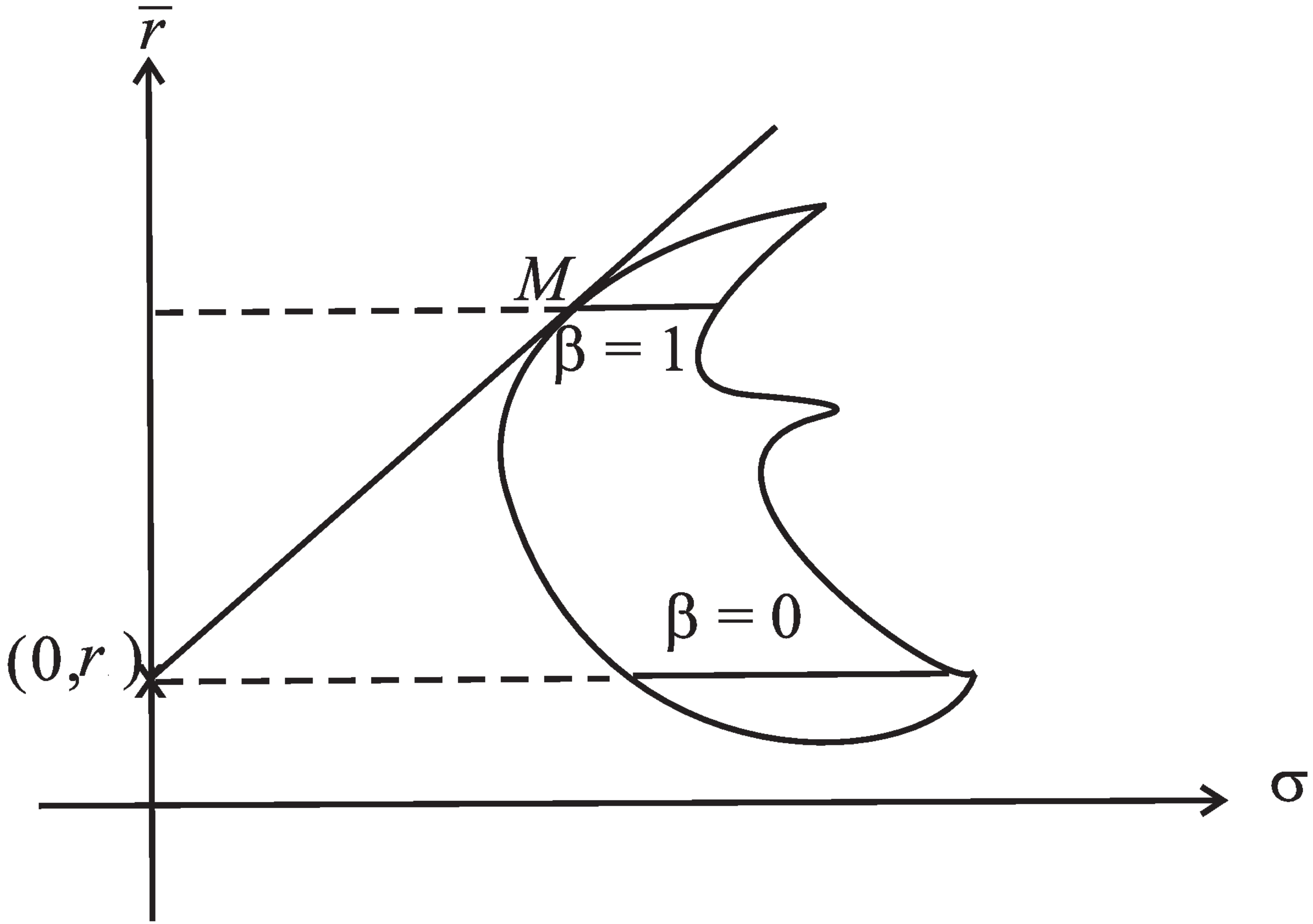
Some special cases of beta values

1. When $\beta_i = 0, \overline{r}_i = r$. A risky asset (with $\sigma_i > 0$) that is uncorrelated with the market portfolio will have an expected rate of return equal to the risk free rate. There is no expected excess return over r even the investor bears some risk in holding a risky asset with zero beta.

2. When $\beta_i = 1, \overline{r}_i = r_M$. The risky asset has the same expected rate of return as that of the market portfolio. The risk as quantified by σ_i is higher than σ_M.

3. When $\beta_i > 1$, the expected excess rate of return of the risky asset is higher than that of market portfolio. It is considered as an *aggressive asset*. When $\beta_i < 1$, the asset is said to be *defensive*.

4. When $\beta_i < 0, \overline{r}_i < r$. Such an asset is considered inferior.



Representation of the risky assets or portfolios of risky assets with $\beta = 0$ and $\beta = 1$ in the $\sigma - \overline{r}$ diagram.

Example

Assume that the expected rate of return on the market portfolio is 12% per annum and the rate of return on the riskfree asset is 7% per annum. The standard deviation of the market portfolio is 32% per annum.

(a) What is the equation of the capital market line?

CML is given by

$$\overline{r} = r + \left(\frac{\overline{r}_M - r}{\sigma_M}\right)\sigma = 0.07 + 0.1562\sigma.$$

(b) (i) If an expected return of 18% is desired for an efficient portfolio, what is the standard deviation of this portfolio?

Substituting $\overline{r} = 0.18$ into the CML equation, we obtain

$$\sigma = \frac{(0.18 - 0.07)}{0.1562} = 0.704.$$

(ii) If you have $\$1,000$ to invest, how should you allocate the wealth among the market portfolio and the riskfree asset to achieve the above portfolio?

Recall

$$\overline{r}_P = \alpha\overline{r}_M + (1-\alpha)r$$

so that

$$\alpha = \frac{\overline{r}_P - r}{\overline{r}_M - r} = \frac{0.18-0.07}{0.12-0.07} = \frac{0.11}{0.05} = 2.2.$$

Note that $1-\alpha = -1.2$. The investor should short sell $\$1,200$ of the riskfree asset and long $\$2,200$ of the market portfolio.

(iii) What is the beta value of this portfolio?

The beta value equals the weight of investment on the market portfolio in the efficient portfolio, so

$$\beta = \alpha = 2.2.$$

(c) If you invest \$300 in the riskfree asset and \$700 in the market portfolio, how much money should you expect to have at the end of the year?

The expected rate of return per annum is given by

$$E[r_P] = 0.3r + 0.7\overline{r}_M = 0.105.$$

The expected amount of money at the end of the year is

$$(\$300 + \$700)(1 + E[r_P]) = \$1,105.$$

Extension of CAPM – reference to an efficient portfolio

1. Let P be any efficient portfolio lying along the CML and Q be any portfolio. An extension of the CAPM gives

$$\overline{r}_Q - r = \beta_{QP}(\overline{r}_P - r), \quad \beta_{QP} = \frac{\sigma_{QP}}{\sigma_P^2}, \tag{A}$$

   that is, we may replace the market portfolio M by an efficient portfolio P.

2. More generally, the random rates of return r_P and r_Q are related by

$$r_Q - r = \beta_{QP}(r_P - r) + \epsilon_{QP} \tag{B}$$

   with $\text{cov}(r_P, \epsilon_{QP}) = E[\epsilon_{QP}] = 0$. The residual ϵ_{QP} has zero expected value and it is uncorrelated with r_P.

Proof

Since Portfolio P is efficient (lying on the CML), then

$$r_P = \alpha r_M + (1-\alpha)r, \quad \alpha > 0.$$

The first result (A) can be deduced from the CAPM by observing

$$\begin{aligned} \sigma_{QP} &= \text{cov}(r_Q, \alpha r_M + (1-\alpha)r) = \alpha\text{cov}(r_Q, r_M) = \alpha\sigma_{QM}, \quad \alpha > 0 \\ \sigma_P^2 &= \alpha^2\sigma_M^2 \quad \text{and} \quad \overline{r}_P - r = \alpha(\overline{r}_M - r). \end{aligned}$$

Putting the results together, we have

$$\begin{aligned} \overline{r}_Q - r &= \beta_{MQ}(\overline{r}_M - r) = \frac{\sigma_{QM}}{\sigma_M^2}(\overline{r}_M - r) \\ &= \frac{\sigma_{QP}/\alpha}{\sigma_P^2/\alpha^2}(\overline{r}_P - r)/\alpha = \beta_{QP}(\overline{r}_P - r). \end{aligned}$$

By performing the linear regression of r_Q on r_P, the relationship among r_Q and r_P can be formally expressed as

$$r_Q = \widehat{\alpha} + \widehat{\beta} r_P + \epsilon_{QP}, \qquad (C)$$

where $\widehat{\alpha}$ and $\widehat{\beta}$ are the resulting coefficients estimated from the linear regression. The residual ϵ_{QP} is taken to have zero expected value. If not, the mean of E_{QP} can be absorbed into $\widehat{\alpha}$. Observe that

$$\overline{r}_Q = \widehat{\alpha} + \widehat{\beta}\overline{r}_P$$

and from result (A), we obtain

$$\overline{r}_Q = \beta_{QP}\overline{r}_P + r(1 - \beta_{QP})$$

so that

$$\widehat{\alpha} = r(1 - \beta_{QP}) \quad \text{and} \quad \widehat{\beta} = \beta_{PQ}.$$

Hence, we obtain result (B).

The use of the term β in CAPM arises from the terminology in linear regression. Mutual fund managers are considered to be α (alpha) seekers since they try hard to raise α. Considering $\text{cov}(r_P, r_Q)$ and noting

$$\beta_{QP} = \text{cov}(r_P, r_Q)/\text{var}(r_P),$$

together with eq. (C), we obtain

$$\text{cov}(r_P, r_Q) = \beta_{QP}\text{var}(r_P) + \text{cov}(r_P, \epsilon_{QP})$$

so that $\text{cov}(r_P, \epsilon_{QP}) = 0$. The residual ϵ_{QP} has zero mean and it is uncorrelated with r_P. This is expected since the residual ϵ_{QP} is the component of r_Q that has no linear dependence on r_P.

Zero-beta CAPM: no reference to the risk free asset

There exists a minimum variance portfolio Z_M whose beta is zero. Since $\beta_{MZ_M} = 0$, we have $\overline{r}_{Z_M} = r$. Consider the following relation from CAPM

$$\overline{r}_Q = r + \beta_{QM}(\overline{r}_M - r),$$

it can be expressed in terms of the market portfolio M and its zero-beta counterpart Z_M as follows

$$\overline{r}_Q = \overline{r}_{Z_M} + \beta_{QM}(\overline{r}_M - \overline{r}_{Z_M}).$$

In this form, the role of the riskfree asset is replaced by the zero-beta portfolio Z_M. However, this version of the CAPM formula is still referencing the market portfolio.

The more general version of the CAPM allows the choice of *any* efficient (mean-variance) portfolio and its zero-beta counterpart.

The generalized CAPM (in terms of the given efficient portfolio of risky assets only and its uncorrelated counterpart) is given by

$$\mu_Q - \mu_{Z_P} = \beta_{QP}(\mu_P - \mu_{Z_P}).$$

This generalized CAPM shows a two-step extension of the classical CAPM:

1. Use any efficient portfolio P (consisting of risky assets only) to replace the market portfolio M.

2. Use the expected rate of return of the uncorrelated counterpart Z_P instead of the riskfree rate of return r.

Zero-beta counterpart of a given efficient portfolio

Let P and Q be any two frontier portfolios of risky assets. Since both P and Q are frontier portfolios, they admit the following representation:

$$\boldsymbol{w}_P^* = \Omega^{-1}(\lambda_1^P \mathbf{1} + \lambda_2^P \boldsymbol{\mu}) \quad \text{and} \quad \boldsymbol{w}_Q^* = \Omega^{-1}(\lambda_1^Q \mathbf{1} + \lambda_2^Q \boldsymbol{\mu})$$

where

$$\lambda_1^P = \frac{c - b\mu_P}{\Delta}, \quad \lambda_2^P = \frac{a\mu_P - b}{\Delta}, \quad \lambda_1^Q = \frac{c - b\mu_Q}{\Delta}, \quad \lambda_2^Q = \frac{a\mu_Q - b}{\Delta},$$
$$a = \mathbf{1}^T \Omega^{-1} \mathbf{1}, \quad b = \mathbf{1}^T \Omega^{-1} \boldsymbol{\mu}, \quad c = \boldsymbol{\mu}^T \Omega^{-1} \boldsymbol{\mu}, \quad \Delta = ac - b^2.$$

The covariance between r_P and r_Q is given by

$$\begin{aligned}
\text{cov}(r_P, r_Q) &= \boldsymbol{w}_P^{*T} \Omega \boldsymbol{w}_Q^* = \left[\Omega^{-1}(\lambda_1^P \mathbf{1} + \lambda_2^P \boldsymbol{\mu})\right]^T (\lambda_1^Q \mathbf{1} + \lambda_2^Q \boldsymbol{\mu}) \\
&= \lambda_1^P \lambda_1^Q a + (\lambda_1^P \lambda_2^Q + \lambda_1^Q \lambda_2^P) b + \lambda_2^P \lambda_2^Q c \\
&= \frac{a}{\Delta}\left(\mu_P - \frac{b}{a}\right)\left(\mu_Q - \frac{b}{a}\right) + \frac{1}{a}. \qquad (A)
\end{aligned}$$

We are able to express $\text{cov}(r_P, r_Q)$ in terms of μ_P and μ_Q. This is in similar spirit as the CAPM, where covariance of a pair of portfolios is related to expected rates of return.

Setting Q to be P, we obtain

$$\sigma_P^2 = \frac{a}{\Delta}\left(\mu_P - \frac{b}{a}\right)^2 + \frac{1}{a} = \frac{a\mu_P^2 - 2b\mu_P + c}{\Delta}.$$

This is the familiar equation that relates μ_P and σ_P^2 for any efficient portfolio P. We find the frontier portfolio Z such that $\text{cov}(r_P, r_Z) = 0$. The corresponding μ_Z is given by [see Eq. (A)]

$$\frac{a}{\Delta}\left(\mu_P - \frac{b}{a}\right)\left(\mu_Z - \frac{b}{a}\right) + \frac{1}{a} = 0.$$

This gives

$$\mu_Z = \frac{b}{a} - \frac{\frac{\Delta}{a^2}}{\mu_P - \frac{b}{a}}.$$

- μ_Z is defined provided that $\mu_P \neq \mu_g = b/a$.

- Since $(\mu_P - \mu_g)(\mu_Z - \mu_g) = -\dfrac{\Delta}{a^2} < 0$, where $\mu_g = \dfrac{b}{a}$, so when one portfolio is efficient, then its zero-covariance counterpart is non-efficient.

