MATH 4512
Fundamentals of Mathematical Finance
Final Examination – Spring 2016

Time allowed: 135 minutes *Course instructor: Prof. Y. K. Kwok*

[points]

1. Consider a portfolio with N risky assets and a riskfree asset whose rate of return is r. The covariance matrix and expected rate of return vector of the N risky assets are Ω and $\boldsymbol{\mu}$, respectively.

   (a) Explain why the set of efficient portfolios in the $\sigma_P - \mu_P$ diagram lie on the tangent line from the riskfree point $(0, r)$ to the hyperbolic arc that bounds the feasible region of the N risky assets. **[3]**

   (b) Recall that the weight vector of the N risky assets of an efficient portfolio on the tangent line takes the form

   $$\boldsymbol{w} = \lambda \Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1}),$$

   where λ is a multiplier and $\mathbf{1} = (1 \ \ldots \ 1)^T$. Define $a = \mathbf{1}^T \Omega^{-1} \mathbf{1}$, $b = \boldsymbol{\mu}^T \Omega^{-1} \mathbf{1}$ and $c = \boldsymbol{\mu}^T \Omega^{-1} \boldsymbol{\mu}$ and assume $r \neq \frac{b}{a}$, find the weight vector of the tangency portfolio. Find the expected rate of return and variance of the portfolio return of the tangency portfolio. Then deduce the equation of the tangent line in the $\sigma_P - \mu_P$ diagram. **[6]**

   Hint: This tangency fund has sum of weights of the risky assets equal one.

   (c) When $r = \frac{b}{a}$, explain why the tangency portfolio does not exist. We define μ_P to be the target expected rate of return of an efficient portfolio. Show that the corresponding weight vector of the efficient portfolio meeting the target level μ_P is given by

   $$\boldsymbol{w} = \frac{a(\mu_P - r)}{ac - b^2} \Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1}).$$ **[5]**

2. Let M be the market portfolio and E be an efficient portfolio that lies on the capital market line. Let β_E be the beta value of E, r_M and r_E be the random rate of return of M and E, respectively.

   (a) Define the idiosyncratic risk of a portfolio and explain why E has zero idiosyncratic risk. **[3]**

   (b) Show that

   $$r_E = \beta_E r_M + (1 - \beta_E) r,$$

   where r is the riskless interest rate. **[3]**

   (c) Let P be any portfolio and r_P denote the random rate of return of P. Define β_{PE} to be

   $$\beta_{PE} = \frac{\mathrm{cov}(r_P, r_E)}{\mathrm{var}(r_E)}.$$

   Suppose the equilibrium condition under the Capital Asset Pricing Model prevails, show that

   $$\overline{r}_P - r = \beta_{PE}(\overline{r}_E - r),$$

   where $\overline{r}_P$ and $\overline{r}_E$ are the expected rate of return of P and E, respectively. **[5]**

3. Let P_0 be the