Geometric construction of the zero-covariance counterpart Z

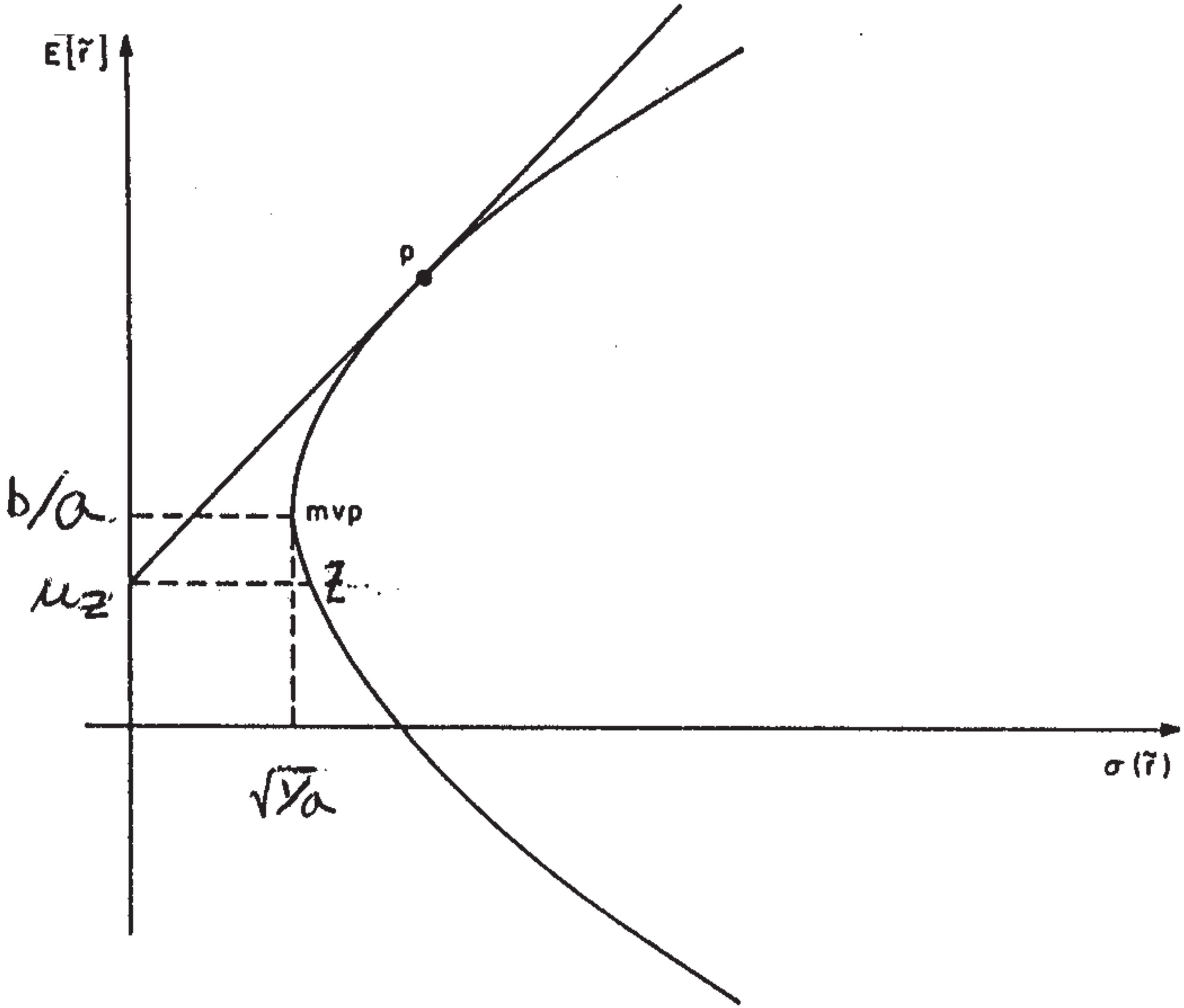
Slope of the tangent at P to the frontier curve:

$$\frac{d\mu_P}{d\sigma_P} = \frac{\Delta\sigma_P}{a\mu_P - b}.$$

The intercept of the tangent line at the vertical axis is

$$\begin{aligned} \mu_P - \frac{d\mu_P}{d\sigma_P}\sigma_P &= \mu_P - \frac{\Delta\sigma_P^2}{a\mu_P - b} \\ &= \mu_P - \frac{a\mu_P^2 - 2b\mu_P + c}{a\mu_P - b} = \frac{b}{a} - \frac{\Delta/a^2}{\mu_P - b/a} = \mu_Z. \end{aligned}$$

These calculations verify that the uncorrelated counterpart Z can be obtained by drawing a tangent to the frontier curve at P and finding the intercept of the tangent line at the vertical axis. Draw a horizontal line from the intercept to hit the frontier curve at Z.



The Location of a Zero Covariance Portfolio in the $\sigma(\tilde{r})$–$E[\tilde{r}]$ Space

Intuition behind the geometric construction of the uncorrelated counterpart of a frontier portfolio

- Given the riskfree point, we determine the market portfolio by the tangency method. Subsequently, all zero-beta funds (uncorrelated with the market portfolio) lie on the same horizonal line through the riskfree point in the $\sigma - \overline{\mu}$ diagram.

- Conversely, we consider the scenario where the riskfree point is NOT specified. Actually, the riskfree asset is absent in the present context. Apparently, given an efficient fund, we determine the corresponding "riskfree point" such that the efficient fund is the market portfolio with reference to the riskfree point. In this case, the frontier fund with the same return as this pseudo "riskfree point" will have its random rate of return uncorrelated with that of the efficient fund. The pseudo "riskfree point" and this uncorrelated counterpart (itself is a minimum variance portfolio) lie on the same horizontal line in the σ-$\overline{r}$ diagram.

Let P be a frontier portfolio other than the global minimum variance portfolio and Q be any portfolio, then

$$\begin{aligned}\text{cov}(r_P, r_Q) &= \boldsymbol{w}_P^T \Omega \boldsymbol{w}_Q = \left[\Omega^{-1}\left(\lambda_1^P \mathbf{1} + \lambda_2^P \boldsymbol{\mu}\right)\right]^T \Omega \boldsymbol{w}_Q \\ &= \lambda_1^P \mathbf{1}^T \boldsymbol{w}_Q + \lambda_2^P \boldsymbol{\mu}^T \boldsymbol{w}_Q = \lambda_1^P + \lambda_2^P \mu_Q.\end{aligned}$$

Solving for μ_Q and substituting $\lambda_1^P = \dfrac{c - b\mu_P}{\Delta}$ and $\lambda_2^P = \dfrac{a\mu_P - b}{\Delta}$, we obtain

$$\mu_Q = \frac{b\mu_P - c}{a\mu_P - b} + \text{cov}(r_P, r_Q)\frac{\Delta}{a\mu_P - b}.$$

We then recall the definition of β_{QP} in terms of $\text{cov}(r_P, r_Q)$ and σ_P^2 and substitute the relation between σ_P^2 and μ_P. This gives

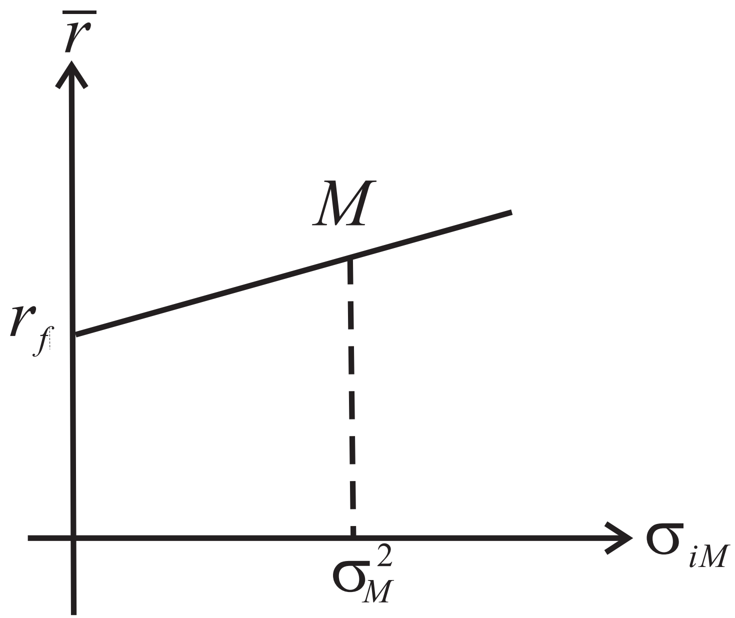
$$\begin{aligned}\mu_Q &= \frac{b}{a} - \frac{\Delta/a^2}{\mu_P - b/a} + \frac{\text{cov}(r_P, r_Q)}{\sigma_P^2}\left[\frac{\left(\mu_P - \frac{b}{a}\right)^2}{\Delta/a} + \frac{1}{a}\right]\frac{\Delta}{a\mu_P - b} \\ &= \mu_{Z_P} + \beta_{QP}\left(\mu_P - \frac{b}{a} + \frac{\Delta/a^2}{\mu_P - b/a}\right) \\ &= \mu_{Z_P} + \beta_{QP}(\mu_P - \mu_{Z_P}).\end{aligned}$$

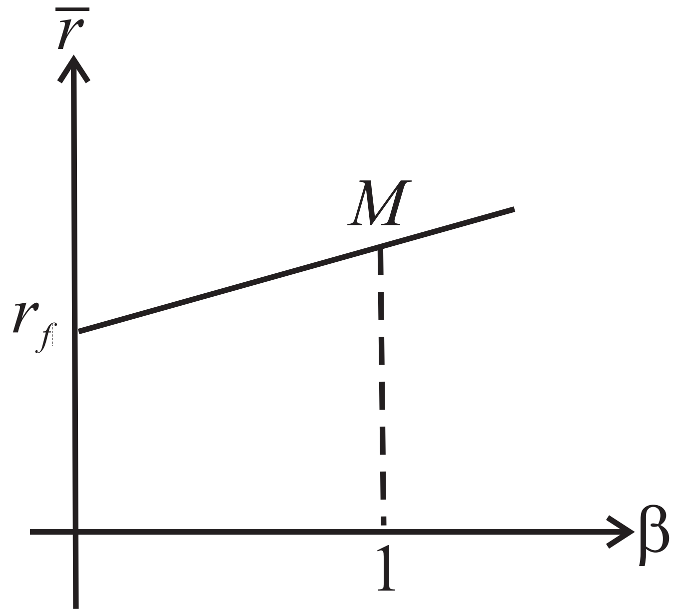
3.2 Interpretation and uses of the capital asset pricing model

Security market line (SML)

From the two relations: $\begin{cases} \overline{r} = r_f + \frac{\overline{r}_M - r_f}{\sigma_M^2}\sigma_{iM} \\ \overline{r} = r_f + (\overline{r}_M - r_f)\beta_i \end{cases}$,

we can plot either $\overline{r}$ against σ_{iM} or $\overline{r}$ against β_i.





Example

Consider the following set of data for 3 risky assets, market portfolio and risk free asset; Here, P_0 and P_1 are the price of the asset at $t = 0$ and $t = 1$, respectively; D_1 is the value at $t = 1$ of the dividend paid during the investment period.

portfolio/security	σ_i	ρ_{iM}	β_i	actual expected rate of return $= \frac{E[P_1 + D_1]}{P_0} - 1.0$
1	10%	1.0	0.5	13%
2	20%	0.9	0.9	15.4%
3	20%	0.5	0.5	13%
market portfolio	20%	1.0	1.0	16%
risk free asset	0	0.0	0.0	10%

- Note that β can be computed using the data given for ρ_{iM}, σ_i and σ_M. For example, $\beta_1 = \rho_{1M}\sigma_1/\sigma_M = 0.5$. Also, recall the CAPM formula:

$$\frac{\overline{r}_i - r}{\sigma_i} = \rho_{iM}\frac{\overline{r}_M - r}{\sigma_M}.$$

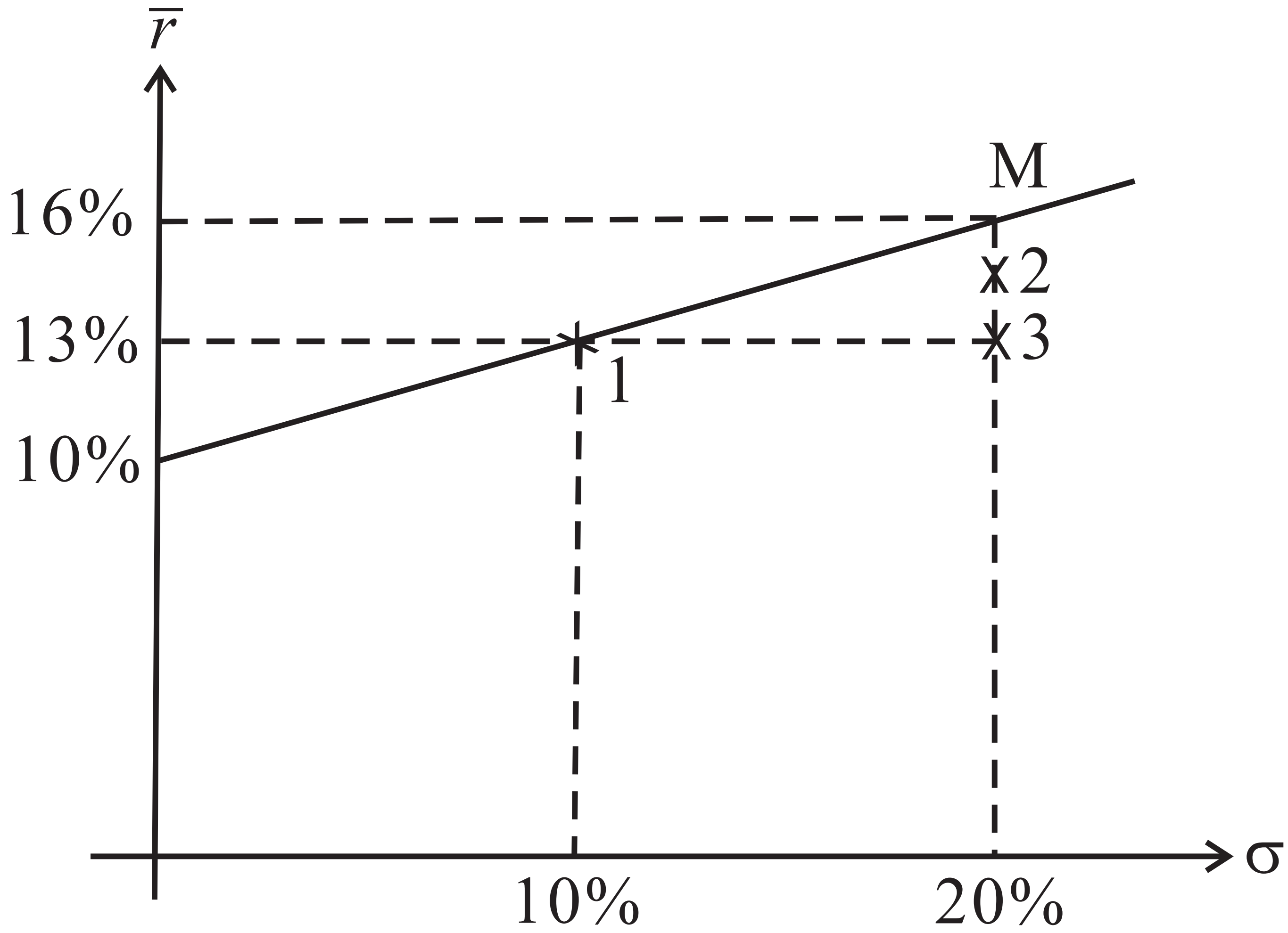
Use of the CML

The CML identifies expected rates of return which are available for *efficient portfolios* at all possible risk levels. Portfolios 2 and 3 lie below the CML. The market portfolio, the risk free asset and Portfolio 1 all lie on the CML. Hence, Portfolio 1 is efficient while Portfolios 2 and 3 are non-efficient.

For Portfolio 1, we observe $\rho_{1M} = 1$. For $\sigma = 10\%$, so

$$\overline{r} = \underbrace{10\%}_{r_f} + \underbrace{10\%}_{\sigma} \times \underbrace{\frac{(16-10)\%}{20\%}}_{(\overline{r}_M - r_f)/\sigma_M} = 13\%.$$

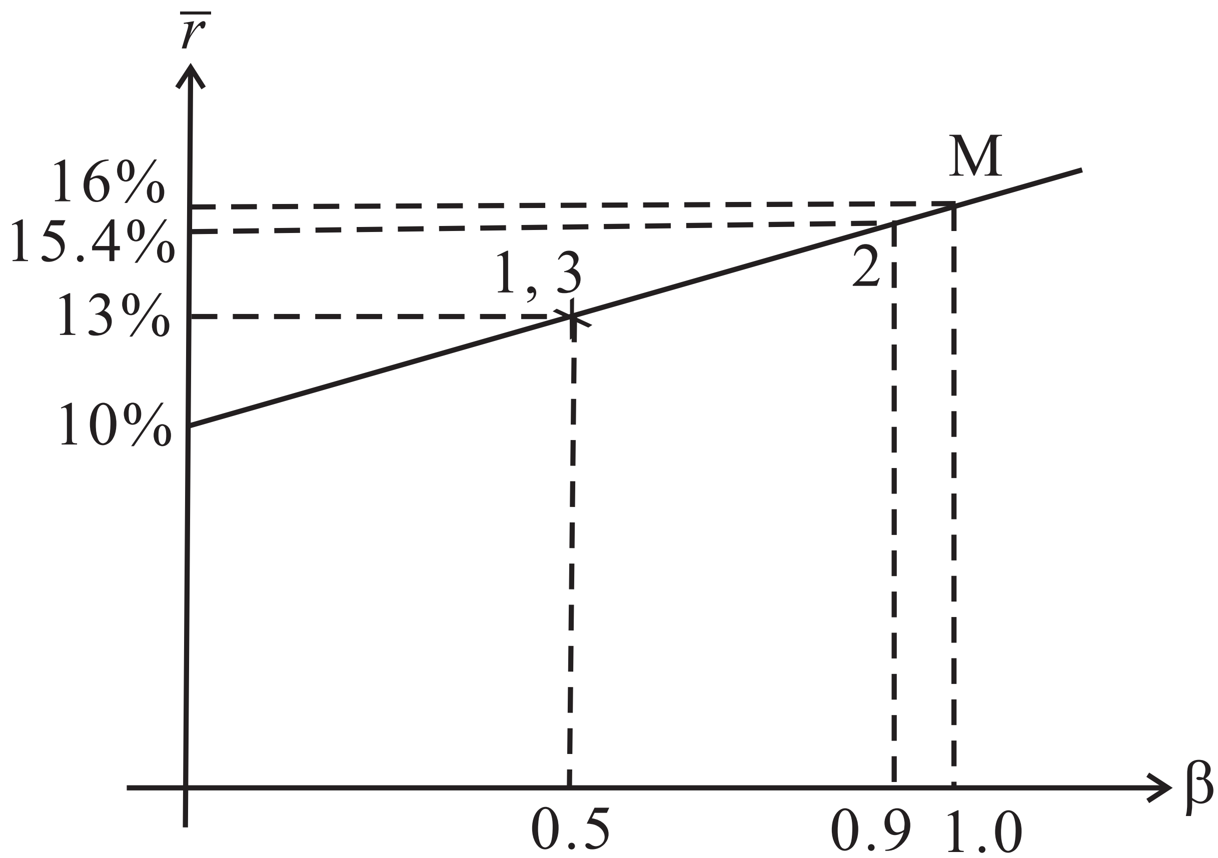
Portfolio 1 and portfolio 3 have the same expected rates of return, while $\sigma_1 = 10\%$ is less than $\sigma_3 = 20\%$.



Note that Asset 2 is closer to the CML since ρ_{2M} is 0.9, which is sufficiently close to 1. Asset 3 has high non-systematic or firm specific risk (risk that does not contribute to expected return) as $\rho_{3M} = 0.5$ is seen to have a low value.

Use of the SML

The SML asks whether the portfolio provides a return equal to what equilibrium conditions suggest should be earned.



The expected rates of return of the portfolios for the given values of beta are given by

$$\begin{aligned}\overline{r}_1 &= \overline{r}_3 = \underbrace{10\%}_{r} + \underbrace{0.5}_{\beta} \times (\underbrace{16\% - 10\%}_{\overline{r}_M - r}) = 13\% \\ \overline{r}_2 &= 10\% + 0.9 \times (16\% - 10\%) = 15.4\%.\end{aligned}$$

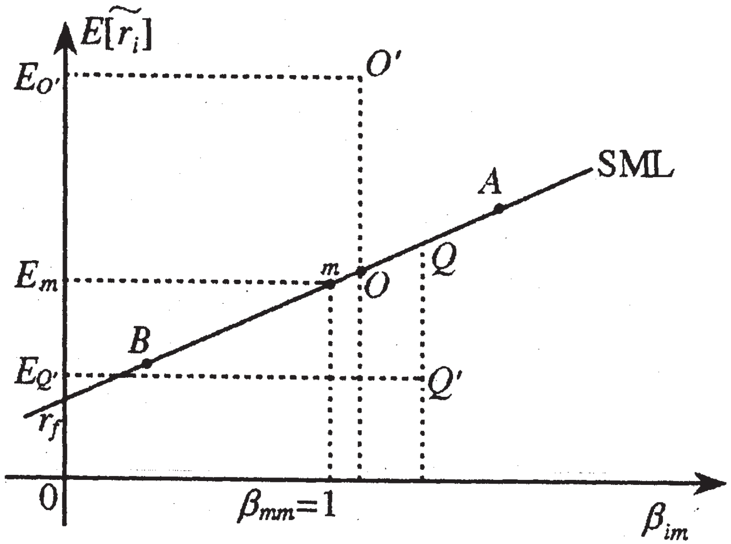
These expected rates of return suggested by the SML agree with the actual expected rates of return. Hence, each investment is fairly priced.

- Portfolio 1 has unit value of ρ_{iM}, that is, it is perfectly correlated with the market portfolio.

- Portfolios 2 and 3 both have ρ_{iM} less than one. Portfolio 2 has ρ_{iM} closer to one and so it lies closer to the CML.

Under the equilibrium conditions assumed by the CAPM, every asset should fall on the SML. The SML expresses the risk reward structure of assets against risk (quantified as β_i or σ_{iM}) according to the CAPM.

- Point O' represents an under-priced security. This is because the expected return is higher than the return with reference to the risk. In this case, the demand for such security will increase and this results in price increase and lowering of the expected return.



Regression and characteristic line

We perform the regression of $r_i - r$ on $r_M - r$ (rate of returns in excess of the riskfree rate are used). The regression line is called the characteristic line, which shows the sensitivity of an asset's excess return to market's excess return. Formally, we write

$$r_i - r = \alpha_i + \beta_i(r_M - r) + \epsilon_i,$$

where ϵ_i is the deviation from the line. The error term ϵ_i is a random variable with mean $E[\epsilon_i] = 0$ and variance $\sigma^2_{\epsilon_i}$. Note that β_i is the slope of the characteristic (regression) line and α_i is the intercept of the characteristic line.

Taking the expectation, we obtain

$$\alpha_i = (\overline{r}_i - r) - \beta_i(\overline{r}_M - r)$$

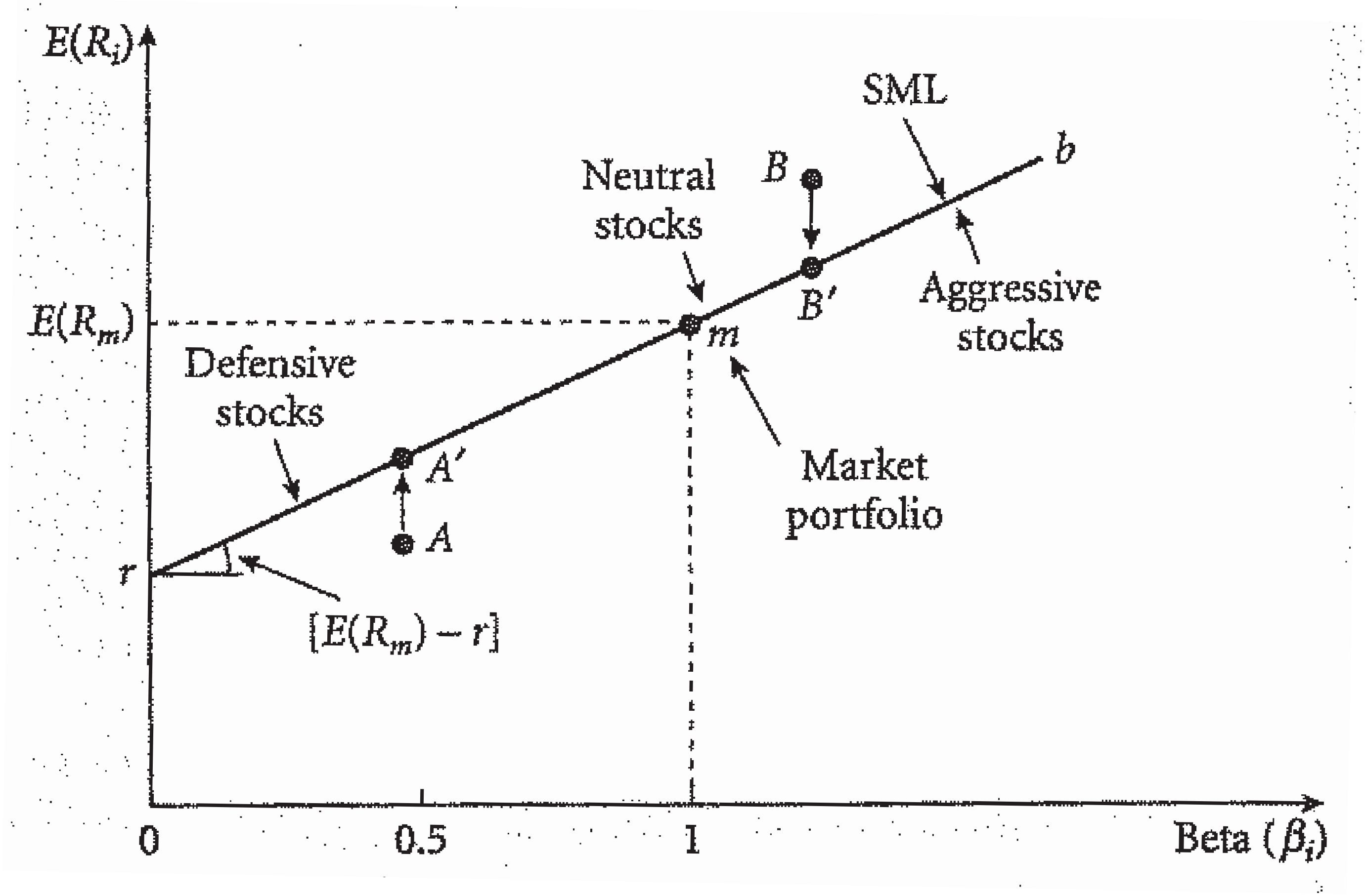
since $E[\epsilon_i] = 0$. Hence, α_i measures the abnormal return or pricing error above the expected rate of return based on the CAPM (or the normal return).

The SML implies that all alphas should equal zero: $\alpha_i = 0$ for all asset i.

If $\alpha_i > 0$, then the asset lies above the SML and it has a positive pricing error. In other words, the asset is underpriced; the expected return is too high and the price should rise. By contrast, if $\alpha_i < 0$, then the asset lies below the SML and it is overpriced.

In order for the market portfolio to be mean-variance efficient, the individual assets should lie on the SML. Similarly, if at least one asset lies above or below the SML, then the market is inefficient. Put differently, the SML gives a necessary and sufficient condition for mean-variance efficiency of risky assets in the market.

Plot of the SML



The SML represents equilibrium, i.e. the situation where all investors hold their optimal portfolio and hence there is no reason to further demand or supply assets. If some assets deviate from the SML, then the market is in disequilibrium.

For example, asset A lies below the SML, its expected return is too low relative to its beta, while the expected return of asset B is too high. The vertical distance of the assets from the SML measures the alpha or the pricing error, the difference between the actual expected rate of return and the equilibrium rate.

Investors can improve the risk-return characteristics of their portfolio by selling asset A (which has negative alpha and is overpriced) and buying asset B (which has a positive alpha and is underpriced). By selling asset A, investors will lower the price and raise the expected return of asset A; by buying asset B, they will raise the price and lower the expected return of asset B.

The selling and buying will continue until further transactions do not improve the risk-return characteristics of the investors' portfolios. In this case, all assets lie on the SML, the alphas are zero and the capital market is in equilibrium.

Decomposition of risks

Suppose we write the random rate of return r_i of asset i formally as

$$r_i = r + \beta_i(r_M - r) + \epsilon_i.$$

The CAPM tells us something about the residual term ϵ_i.

(i) Taking expectation on both sides

$$E[r_i] = r + \beta_i(\overline{r}_M - r) + E[\epsilon_i]$$

while $\overline{r}_i = r + \beta_i(\overline{r}_M - r)$ so that $E[\epsilon_i] = 0$.

(ii) Taking the covariance of r_i with r_M

$$\begin{aligned} \text{cov}(r_i, r_M) &= \overbrace{\text{cov}(r, r_M)}^{\text{zero}} + \beta_i \left[\text{cov}(r_M, r_M) - \underbrace{\text{cov}(r, r_M)}_{\text{zero}} \right] \\ &\quad + \text{cov}(\epsilon_i, r_M) \end{aligned}$$

so that $\text{cov}(\epsilon_i, r_M) = 0$.

(iii) Consider the variance of r_i

$$\text{var}(r_i) = \beta_i^2 \underbrace{\text{cov}(r_M - r, r_M - r)}_{\text{var}(r_M)} + \text{var}(\epsilon_i)$$

so that $\sigma_i^2 = \beta_i^2 \sigma_M^2 + \text{var}(\epsilon_i)$.

The total risk consists of systematic risk $\beta_i^2 \sigma_M^2$ and firm-specific (idiosyncratic) risk var(ϵ_i).

Systematic risk $= \beta_i^2 \sigma_M^2$, this risk cannot be reduced by diversification because every asset with nonzero beta contains this risk.

It is the systematic risk where the investor is rewarded for excess expected return above the riskfree rate.

Efficient portfolios: zero non-systematic risk

Consider a portfolio P formed by the combination of the market portfolio and the risk free asset. This portfolio is an efficient portfolio (one fund theorem) and it lies on the CML with a beta value equal to β_P (say). Recall that β_P equals the weight of M in P, so its rate of return can be expressed as

$$r_P = (1 - \beta_P)r + \beta_P r_M = r + \beta_P(r_M - r)$$

so that $\epsilon_P = 0$. The portfolio variance is $\beta_P^2 \sigma_M^2$. This portfolio has only systematic risk (zero non-systematic risk).

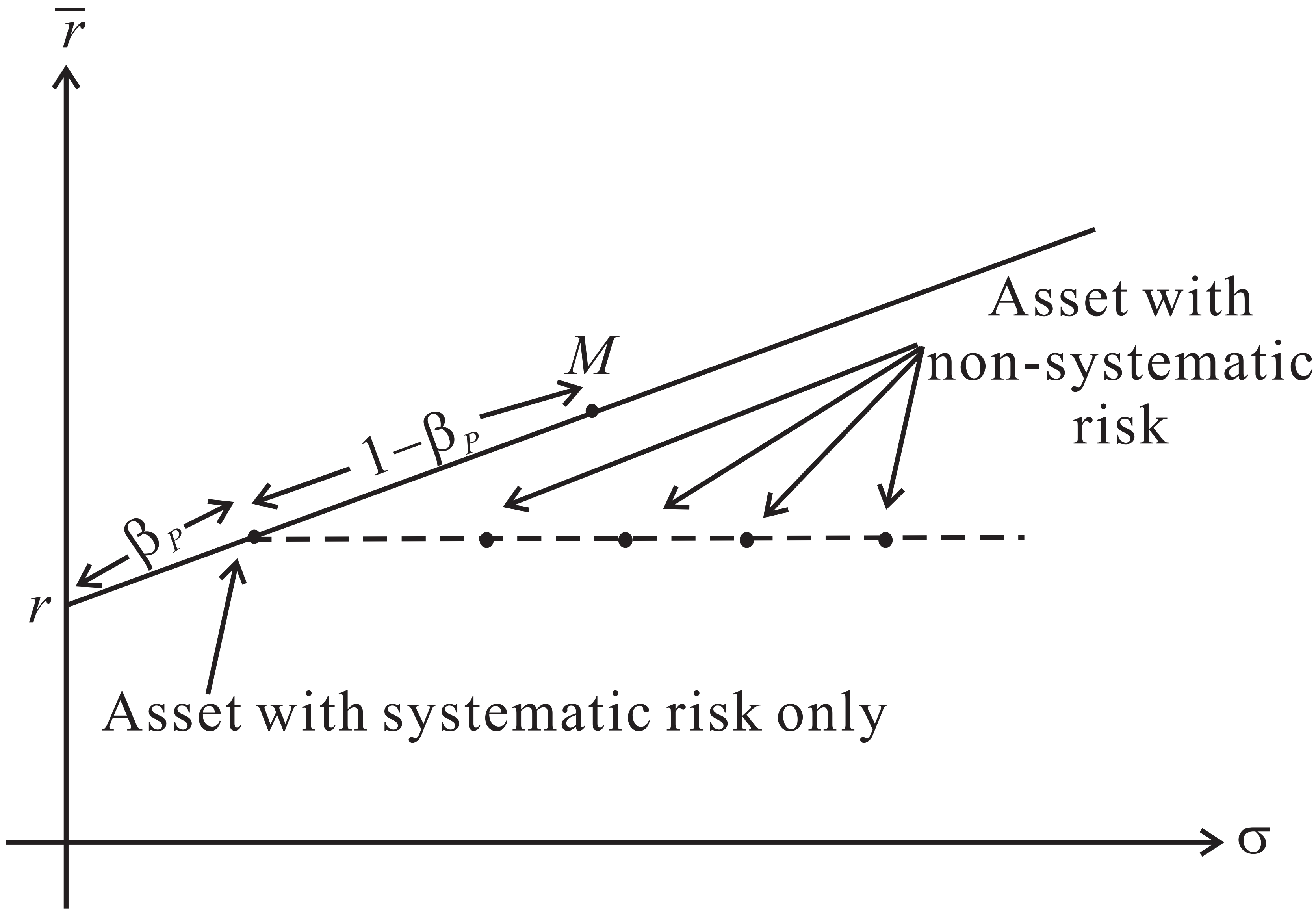
For an efficient portfolio P, we have $\rho_{PM} = 1$ so that $\beta_P = \dfrac{\sigma_P}{\sigma_M}$.

Portfolios not on the CML – non-efficient portfolios

For other portfolios with the same value of β_P but not lying on the CML, they lie below the CML since they are non-efficient portfolios. With the same value of β_P, they all have the same expected rate of return given by

$$\overline{r} = r + \beta_P(\overline{r}_M - r)$$

but the portfolio variance is greater than $\beta_P^2\sigma_M^2$. The extra part of the portfolio variance is var(ϵ_i), which is called the firm-specific or idiosyncratic risk.



equation of CML: $\overline{r} = r + \frac{\overline{r}_M - r}{\sigma_M}\sigma$

Diversification effect

Note that ϵ_i is uncorrelated with r_M as revealed by $\text{cov}(\epsilon_i, r_M) = 0$. The term $\text{var}(\epsilon_i)$ is called the *non-systematic* or *specific* risk of asset i. This risk can be reduced by diversification. Consider $r_i = (1 - \beta_{iM})r + \beta_{iM} r_M + \epsilon_i$ and observe $\text{cov}(\epsilon_i, \epsilon_j) \approx 0$ for $i \neq j$ (since ϵ_i and ϵ_j are firm specific risks) and $\text{cov}(r_M, \epsilon_i) = 0$ for all i, then

$$\begin{aligned}
\sigma_P^2 &= \text{cov}\left(\sum_{i=1}^n w_i \beta_{iM} r_M, \; \sum_{j=1}^n w_j \beta_{jM} r_M\right) + \text{cov}\left(\sum_{i=1}^n w_i \epsilon_i, \; \sum_{j=1}^n w_j \epsilon_j\right) \\
&\approx \left(\sum_{i=1}^n w_i \beta_{iM}\right)\left(\sum_{j=1}^n w_j \beta_{jM}\right)\sigma_M^2 + \sum_{i=1}^n w_i^2 \sigma_{\epsilon_i}^2.
\end{aligned}$$

Recall $\beta_{PM} = \sum_{i=1}^n w_i \beta_{iM}$, so

$$\sigma_P^2 = \beta_{PM}^2 \sigma_M^2 + \sum_{i=1}^n w_i^2 \sigma_{\epsilon_i}^2.$$

To illustrate the diversification effect, suppose we take $w_i = 1/n$ so that

$$\sigma_P^2 = \beta_{PM}^2 \sigma_M^2 + \frac{1}{n^2} \sum_{i=1}^{n} \sigma_{\epsilon_i}^2 = \beta_{PM}^2 \sigma_M^2 + \overline{\sigma}^2/n,$$

where $\overline{\sigma}^2$ is the average of $\sigma_{\epsilon_1}^2, \cdots, \sigma_{\epsilon_n}^2$. When n is sufficiently large, we obtain

$$\sigma_P \to \left(\sum_{i=1}^{n} w_i \beta_{iM} \right) \sigma_M = \beta_{PM} \sigma_M.$$

- The contribution of asset i to the portfolio standard deviation σ_P is $w_i \beta_{iM} \sigma_M$, $i = 1, 2, \ldots, n$. The contribution from the residual risk $\sigma_{\epsilon_i}^2$ to the portfolio variance σ_P^2 goes to zero as $n \to \infty$.

- Suppose the covariance terms, cov(ϵ_i, ϵ_j), $i, j = 1, 2, \ldots, n$, are finite, we then have n^2 covariance terms. The sum of these n^2 terms would not become vanishingly small when $n \to \infty$.

Systematic risk

- The variance of a security's returns stems from the overall market movement and is measured by beta. It is only this risk that investors are rewarded for bearing (through earning excess expected rate of return above the riskfree rate).
- Systematic risk is given by $\beta^2 \sigma_M^2$.

Nonsystematic (firm specific or idiosyncratic) risk

Diversifiable risk that is unique to a particular stock / portfolio. The residual risks are uncorrelated to the market portfolio.

Example

Suppose that the relevant equilibrium model is the CAPM with unlimited borrowing and lending at the riskless rate of interest. Complete the blanks in the following table.

Stock	Expected Return	Standard Deviation	Beta	Residual Variance
1	0.15	—	2.00	0.10
2	—	0.25	0.75	0.04
3	0.09	—	0.50	0.17

Solution

Under the CAPM assumptions, the relationship between the expected excess rate of return and beta is given by

$$E[r_M] - r = \frac{E[r_i] - r}{\beta_i} = \frac{E[r_j] - r}{\beta_j}.$$

From the information for stock 1 and 3, we obtain

$$E[r_M] - r = \frac{E[r_1] - E[r_3]}{\beta_1 - \beta_3} = \frac{0.15 - 0.09}{2.0 - 0.5} = 0.04.$$

Once we obtain $E[r_M] - r$, we can use the information for stock 1 to find the risk-free rate

$$r = E[r_1] - (E[r_M] - r)\beta_1 = 0.15 - (0.04)(2.00) = 0.07.$$

Once we know r, we obtain $E[r_M] = r + 0.04 = 0.11$. The expected rate of return for stock 2 is

$$E[r_2] = r + (E[r_M] - r)\beta_2 = 0.07 + (0.04)(0.75) = 0.10.$$

Calculation of $\sigma^2(r_M)$

The information given for stock 2 allows us to estimate the variance of returns of the market:

$$\begin{aligned}
\sigma^2(r_2) &= \beta_2^2\sigma^2(r_M) + \sigma^2(\epsilon_2) \\
\sigma^2(r_M) &= \frac{\sigma^2(r_2) - \sigma^2(\epsilon_2)}{\beta_2^2} = \frac{(0.25)^2 - (0.04)}{(0.75)^2} = 0.04.
\end{aligned}$$

The standard deviations of stock 1 and 3 can now be found:

$$\sigma^2(r_1) = (2.0)^2(0.04) + 0.10 = 0.26; \sigma(r_1) = 0.5099.$$
$$\sigma^2(r_3) = (0.5)^2(0.04) + 0.17 = 0.18; \sigma(r_3) = 0.4243.$$

Stock	Expected Return	Standard Deviation	Beta	Residual Variance
1	0.15	**0.51**	2.00	0.10
2	**0.10**	0.25	0.75	0.04
3	0.09	**0.42**	0.5	0.17
risk free asset	**0.07**	0	0	0
market portfolio	**0.11**	**0.2**	1.00	0

Stock 3 has very high firm specific risk; $\sigma(r_3) = 0.4243$ is much higher than $\sigma(r_M) = 0.2$ but the expected rate of return is only 9% as compared to $E[r_M] = 11\%$. This represents an inferior stock.

The link between the security market line and the capital market line

The SML, which is the relationship between mean and beta, applies to all individual assets as well as all portfolios, regardless of whether they are efficient.

By contrast, the CML, which is the relationship between mean and standard deviation, applies only for efficient portfolios. The CML does not apply to individual assets or to portfolios that are inefficient, because investors would not receive compensation on expected rate of return for non-systematic risk.

Recall that

$$\beta_i = \frac{\sqrt{\sigma_i^2 - \sigma_{\epsilon_i}^2}}{\sigma_M},$$

and by the CAPM formula:

$$\begin{aligned} \overline{r}_i &= r + \beta_i(\overline{r}_M - r) \\ &= r + \frac{\sqrt{\sigma_i^2 - \sigma_{\epsilon_i}^2}}{\sigma_M}(\overline{r}_M - r). \end{aligned}$$

Efficient portfolios have no non-systematic risk, so $\sigma_{e_i}^2 = 0$. Hence, the above equation reduces to the equation of the CML:

$$\overline{r}_i = r + \frac{\overline{r}_M - r}{\sigma_M}\sigma_i.$$

when $\sigma_{e_i}^2 = 0$, we have $\sigma_P = \beta_P \sigma_M$.

For efficient portfolios, we can measure risk as beta (SML) or as variance or standard deviation (CML). For inefficient portfolios ($\sigma_{\epsilon_i}^2 > 0$), only the SML applies.

Excess expected rate of return $E[r_i]-r$ normalized by risk (quantified by either β_i or σ_i)

The CAPM predicts that under equilibrium the excess expected return on any stock (portfolio) adjusted for the risk on that stock (portfolio) should be the same

$$\frac{E[r_i]-r}{\beta_i} = \frac{E[r_j]-r}{\beta_j}. \qquad (A)$$

This is in contrast to the Sharpe ratio, where

$$\frac{E[r_i]-r}{\sigma_i} \gtreqless \frac{E[r_j]-r}{\sigma_j}. \qquad (B)$$

The asset with a lower value of Sharpe ratio is considered inferior.

Recall $\sigma_i^2 = \beta_i^2\sigma_M^2 + \text{var}(\varepsilon_i)$ and $\sigma_j^2 = \beta_j^2\sigma_M^2 + \text{var}(\varepsilon_j)$. More precisely, asset i is inferior compared to asset j when its residual variance normalized by beta squared is higher, that is

$$\frac{\text{var}(\epsilon_i)}{\beta_i^2} > \frac{\text{var}(\epsilon_j)}{\beta_j^2}.$$

To show the claim, given the above condition on residual variances, it suffices to show that asset i has a lower sharpe ratio. This is verified as follows:

$$\begin{aligned}\frac{E[r_i]-r}{\sqrt{\beta_i^2\sigma_M^2+\text{var}(\epsilon_i)}} &= \frac{E[r_i]-r}{\beta_i\sqrt{\sigma_M^2+\frac{\text{var}(\epsilon_i)}{\beta_i^2}}} = \frac{E[r_j]-r}{\beta_j\sqrt{\sigma_M^2+\frac{\text{var}(\epsilon_i)}{\beta_i^2}}} \\ &< \frac{E[r_j]-r}{\beta_j\sqrt{\sigma_M^2+\frac{\text{var}(\epsilon_j)}{\beta_j^2}}} = \frac{E[r_j]-r}{\sqrt{\beta_j^2\sigma_M^2+\text{var}(\epsilon_j)}}.\end{aligned}$$

Using financial intuition, the asset with a higher value of

$$\text{specific risk/systematic risk } = \text{var}(\epsilon)/\beta^2\sigma_M^2$$

is considered to be inferior.

CAPM as a pricing formula

Suppose an asset is purchased at P and later sold at Q. The rate of return is $\dfrac{Q-P}{P}$, P is known and Q is random. Using the CAPM,

$$\frac{\overline{Q}-P}{P} = r + \beta(\overline{r}_M - r) \text{ so that } P = \frac{\overline{Q}}{1+r+\beta(\overline{r}_M - r)}.$$

Here, P gives the fair price of the asset with expected value $\overline{Q}$ and beta β.

The factor $\dfrac{1}{1+r+\beta(\overline{r}_M - r)}$ can be regarded as the *risk adjusted discount rate*. All risky assets with the same β (in general with differing levels of idiosyncratic risk) has the same risk adjusted discount rate.

Implicitly, β also involves P since $\beta = \text{cov}\left(\dfrac{Q}{P} - 1, r_M\right) \Big/ \sigma_M^2$ so that $\beta = \dfrac{\text{cov}(Q, r_M)}{P\sigma_M^2}$.

We rearrange the terms in the CAPM pricing formula to solve for P explicitly

$$1 = \frac{\overline{Q}}{P(1+r) + \text{cov}(Q, r_M)(\overline{r}_M - r)/\sigma_M^2}$$

so that the fair price based on the CAPM is

$$P = \frac{1}{1+r}\left[\overline{Q} - \frac{\text{cov}(Q, r_M)(\overline{r}_M - r)}{\sigma_M^2}\right].$$

In this new form, the riskfree discount factor $\dfrac{1}{1+r}$ is applied on the *certainty equivalent*, which is defined as $\overline{Q}$ minus dollar discount. The amount of underpricing of the asset is the difference between the fair price and the observed price, which is then given by

$$-P_{\text{obs}} + \frac{1}{1+r}\left[\overline{Q} - \frac{\text{cov}(Q, r_M)(\overline{r}_M - r)}{\sigma_M^2}\right].$$

Example (Investment in a mutual fund)

A mutual fund invests 10% of its funds at the risk free rate of 7% and the remaining 90% at a widely diversified portfolio with asymptotically low level of idiosyncratic risk and $\overline{r}_P = 15\%$. The beta with reference to this "almost" efficient fund is then equal to 0.9. Recall that the CAPM formula remains valid when the market portfolio is replaced by an efficient portfolio (beta is with reference to the efficient portfolio). Suppose the expected value of one share of the fund one year later is $\overline{Q} = \$110$, what should be the fair price of one share of the fund now?

According to the pricing formula of the CAPM, the current fair price of one share $= \dfrac{\$110}{1 + 7\% + 0.9 \times (15 - 7)\%} = \dfrac{\$110}{1.142} = \$96.3.$

Note that $\$96.3 \times 1.07 = \$103.04$ is the certainty equivalent, so the dollar discount is $\$110 - \$103.04 = \$6.96$.

Difficulties with the mean-variance approach

1. The application of the mean-variance theory requires the determination of the parameter values: mean values of the asset returns and the covariances among them. Suppose there are n assets, then there are n mean values, n variances and $\frac{n(n-1)}{2}$ covariances. For example, when $n = 1,000$, the number of parameter values required $= 501,500$.

2. In the CAPM, there is really only one risk factor that influences the expected return, namely, $r_M - r$.

The assumption of investors utilizing a mean-variance framework is replaced by an assumption that security returns are generated by a set of risk factors. The challenge is to find these risk factors that explain the asset returns.

3.3 Arbitrage pricing theory (APT) and factor models

Law of one price and arbitrage

The law of one price states that portfolios with the same payoff have the same price. Arbitrage opportunities arise when two securities with the same payoff have different prices – buy the cheap one and sell the expensive one to secure a risk free profit. Since the violation of law of one price implies presence of arbitrage, so

$$\text{absence of arbitrage} \Rightarrow \text{law of one price.}$$

Example

	securities		
State of economy	A	B	C
recession	-2	-4	0
stable	6	4	10
boom	10	16	6

Assume that the current prices of the 3 securities are the same, while their random terminal payoff vectors depend on the state of the economy.

Let

$$P_A = \begin{pmatrix} -2 \\ 6 \\ 10 \end{pmatrix}, \quad P_B = \begin{pmatrix} -4 \\ 4 \\ 16 \end{pmatrix} \quad \text{and} \quad P_C = \begin{pmatrix} 0 \\ 10 \\ 6 \end{pmatrix}$$

and note that $2P_A \leq P_B + P_C$.

Arbitrage opportunity exists since one can long one unit for both security B and C and short 2 units of security B. There is zero cost in initial investment but this strategy guarantees non-negative terminal payoff in all states and positive at least for one state of the economy.

Factor models

Randomness displayed by the returns of n assets can be traced back to a smaller number of underlying basic sources of randomness (factors). This would lead to a simpler covariance structure.

The return on a security can be broken down into an expected return and an unexpected (or surprise) component. The multi-factor model assumes that the random return rate of any stock be linearly related to a number of risk factors.

Single-factor model

The random rate of return r_i of asset i and the factor f are assumed to be linearly related by

$$r_i = a_i + b_i f + e_i, \quad i = 1, 2, \cdots, n.$$

Here, f is the single random factor shared by all assets, a_i and b_i are fixed constants, e_i's are random errors (we can always take $E[e_i] = 0$). Here, $b_i =$ factor loading; which measures the sensitivity of the return r_i to the factor. Further, we assume

$$\text{cov}(e_i, f) = 0 \quad \text{and} \quad E[e_i e_j] = 0, \quad i \neq j.$$

One can interpret the CAPM model in terms of excess returns $r_i - r$ of any risky asset and $r_M - r$ of the market portfolio, where

$$r_i - r = \beta_i(r_M - r) + e_i.$$

Here, $r_M - r$ is the single random factor that drives the asset return. The factor loading is β_i.

Specifying the factors (macroeconomic state variables) that affect the return-generating process

1. Inflation

   Inflation impacts both the level of the discount rate.

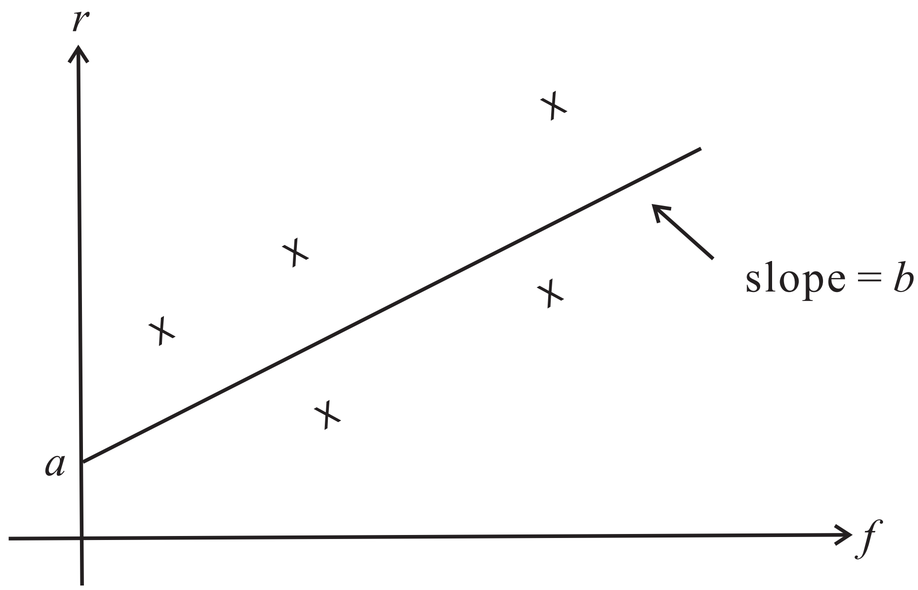
2. Risk premia

   Differences between the return on safe bonds and more risky bonds are used to measure the market's reaction to risk.

3. Industrial production

   Changes in industrial production affect the opportunities facing investors.

Most of the empirical APT research works have focused on the identification of these factors. For example, industrial stocks are more sensitive to oil prices fluctuation as compared to real estate stocks.



⋆ Different data sets (say, past one month or two months data) may lead to different estimated values.

From $r_i = a_i + b_i f + e_i$, we deduce that

$$\begin{aligned} \overline{r}_i &= a_i + b_i \overline{f} \\ \sigma_i^2 &= b_i^2 \sigma_f^2 + \sigma_{e_i}^2 && \text{[using cov}(f, e_i) = 0] \\ \sigma_{ij} &= b_i b_j \sigma_f^2, \quad i \neq j && \text{[using cov}(e_i, e_j) = 0 \text{ in addition]} \\ b_i &= \text{cov}(r_i, f)/\sigma_f^2. \end{aligned}$$

Example (Four stocks and one index)

Historical rates of return for four stocks over 10 years, record of industrial price index over the same period.

Estimate of $\overline{r}_i$ is $\widehat{\overline{r}}_i = \dfrac{1}{10}\sum_{k=1}^{10} r_i^k$, where r_i^k is the observed rate of return of asset i in the k^{th} year. The estimated variances and co-variances are given by

$$\widehat{\text{var}}(r_i) = \frac{1}{9}\sum_{k=1}^{10}(r_i^k - \widehat{\overline{r}}_i)^2$$

$$\widehat{\text{cov}}(r_i, f) = \frac{1}{9}\sum_{k=1}^{10}(r_i^k - \widehat{\overline{r}}_i)(f^k - \widehat{\overline{f}}).$$

Once the covariances have been estimated, b_i and a_i are found by

$$b_i = \frac{\widehat{\text{cov}}(r_i, f)}{\widehat{\text{var}}(f)} \quad \text{and} \quad a_i = \widehat{\overline{r}}_i - b_i\widehat{\overline{f}}.$$

Also, e_i can be estimated once the estimated values of a_i and b_i are known.

We estimate the variance of the error under the assumption that these errors are uncorrelated with each other and with the index. The formula to be used is

$$\text{var}(e_i) = \text{var}(r_i) - b_i^2 \text{var}(f).$$

In addition, we estimate $\text{cov}(e_i, e_j)$ by following similar calculations as those for $\widehat{\text{cov}}(r_i, f)$.

- Unfortunately, the error variances are almost as large as the variances of the stock returns.
- There is a high non-systematic risk, so the choice of this factor does not explain much of the variation in returns.
- Further, $\text{cov}(e_i, e_j)$ for $i \neq j$ are not small so that the errors are highly correlated. We have

  $$\text{cov}(e_1, e_2) = 44 \quad \text{and} \quad \text{cov}(e_2, e_3) = 91.$$

  Recall that the factor model is constructed under the assumption of zero error covariances. We observe inconsistency of the calculated results and assumption made in the factor model.

Year	Stock 1	Stock 2	Stock 3	Stock 4	Index
1	11.91	29.59	23.27	27.24	12.30
2	18.37	15.25	19.47	17.05	5.50
3	3.64	3.53	−6.58	10.20	4.30
4	24.37	17.67	15.08	20.26	6.70
5	30.42	12.74	16.24	19.84	9.70
6	−1.45	−2.56	−15.05	1.51	8.30
7	20.11	25.46	17.80	12.24	5.60
8	9.28	6.92	18.82	16.12	5.70
9	17.63	9.73	3.05	22.93	5.70
10	15.71	25.09	16.94	3.49	3.60
aver	15.00	14.34	10.90	15.09	6.74
var	90.28	107.24	162.19	68.27	6.99
cov	2.34	4.99	5.45	11.13	6.99
b	0.33	0.71	0.78	1.59	1.00
a	12.74	9.53	5.65	4.36	0.00
e-var	89.49	103.68	157.95	50.55	

The record of the rates of return for four stocks and an index of industrial prices are shown. The averages and variances are all computed, as well as the covariance of each with the index. From these quantities, the b_i''s and the a_i's are calculated. Finally, the computed error variances are also shown. The index does not explain the stock price variations very well.

Portfolio risk under single-factor models – systematic and non-systematic risks

Let w_i denote the weight for asset $i, i = 1, 2, \cdots, n$.

$$r_P = \sum_{i=1}^{n} w_i a_i + \sum_{i=1}^{n} w_i b_i f + \sum_{i=1}^{n} w_i e_i$$

so that $r_P = a + bf + e$, where

$$a = \sum_{i=1}^{n} w_i a_i, \quad b = \sum_{i=1}^{n} w_i b_i \quad \text{and} \quad e = \sum_{i=1}^{n} w_i e_i.$$

Further, since $E[e_i] = 0, E[(f - \overline{f})e_i] = 0$ so that

$$E[e] = 0 \quad \text{and} \quad E[(f - \overline{f})e] = 0;$$

e and f are uncorrelated. Also, $\text{cov}(e_i, e_j) = 0, i \neq j$, so that $\sigma_e^2 = \sum_{i=1}^{n} w_i^2 \sigma_{e_i}^2$. The overall variance of portfolio is given by

$$\sigma^2 = b^2 \sigma_f^2 + \sigma_e^2.$$

Let $\overline{\sigma_e^2}$ denote the average of $\sigma_{e_i}^2$ and take $w_i = 1/n$ for all i so that $\sigma_e^2 = \dfrac{\overline{\sigma_e^2}}{n}$. As $n \to \infty, \sigma_e^2 \to 0$. The overall variance of portfolio σ^2 tends to decrease as n increases since σ_e^2 goes to zero, but σ^2 does not go to zero since $b^2\sigma_f^2$ remains finite.

- The risk due to e_i is said to be *diversifiable* since its contribution to the overall risk is essentially zero in a well-diversified portfolio. This is because e_i's are uncorrelated and so each can be reduced by diversification.

- The risk due to $b_i f$ is said to be systematic since it is present even in a diversified portfolio.

The random return on the portfolio is made up of the expected returns on the individual securities and the random component arising from the single risk factor f.

Single-factor models with zero residual risk

Assume zero idiosyncratic (asset-specific) risk, the rate of return of the i^{th} asset is characterized by

$$r_i = a_i + b_i f, \quad i = 1, 2, \cdots, n,$$

where the factor f is chosen to satisfy $E[f] = 0$ for convenience (with no loss of generality) so that $\overline{r}_i = a_i$.

Consider two assets which have two different factor loading b_i's, what should be the relation between their expected returns under the assumption of no arbitrage?

Consider a portfolio with weight w in asset i and weight $1 - w$ in asset j. The portfolio return is

$$r_P = w(a_i - a_j) + a_j + [w(b_i - b_j) + b_j]f.$$

By choosing $w^* = \dfrac{b_j}{b_j - b_i}$, the portfolio becomes risk free and

$$r_P^* = \frac{b_j(a_i - a_j)}{b_j - b_i} + a_j.$$

This must be equal to the return of the risk free asset, denoted by r. If otherwise, arbitrage opportunities arise. Suppose the risk free two-asset portfolio has a return higher than that of the riskfree asset, we then short sell the riskfree asset and long hold the risk free portfolio. We write the above relation as

$$\frac{a_j - r}{b_j} = \frac{a_j - a_i}{b_j - b_i} = \frac{a_i - r}{b_i} = \lambda.$$

↑ set

Hence, $\overline{r}_i = r + b_i\lambda$, where λ is the factor risk premium. Note that when two assets have the same factor loading b, they have the same expected return.

1. The risk free return r is the expected return on a portfolio with zero factor loading.

2. In general, the term *risk premium* refers to the excess return above the riskfree rate of return demanded by an investor who bears the risk of the investment. The factor risk premium λ gives the extra return above r per unit loading of the risk factor,

$$\lambda = (\overline{r}_i - r)|_{b_i=1}.$$

3. Under the general single-factor model, where

$$r_i = a_i + b_i f + e_i,$$

$$\text{cov}(r_i, r_j) = \text{cov}(a_i + b_i f, a_j + b_j f) = b_i b_j \text{var}(f) = b_i b_j \sigma_f^2,$$

The above result is obtained based on the usual assumption that the asset-specific risks are assumed to be uncorrelated with the factor risk and among themselves, where

$$\text{cov}(e_i, f) = \text{cov}(e_j, f) = \text{cov}(e_i, e_j) = 0.$$

Numerical example

Given $a_1 = 0.10, b_1 = 2, a_2 = 0.08$ and $b_2 = 1$, and assuming $E[f] = e_1 = e_2 = 0$ for the two assets under the single-factor model, find the factor risk premium λ. How to construct the zero-beta portfolio from these two risky assets?

The two unknowns r and λ are determined from the no-arbitrage relation:

$$\frac{0.10 - r}{2} = \frac{0.08 - r}{1} = \lambda$$

so that $r = 0.06$ and $\lambda = 0.02$. The expected rate of return of the two assets are given by $\overline{r}_1 = r + 2\lambda = 0.10$ and $\overline{r}_2 = r + \lambda = 0.08$.

To construct a zero-beta portfolio, we long two units of asset 2 and short one unit of asset 1 so that

$$r_P = 2r_2 - r_1 = 2(0.08 + f) - (0.10 + 2f) = 0.06.$$

Two-factor extension

Consider the two-factor model

$$r_i = a_i + b_{i1} f_1 + b_{i2} f_2, \quad i = 1, 2, \cdots, n,$$

where the factors f_1 and f_2 are chosen such that $E[f_1] = E[f_2] = 0$. Consider a 3-asset portfolio, with the assumption that $\mathbf{1}, \boldsymbol{b}_1 = \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix}$ and $\boldsymbol{b}_2 = \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix}$ are linearly independent. Form the portfolio with weights w_1, w_2 and w_3 so that the portfolio

$$r_P = \sum_{i=1}^{3} w_i a_i + f_1 \sum_{i=1}^{3} w_i b_{i1} + f_2 \sum_{i=1}^{3} w_i b_{i2}$$

becomes riskfree (independent of the random factors f_1 and f_2). This requires

$$w_1 + w_2 + w_3 = 1, \ \sum_{i=1}^{3} w_i b_{i1} = 0 \ \text{ and } \ \sum_{i=1}^{3} w_i b_{i2} = 0.$$

Since $\mathbf{1}, \boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are independent, the following system of equations

$$\begin{pmatrix} 1 & 1 & 1 \\ b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad (A)$$

always has unique solution. By choosing this set of values for $w_i, i = 1, 2, 3$, the portfolio becomes riskfree. By applying the no-arbitrage argument again, the risk free portfolio should earn the return same as that of the riskfree asset, thus

$$r_P = \sum_{i=1}^{3} w_i a_i = r.$$

Rearranging, we obtain a new relation between w_1, w_2 and w_3:

$$\sum_{i=1}^{3} (a_i - r) w_i = 0. \qquad (B)$$

This implies that there exists a non-trivial solution to the following homogeneous system of linear equations:

$$\begin{pmatrix} a_1 - r & a_2 - r & a_3 - r \\ b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The above coefficient matrix must be singular. Since the second and third rows are independent, it must occur that the first row is formed by some linear combination of the second and third rows.

This gives

$$a_i - r = \overline{r}_i - r = \lambda_1 b_{i1} + \lambda_2 b_{i2}, \quad i = 1, 2, 3,$$

for some constant parameters λ_1 and λ_2.

Remark

The no-arbitrage pricing approach is based on the observation where securities that share the same set of risk factors are hedgeable.

Remark

What happens if $\mathbf{1}, \boldsymbol{b}_1$ and $\boldsymbol{b}_2$ are not independent? In this case, we cannot form a riskfree portfolio using the 3 given assets as there is no solution to the linear system (A).

Factor risk premium: λ_1 *and* λ_2

– interpreted as the excess expected return per unit loading associated with the factors f_1 and f_2.

For example, $\lambda_1 = 3\%, \lambda_2 = 4\%$, factor loadings are $b_{i1} = 1.2, b_{i2} = 0.7, r = 7\%$, then

$$\overline{r}_i = 7\% + 1.2 \times 3\% + 0.7 \times 4\% = 13.6\%.$$

Absence of the riskfree asset

$$\overline{r}_i - r = \lambda_1 b_{i1} + \lambda_2 b_{i2}, \quad i = 1, 2, \cdots, n.$$

If the risk free asset does not exist, then we replace r by λ_0, where λ_0 is the return of the zero-beta asset (whose factor loadings are all zero). Note that the zero-beta asset is riskfree. Once λ_0, λ_1 and λ_2 are known, the expected return of an asset is completely determined by the factor loadings b_{i1} and b_{i2}. Theoretically, a riskless portfolio can be constructed from any three risky assets and λ_0 can be determined accordingly.

Indeed, we choose a solution $\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ that satisfies Eq. (A), we obtain a risk free portfolio. We define λ_0 to be the expected rate of return of this riskfree portfolio, where

$$\lambda_0 = \sum_{i=1}^{3} w_i a_i.$$

The expected rate of return becomes $\overline{r}_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2}$.

Numerical example

Consider 3 assets whose random rates of return are governed by

$$\begin{aligned} r_1 &= 5 + 2f_1 + 3f_2 \\ r_2 &= 6 + f_1 + 2f_2 \\ r_3 &= 4 + 6f_1 + 10f_2, \end{aligned}$$

where f_1 and f_2 are the risk factors observing $E[f_1] = E[f_2] = 0$. We can form a riskfree portfolio by assigning weights w_1, w_2 and w_3, which can obtained by solving

$$\begin{aligned} w_1 + w_2 + w_3 &= 1 \\ 2w_1 + w_2 + 6w_3 &= 0 \\ 3w_1 + 2w_2 + 10w_3 &= 0. \end{aligned}$$

The solution of the above system of equations gives $w_1 = w_2 = \frac{2}{3}$ and $w_3 = -\frac{1}{3}$.

This riskfree portfolio has zero factor loading (or called zero-beta portfolio). Its deterministic rate of return $= \sum_{i=1}^{3} w_i a_i = 5w_1 + 6w_2 + 4w_3 = 6$. This may be considered as the proxy riskfree rate, and it is called λ_0. To determine the factor risk premia λ_1 and λ_2, we observe

$$\begin{pmatrix} -1 & 0 & -2 \\ 2 & 1 & 6 \\ 3 & 2 & 10 \end{pmatrix} \begin{pmatrix} 2/3 \\ 2/3 \\ -1/3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Note that the first row is given by

$$\begin{pmatrix} a_1 - \lambda_0 & a_2 - \lambda_0 & a_3 - \lambda_0 \end{pmatrix} = \begin{pmatrix} 5 & 6 & 4 \end{pmatrix} - 6 \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -2 \end{pmatrix}.$$

The first row can be written as (-2) times the second row plus the third row, so $\lambda_1 = -2$ and $\lambda_2 = 1$. We check that

$$\begin{aligned} \overline{r}_1 &= \lambda_0 + 2\lambda_1 + 3\lambda_2 = 6 - 4 + 3 = 5; \\ \overline{r}_2 &= \lambda_0 + \lambda_1 + 2\lambda_2 = 6 - 2 + 2 = 6; \\ \overline{r}_3 &= \lambda_0 + 6\lambda_1 + 10\lambda_2 = 6 - 12 + 10 = 4. \end{aligned}$$

Remarks

1. In this example, we obtain $\lambda_1 = -2$. This is because r_3 has a low value of expected value ($\overline{r}_3 = 4$), though r_3 has high factor loading in f_1. This leads to negative risk premium value for f_1.

   Suppose we modify the expected return values to assume some higher numerical values; for example

   $$r_1 = 13+2f_1+3f_2, \quad r_2 = 10+f_1+2f_2, \quad r_3 = 28+6f_1+10f_2.$$

   The same portfolio weights on the 3 assets can be used to form a riskfree portfolio since the factor loadings of the 3 assets remain the same. The new $\lambda_0 = \frac{2}{3} \times 13 + \frac{2}{3} \times 10 - \frac{1}{3} \times 28 = 6$. The new first row $= (13-\lambda_0 \quad 10-\lambda_0 \quad 28-\lambda_0) = (7 \quad 4 \quad 22)$, which can be expressed as 2 times the second row $= (2 \quad 1 \quad 6)$ plus the third row $= (3 \quad 2 \quad 10)$. We obtain $\lambda_1 = 2$ and $\lambda_2 = 1$.

2. Suppose we modify the risk factors by some scalar multiples, say, new factors $\widetilde{f}_1$ and $\widetilde{f}_2$ are chosen to be $\widetilde{f}_1 = 2f_1$ and $\widetilde{f}_2 = 3f_2$. The factor loading b_{i1} and b_{i2} are reduced by a factor of $\frac{1}{2}$ and $\frac{1}{3}$, respectively. We now have

$$\begin{aligned} r_1 &= 5 + \widetilde{f}_1 + \widetilde{f}_2 \\ r_2 &= 6 + \frac{\widetilde{f}_1}{2} + \frac{2}{3}\widetilde{f}_2 \\ r_3 &= 4 + 3\widetilde{f}_1 + \frac{10}{3}\widetilde{f}_2. \end{aligned}$$

The new factor risk premia become $\widetilde{\lambda}_1 = 2\lambda_1 = -4$ and $\widetilde{\lambda}_2 = 3\lambda_2 = 3$. Not surprisingly, we obtain the same results for the expected rates of return of the assets.

3. In the derivation of the factor risk premia, we have assumed zero idiosyncratic risk for all asset returns; that is, $e_j = 0$ for all assets. When idiosyncratic risks are present, we obtain the same result for the factor risk premia for a well diversified portfolio (under the notion of so-called asymptotic arbitrage).

Summary

The expected excess rate of return above the riskfree rate is given by the sum of the product of the factor loading and factor risk premium for each risk factor. With m risky factors, we have

$$\overline{r}_j - r = \sum_{k=1}^{m} b_{jk}\lambda_k.$$

Expected excess return in terms of the expected excess return of two portfolios

Given any two portfolios P and M with $\frac{b_{P1}}{b_{P2}} \neq \frac{b_{M1}}{b_{M2}}$, we can solve for λ_1 and λ_2 in terms of the expected excess return on these two portfolios: $\overline{r}_M - r$ and $\overline{r}_P - r$. The governing equations for the determination of λ_1 and λ_2 are

$$\begin{aligned}\overline{r}_P - r &= \lambda_1 b_{P1} + \lambda_2 b_{P2} \\ \overline{r}_M - r &= \lambda_1 b_{M1} + \lambda_2 b_{M2}.\end{aligned}$$

Once λ_1 and λ_2 are obtained in terms of $\overline{r}_P - r, \overline{r}_M - r$ and factor loading coefficients, we then have the following CAPM-like formula:

$$\overline{r}_i = r + \lambda_1 b_{i1} + \lambda_2 b_{i2} = r + b'_{i1}(\overline{r}_M - r) + b'_{i2}(\overline{r}_P - r)$$

where

$$b'_{i1} = \frac{b_{i1}b_{P2} - b_{i2}b_{P1}}{b_{M1}b_{P2} - b_{M2}b_{P1}}, \quad b'_{i2} = \frac{b_{i2}b_{M1} - b_{i1}b_{M2}}{b_{M1}b_{P2} - b_{M2}b_{P1}}.$$

Proof

Consider $\bar{r}_i = r + \begin{pmatrix} b_{i1} & b_{i2} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ and $\begin{pmatrix} b_{M1} & b_{M2} \\ b_{P1} & b_{P2} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \bar{r}_M - r \\ \bar{r}_P - r \end{pmatrix}$,

so that

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{b_{M1}b_{P2} - b_{M2}b_{P1}} \begin{pmatrix} b_{P2} & -b_{M2} \\ -b_{P1} & b_{M1} \end{pmatrix} \begin{pmatrix} \bar{r}_M - r \\ \bar{r}_P - r \end{pmatrix}.$$

Therefore, we obtain

$$\begin{aligned} \bar{r}_i &= r + \frac{1}{b_{M1}b_{P2} - b_{M2}b_{P1}} \begin{pmatrix} b_{i1} & b_{i2} \end{pmatrix} \begin{pmatrix} b_{P2} & -b_{M2} \\ -b_{P1} & b_{M1} \end{pmatrix} \begin{pmatrix} \bar{r}_M - r \\ \bar{r}_P - r \end{pmatrix} \\ &= r + \begin{pmatrix} \frac{b_{i1}b_{P2} - b_{i2}b_{P1}}{b_{M1}b_{P2} - b_{M2}b_{P1}} & \frac{b_{i2}b_{M1} - b_{i1}b_{M2}}{b_{M1}b_{P2} - b_{M2}b_{P1}} \end{pmatrix} \begin{pmatrix} \bar{r}_M - r \\ \bar{r}_P - r \end{pmatrix}. \end{aligned}$$

Numerical example

Consider the previous example with the following 2 assets:

$$r_1 = 5 + 2f_1 + 3f_2, \quad r_2 = 6 + f_1 + 2f_2;$$

with riskfree rate $r = 6$. Here, r is given since we cannot determine r (or λ_0) with the information of only 2 risky assets. Now, λ_1 and λ_2 are governed by

$$\overline{r}_1 - 6 = 2\lambda_1 + 3\lambda_2, \quad \overline{r}_2 - 6 = \lambda_1 + 2\lambda_2;$$

so that

$$\lambda_1 = \frac{\begin{vmatrix} \overline{r}_1 - 6 & 3 \\ \overline{r}_2 - 6 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}} = 2(\overline{r}_1 - 6) - 3(\overline{r}_2 - 6)$$

$$\lambda_2 = \frac{\begin{vmatrix} 2 & \overline{r}_1 - 6 \\ 1 & \overline{r}_2 - 6 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix}} = 2(\overline{r}_2 - 6) - (\overline{r}_1 - 6).$$

Lastly, we express the expected excess of the third asset in CAPM-like form:

$$\overline{r}_3 = r + 6\lambda_1 + 10\lambda_2 = 6 + 2(\overline{r}_1 - 6) + 2(\overline{r}_2 - 6).$$

Remark

With n risk factors, we may write the excess expected return above r of the asset j as sum of scalar multiples of known values of excess expected return above r of n assets $\overline{r}_1 - r, \overline{r}_2 - r, \cdots, \overline{r}_n - r$. That is,

$$\overline{r}_j - r = \sum_{k=1}^{n} b'_{jk}(\overline{r}_k - r), \qquad j \neq 1, 2, \ldots, n,$$

where

$$\begin{pmatrix} b'_{j1} & \cdots & b'_{jn} \end{pmatrix} = \begin{pmatrix} b_{j1} & \cdots & b_{jn} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ b_{21} & & b_{2n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}^{-1}.$$

The derivation procedure follows the same steps as the two-factor case shown on p.93-94.

Compare and contrast the CAPM and the APT

- Unlike the CAPM, the APT does not assume that investors make decisions according to the mean-variance rule.

- The primary assumption of the APT is that security returns are generated by a linear factor model. The APT is the return-risk relationship that applies in the equilibrium situation with no arbitrage opportunities.

- The single-factor model drastically reduces the inputs needed in solving for the optimum portfolios in the efficient frontier, since the covariances can be calculated easily: $\text{cov}(r_i, r_j) = \beta_i \beta_j \sigma_f^2$.

Example

Assume that a three-factor model is appropriate, and there are an infinite number of assets. The expected return on a portfolio with zero beta values is 5 percent. You are interested in an equally weighted portfolio of two stocks, A and B. The factor risk premiums are indicated in the accompanying table, along with the factor betas for A and B. Compute the approximate expected return of the portfolio.

Factor i	β_{Ai}	β_{Bi}	Factor risk premium
1	0.3	0.5	0.07
2	0.2	0.6	0.09
3	1.0	0.7	0.02

The zero-beta portfolio can be obtained by constructing an appropriate portfolio of 4 stocks whose returns are driven by these 3 risk factors.

Solution:

By APT, the expected return of a portfolio is given by

$$\overline{r}_P = \lambda_0 + \lambda_1\beta_{P1} + \lambda_2\beta_{P2} + \lambda_3\beta_{P3}.$$

Here, $\lambda_0 = 5\%$, and the beta values for the three factors are

$$\begin{aligned}
\beta_{P1} &= \frac{1}{2}(\beta_{A1} + \beta_{B1}) = \frac{1}{2}(0.3 + 0.5) = 0.4, \\
\beta_{P2} &= \frac{1}{2}(\beta_{A2} + \beta_{B2}) = \frac{1}{2}(0.2 + 0.6) = 0.4, \\
\beta_{P3} &= \frac{1}{2}(\beta_{A3} + \beta_{B3}) = \frac{1}{2}(1.0 + 0.7) = 0.85.
\end{aligned}$$

Given $\lambda_1 = 0.07, \lambda_2 = 0.09, \lambda_3 = 0.02$, so

$$\overline{r}_P = 5\% + 0.07 \times 0.4 + 0.09 \times 0.4 + 0.02 \times 0.85 = 13.1\%.$$

The majority of the quantitative equity portfolio managers employ some form of factor models. Factors are the key ingredients of these models. Factors come in many varieties: fundamental, technical, macroeconomic, etc. Good factors exhibit relationships with stock return that are not only stable and consistent but also can be explained by economic theory.

Stability means the factor loadings of the stocks are stable with respect to estimations obtained from returns data over different investment time periods. Consistency means the factor risk premiums of the risk factors as estimated from different sets of stocks give consistent values.

We have identified four factors in the return-generating model

I_1 = random change in inflation, denoted by I_I

I_2 = random change in aggregate sales, denoted by I_S

I_3 = random change in oil prices, denoted by I_O

I_4 = random return in the S&P index constructed to be uncorrelated to the other factors, denoted by I_M.

Furthermore, assume that the oil risk is not priced, $\lambda_O = 0$; then

$$\overline{r}_i - r = \lambda_I b_{iI} + \lambda_S b_{iS} + \lambda_M b_{iM}.$$

Factor	b	λ	Contribution to mid-cap Expected Excess Return (%)
Inflation	-0.37	-4.32	$1.59 = (-0.37)(-4.32)$
Sales growth	1.71	1.49	$2.54 = 1.71 \times 1.49$
Oil prices	0.00	0.00	0.00
Market	1.00	3.96	3.96
Expected excess return for mid-cap stock portfolio			8.09

The expected excess return for the mid-cap (company with medium size of capitalization) stock portfolio is 8.09%. Sales growth contributes 2.54% to the expected return for the mid-cap. In other words, sensitivity to sales growth accounts for $2.54 \div 8.09$ or 31.4% of the total expected excess return.

Factor	b	λ	Contribution to Growth Stock Portfolio Expected Excess Return (%)
Inflation	−0.50	−4.32	2.16
Sales growth	2.75	1.49	4.10
Oil prices	−1.00	0.00	0.00
Market	1.30	3.96	5.15
Expected excess return for growth stock portfolio			11.41

- The expected excess return for the growth stock portfolio (11.41%) is higher than it was for the mid-cap (8.09%). The growth stock portfolio has more risk, with respect to each index, than the mid-cap portfolio.

- Individual factors have a different absolute and relative contribution to the expected excess return on a growth stock portfolio than they have on the mid-cap index.

Portfolio management

Factor models are used to estimate short-run expected returns to the asset classes. The factors are usually macroeconomic variables, some of which are list below:

1. The rate of return on a treasury bill (T bill).
2. The difference between the rate of return on a short-term and long-term government bond (term).
3. Unexpected changes in the rate of inflation in consumer prices (inflation).
4. Expected percentage changes in industrial production (individual production).
5. The ratio of dividend to market price for the S&P 500 in the month preceding the return (yield).
6. The difference between the rate of return on a low- and high-quality bond (as a proxy of confidence).
7. Unexpected percentage changes in the price of oil (oil).

Four distinctive phases of the market are identified which are based on the directional momentum in stock prices and earnings per share:

1. The initial phase of a bull market.

2. The intermediate phase of a bull market.

3. The final phase of a bull market.

4. The bear market.

Interestingly, for a given type of stock, the factor sensitivities can change dramatically as the market moves from one phase to the next.

Bear market $\longrightarrow$ *Initial phase of bull market*

The factor sensitivities for large versus small stocks in going from a bear market to the initial phase of a bull market are listed below:

	Phase IV		*Phase I*	
Factor	Small Stocks	Large Stocks	Small Stocks	Large Stocks
T bill	−6.45	−1.21	5.16	5.81
Term	0.34	0.45	0.86	0.92
Inflation	−3.82	−2.45	−3.23	−2.20
Ind. prod.	0.54	0.06	0.00	0.40
Yield	1.51	−0.16	−0.18	0.00
Confidence	−0.63	−0.43	2.46	1.45
Oil	−0.21	−0.07	0.26	0.20

Asset allocation decision procedure

- Identify the current market phase, calculate the factor values typically experienced in such a phase, and make modifications in these factor sensitivities to reflect expectations for the forthcoming period (usually a year).

- Calculate expected returns for the asset classes (such as large and small stocks) on the basis of the factor sensitivities in the phase.

- These expected returns can then be imported to an optimizer to determine the mix of investments that maximizes expected return given risk exposure for the forthcoming year.

3.4 Portfolio performance analysis

- To evaluate the performance of a portfolio, the first thing we would like to know is the rate of return.
- A direct way of computing rate of return: $r = (MV_1 - MV_0)/MV_0$, where MV_0 is beginning market value of the portfolio/fund and MV_1 is the ending market value. This works well for "static" portfolios that have no intermediate cash flows.

Money-Weighted Return (MWR)

A money-weighted return is analogous in concept to an *internal rate of return*. It is the discount rate on which the net present value of inflows is the same as the present value of outflows.

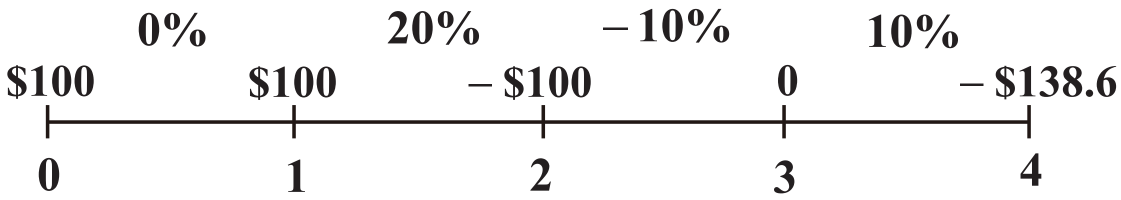
Example 1 *The two funds A and B have identical rates of return at the end of each of the 4 years of investment*

Period	1	2	3	4
Rate of return	0%	20%	-10%	10%
Fund A				
1. Beginning MV	100	200	140	126
2. Appreciation (depreciation) in value	0	40	(14)	12.6
3. Deposit (withdrawal)	100	(100)	0	0
4. Ending MV	200	140	126	138.6
Fund B				
1. Beginning MV	100	100	220	98
2. Appreciation (depreciation) in value	0	20	(22)	9.8
3. Deposit (withdrawal)	0	100	(100)	0
4. Ending MV	100	220	98	107.8

Apparently, for Fund A, we calculate the MWR as

$$100 + \frac{100}{1+r} = \frac{100}{(1+r)^2} + \frac{138.6}{(1+r)^4},$$

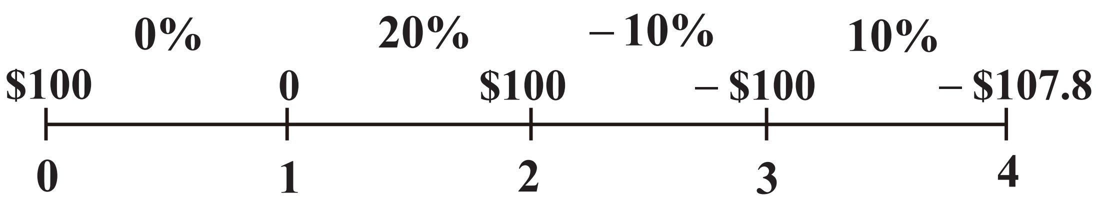
yielding $r \approx 7\%$.



Similarly, for Fund B,

$$100 + \frac{100}{(1+r)^2} = \frac{100}{(1+r)^3} + \frac{107.8}{(1+r)^4},$$

yielding $r \approx 1.9\%$. Is Fund A a better investment than Fund B?



Fund A happens to be lucky in the sense that deposit of $100 occurs right before a good year of 20% rate of return and withdrawal of $100 occurs right before a bad year of -10% rate of return.

Advantages and disadvantages of MWR

- MWR incorporates the size and timing of interim cash flows (e.g. withdrawal/deposit and dividend/interest). A direct measure of investment performance from an investor's standpoint.

- Deposits/withdrawals have significant impact on the MWR, while they are neither under the control of the manager nor has much to do with the management skills. Therefore, it is not a good measure for the investment skills of the fund manager.

Time-Weighted Return (TWR)

The time-weighted return is essentially a geometric mean of a series of holding-period returns (HPR) that are linked together or compounded over time (thus, time-weighted).

$$HPR_1 = (MV_1 - MV_0 + D_1)/MV_0,$$

where D_1 is the dividend/interest inflows.

$$TWR = \sqrt[n]{(1 + HPR_1)(1 + HPR_2)\cdots(1 + HPR_n)} - 1.$$

Example 1 revisited

$$TWR = \sqrt[4]{1 \times 1.2 \times 0.9 \times 1.1} - 1 = 4.4\%.$$

TWR versus MWR

- TWR is useful in determining the management skills and comparing an investment's performance to indices.

- TWR does not take into account deposits, withdrawals or amount of investments, which do have material impact on the final profit and loss of investors. MWR is referred as the "investors' way" of calculating return on investment, and TWR as the "managers' way".

Beyond computing the rate of return over the evaluation period

Suppose that you learned that manager A realized a rate of return of -2% over the past year, while manager B earned $+12\%$. What is the appropriate action? Replace manager A? Give manager B a large bonus? The answer is not clear.

- The $+12\%$ earned by manager B is less impressive if he earned the money with an extremely risky strategy, such as buying a short-run, deep out-of-the-money call option (a very risky investment strategy). How could he or she take such risks with other people's hard-earned money? He or she could have lost it all just as easily as he or she had won it! Surely, then, manager B should not be rewarded. In general, performance evaluation must weigh the realized rate of return against the risk taken in order to achieve this return.

- Performance evaluation needs to separate investment skill from chance. For example, the performance of manager A is put in a different light if we know that he or she consistently outperformed the market in the previous ten years and that the poor performance in the last year is attributable solely to the unexpected bankruptcy of a company that was generally believed to have a high credit quality. Put differently, we need to determine whether the results are significantly better or significantly worse than expected.

- The -2% return of manager A looks very different if we know that the general market went down by 10%; in this case, manager A still outperformed the market by an impressive 8%. In other words, any analysis of investment performance needs to correct for the market conditions faced by the investment manager.

Risk-adjusted performance measures

Rate of return alone is not an appropriate measure of investment performance, as return depends more on (1) the target risk level of the portfolio, (2) the performance of the overall market, than on the skill level of the portfolio manager.

For example, consider two funds A and B with $\beta_B > \beta_A$. In a bull market, Fund B is expected to produce a higher rate of return than Fund A, but it does not mean managers of Fund B is doing a better job. When the market is bearish, Fund A would outperform.

Efficient market hypothesis

The efficient market hypothesis (EMH) states that asset prices fully reflect all available information. The weak form of EMH claims that prices on traded assets already reflect all past publicly available information. The strong form of the EMH additionally claims that prices instantly reflect even hidden "insider" information.

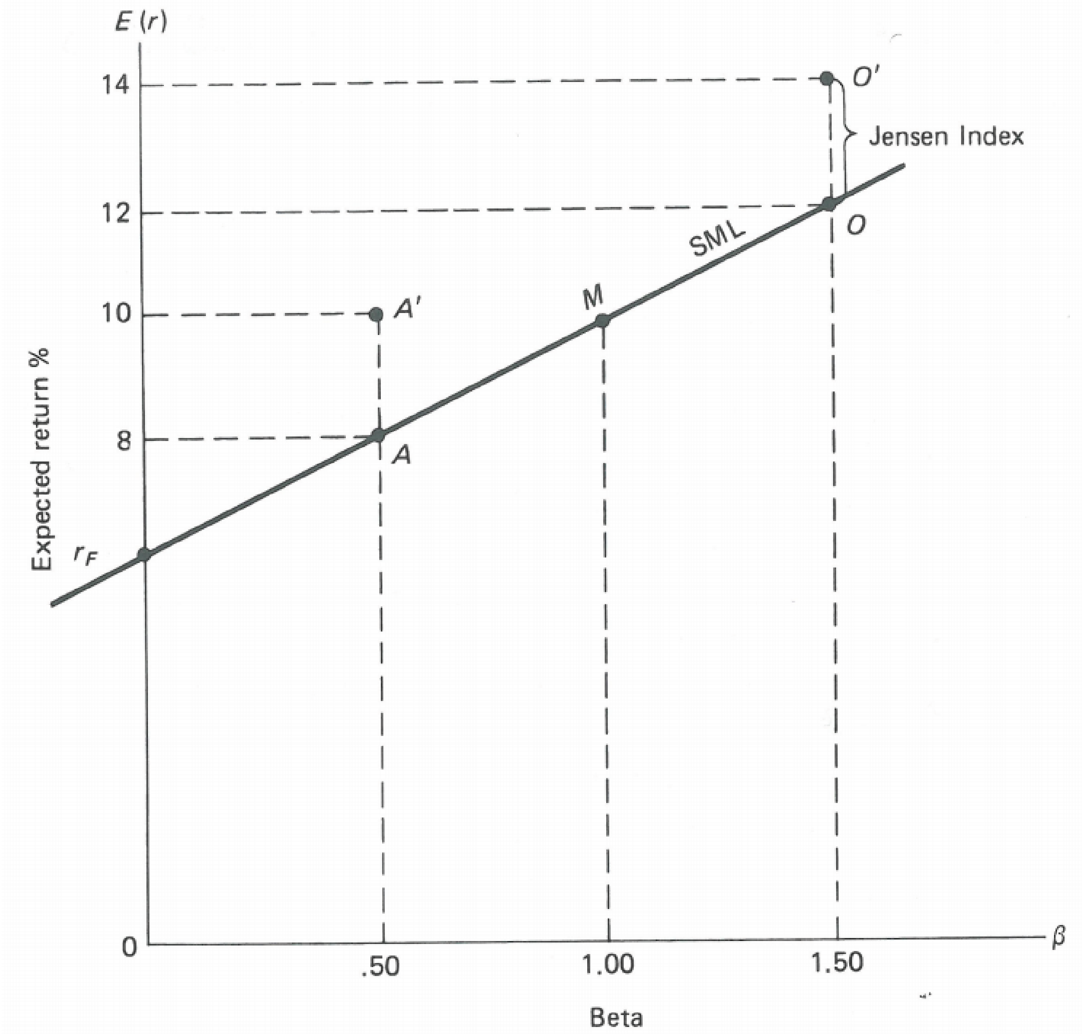
Risk-adjusted performance measures based on the CAPM

It is reasonable to divide the information relevant to the valuation of any stock into two categories: (1) public information, which is freely available to everyone, and (2) private information, which is known to selected individuals.

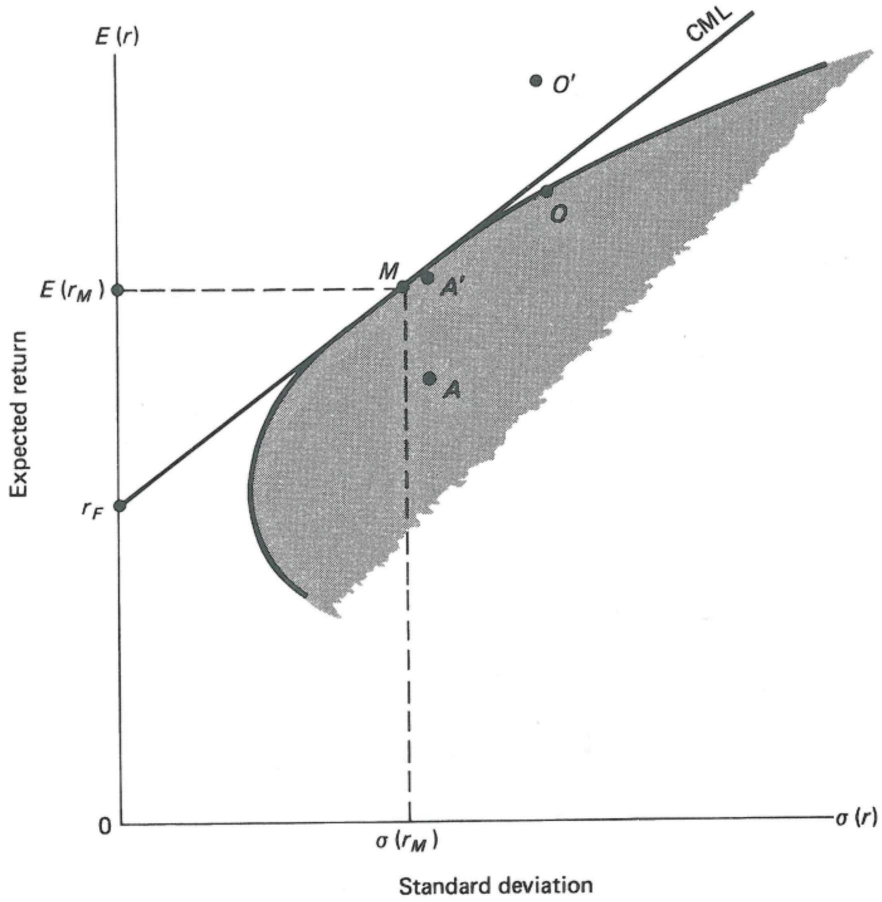
Analysis based on public information only would suggest that every stock and portfolio are positioned on the SML. The CAPM is derived based on common belief on the expected returns and correlation of returns of the risky assets.

Example 2 *Two professionally managed funds: Alpha and Omega*

	Alpha Fund	*Omega Fund*
Stock portfolio	*a single stock*	*multiple stocks*
Public expected return	*8%*	*12%*
Private expected return	*10%*	*14%*
Beta value	*0.5*	*1.5*
Residual variance	*7.5%*	*Well-diversified*



Based on public information, both Alpha Fund and Omega Fund lie on the SML. However, their real positions are A' and O', due to the excess return attributed to private information.



Based on public information, both Alpha Fund and Omega Fund are below the CML as they invest in risky stocks only. But Omega Fund is well-diversified, so it lies on the minimum variance frontier. Their real positions should be A' and O', respectively, if both public and private information are taken into account.

Jensen Alpha

For any portfolio P, the Jensen Alpha is defined by

$$\alpha_P = \overline{r}_P - [r + \beta_P(\overline{r}_M - r)],$$

where $\overline{r}_M$ is the expected rate of return of market portfolio and r is the riskfree rate.

- The term in the squared brackets is the expected rate of return if P were on the SML. The Jensen Alpha α_P measures the difference between the portfolio's actual expected return and the model predicted expected return.

- A statistically significant positive α_P implies that the manager is indeed adding value (perhaps with some private information); a negative one implies that the fund is managed in a systematically "wrong" way. An insignificant α_P implies that the manager knows no more than the herd.

Depth and breadth of investment performance

Example 2 revisited: The Jensen Alpha for the two funds are

$$\begin{aligned}\alpha_A &= 0.10 - [0.06 + (0.1 - 0.06) \times 0.5] = 0.02 \\ \alpha_O &= 0.14 - [0.06 + (0.1 - 0.06) \times 1.5] = 0.02.\end{aligned}$$

It seems that Alpha and Omega are managed equally well.

Investment performance is assessed in two aspects: (1) depth (the magnitude of "alpha" or excess return captured by the manager), and (2) breadth (the number of securities for which a manager can capture excess returns).

Though Alpha Fund and Omega Fund have the same depth, Alpha has less breadth than Omega, due to the fact that Alpha has private information of a single stock while Omega has private information of multiple stocks.

Pros and cons of the Jensen Alpha

- The Jensen Alpha is an improvement over the pure return measures. The benchmark return $[r + \beta_P(\overline{r}_M - r)]$ looks at the expected return generated from bearing the systematic risk (as characterized by the beta value). It is insensitive to non-systematic risk.

- The Jensen Alpha reflects only the *depth* of investment performance, but fails to incorporate the *breadth*. It can be used for performance evaluation for portfolios that are well diversified (good level of breadth) and have equal beta (same level of systematic risk).

Alpha and Omega are considered as doing equally well, even though they bear different levels of systematic risks ($\beta_A \neq \beta_O$).

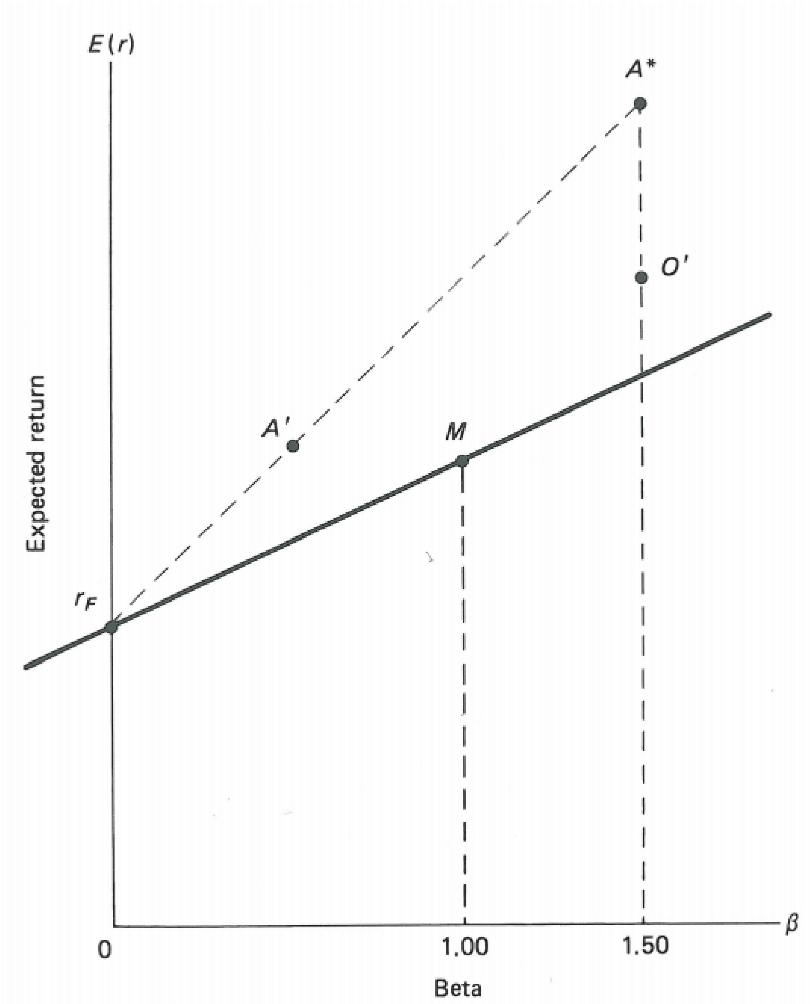
Treynor ratio

For any portfolio P, we define the Treynor ratio by

$$T_P = \frac{\overline{r}_P - r}{\beta_P}.$$

- Excess rate of return earned per unit of risk taken, where risk is measured in terms of the beta factor of the portfolio. Since β_P is used in T_P, only systematic risk is considered.

- Geometrically, T_p is equal to the slope of a straight line connecting the fund with the riskfree asset in $(\beta, \bar{r})$ diagram.



To evaluate the performances of Alpha Fund and Omega Fund by the Treynor ratio, one first constructs a leveraged equivalent A^* such that $T_{A^*} = T_A$ and $\beta_{A^*} = \beta_O$. Here, A^* dominates O^*, implying that Alpha is a more desirable investment.

Relationship between Treynor ratio and Jensen Alpha

Reformulating the equation that defines the Jensen Alpha, we obtain

$$\bar{r}_P - r = \alpha_P + \beta_P(\bar{r}_M - r).$$

Dividing both sides by β_P yields

$$\frac{\bar{r}_P - r}{\beta_P} = \frac{\alpha_P}{\beta_P} + \bar{r}_M - r.$$

Therefore,

$$T_P = \frac{\alpha_P}{\beta_P} + \bar{r}_M - r.$$

Since $\bar{r}_M - r$ is independent of P, T_P gives the same ranking as the beta-adjusted Jensen Alpha, $\frac{\alpha_P}{\beta_P}$ (see the figure on the last page).

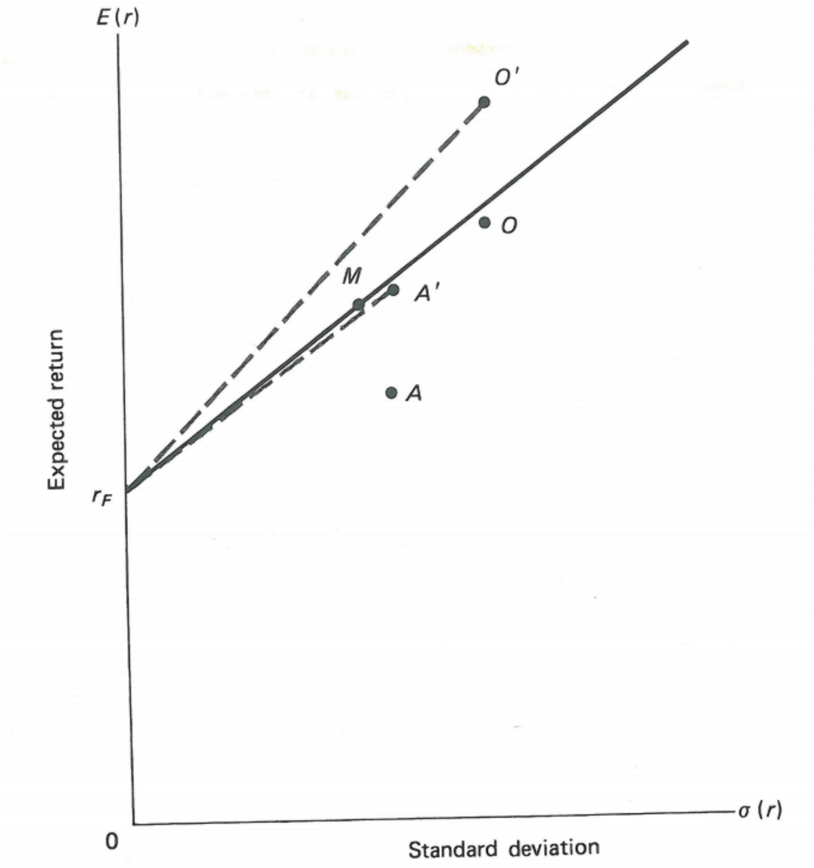
- The Treynor ratio does not reflect breadth. For instance, T_A fails to penalize Alpha Fund for bearing nonzero residual (diversifiable) risk. One may wrongly conclude that Alpha Fund is better managed than Omega Fund based on the Treynor ratio.

Sharpe ratio

Sharpe's reward-to-variability ratio is defined by

$$S_P = \frac{\overline{r}_P - r}{\sigma_P}.$$

- It measures the excess return earned per unit of total risk taken, where total risk is measured by standard deviation σ_P. In the $(\sigma, \overline{r})$ plane, Sharpe ratio is the slope of the a straight line connecting the fund with the riskfree asset.
- A higher Sharpe ratio would indicate a better performance.
- When there is only public information, CAPM tells us that the Sharpe Ratio of any portfolio is capped by the slope of the CML. This is no longer the case when private information is available.



Based on public information only, S_A and S_O are both less than the slope of the CML, but Omega is closer to CML because of its breadth (lower level of diversifiable risk). Both Alpha fund and Omega fund are lifted by 2% due to private information. It is seen that Omega Fund outperforms the market.

Relationships between the Sharpe ratio, Jensen Alpha and Treynor ratio under very low level of diversifiable risk

Recall that

$$\beta_P = \frac{\rho\sigma_P\sigma_M}{\sigma_M^2} = \frac{\rho\sigma_P}{\sigma_M}.$$

If the portfolio is well diversified, then there will be very low level of diversifiable (non-systematic) risk. The portfolio lies closely to the CML, so $\rho \approx 1$. As a result, the Jensen Alpha becomes

$$\overline{r}_P - r = \alpha_P + \beta_P(\overline{r}_M - r) \approx \alpha_P + \frac{\sigma_P}{\sigma_M}(\overline{r}_M - r),$$

which gives the following relation between S_P and α_P

$$S_P = \frac{\overline{r}_P - r}{\sigma_P} \approx \frac{\alpha_P}{\sigma_P} + \frac{\overline{r}_M - r}{\sigma_M}.$$

On the other hand, T_P and S_P are related by

$$T_P = \frac{\overline{r}_P - r}{\beta_P} = \frac{\overline{r}_P - r}{\sigma_P}\frac{\sigma_P}{\beta_P} \approx S_P\sigma_M.$$

For well-diversified portfolios, the Sharpe ratio and Treynor Ratio give consistent rankings.

Pros and cons of the Sharpe ratio

- The Sharpe ratio is based on the total risk, sum of systematic risk and idiosyncratic risk. It gives better performance to fund managers who care about both depth (extracting alpha) and breadth (minimizing diversifiable risk).

- The Sharpe ratio is sensitive to both depth and breadth, where the latter is captured by the diversifiable risk. In Example 2, Alpha Fund is found to be inferior to Omega Fund when the Sharpe ratio is used as the performance measure. Alpha Fund also underperforms the market portfolio, probably because the negative effect of its lack of breadth overrides the positive effect of its depth.

Pitfalls associated with the performance measures

- *Restriction on borrowing of riskfree asset:* When investors cannot borrow at the riskfree rate or can only borrow at a high rate, the graphs of CML and SML will change accordingly.

- *Specification of the market portfolio*: In practice, the market portfolio is not observable. Any errors that result from specifying the market portfolio could lead to substantial difference in performance evaluation implied by these single-index measures.

- *Evaluation period*: With no knowledge of the precise distribution of the return, we have to replace the statistics of return with their samples. Observed samples will vary from period to period.

Summary of key concepts

Construction of the uncorrelated counterpart of an efficient fund

- Recall that zero-beta funds are those funds which lie on the same horizontal line with the riskfree point. Given the riskfree point, we determine the market portfolio by the tangency method. Conversely, given an efficient fund, we find the corresponding "pseudo" riskfree point such that the efficient fund becomes the market portfolio. This is done by drawing a tangent to the efficient frontier at the frontier fund and finding the intercept of this tangent line at the vertical $\overline{r}$-axis.

Capital market line and efficient portfolios

- All portfolios lying on the CML are efficient, and all are composed of various mixes of the market portfolio and the risk free asset.

- The beta value of an efficient portfolio is equal to the proportional weight of market portfolio in the efficient portfolio. This is obvious since the excess return above the riskfree rate is contributed by the portion of market portfolio.

- The CML does not apply to individual asset or portfolios that are inefficient, because investors do not require a compensation for non-systematic risk.

- Efficient portfolios have the same Sharpe ratio as that of the market portfolio.

- All portfolios are on or below the CML. When the correlation coefficient between portfolio's return and market return is closer to 100%, the portfolio is closer to being efficient and comes closer to the CML. The Sharpe ratio of an inefficient portfolio P is $\rho_{PM} \times$ Sharpe ratio of the market portfolio M.

- Efficient portfolios have zero specific (also called diversifiable or non-systematic) risk.

- The extended version of CAPM allows the replacement of the Market Portfolio by any efficient portfolio.

Security market line

- In equilibrium, all assets and portfolios lie on the security market line. All assets are priced correctly and one cannot find bargains. Any derivation from the SML implies that the market is not in the CAPM equilibrium.

- When equilibrium prevails, the expected excess return above the riskfree rate normalized by the beta is constant for all assets / portfolios. That is,

$$\frac{\overline{r}_i - r}{\beta_i} = \frac{\overline{r}_j - r}{\beta_j}$$

  for any pair of asset i and asset j. Therefore, beta is the appropriate measure of risk to compare asset returns under equilibrium conditions.

Beta value

- According to CAPM, the higher the asset risk (beta), the higher the expected rate of return.

- All assets/portfolios with the same beta share the same amount of systematic risk, and they have the same excess return above the riskfree rate. The beta value (not portfolios standard deviation) is used as a measure of risk in CAPM since only the systematic risk is rewarded with extra returns. When the specific risk becomes zero, the portfolio standard deviation equals beta times market portfolio's standard deviation.

APT model

- Returns of assets are driven by a set of macroeconomic factors and asset-specific component.

- The contribution from each risk factor to the expected excess return above the riskfree rate is the product of factor loading and factor risk premium.

- The APT does not require the identification of the market portfolio. Instead it requires the specification of the relevant macroeconomics factors. Much of the empirical APT research has focused on the identification of these factors.

- The APT and CAPM complement each other. They both predict that excess returns are governed by factor sensitivities that move with the market.

Performance indexes

Jensen alpha: $\alpha_P = \overline{r}_P - [r + \beta_P(\overline{r}_M - r)]$

Treynor index: $T_P = \dfrac{\overline{r}_P - r}{\beta_P}$.

Both are based on the security market line (expected rate of return is rewarded by bearing the systematic risk).

- Jensen alpha looks at the amount of expected rate of return above (or below) the SML.
- Treynor index is benchmarked against the slope of SML, or e-quivalently, the Treynor index of the market portfolio, where

$$T_M = \frac{\overline{r}_M - r}{\beta_M}.$$

Shape index: $S_P = \dfrac{\overline{r}_P - r}{\sigma_P}$

It is based on the capital market line. The risk considered is the sum of systematic risk and diversifiable risk. A fund manager who is successful in seeking breadth would have low level of diversifiable risk in his portfolio. Shape index is benchmarked against the slope of CML, which is the Sharpe ratio of the market portfolio.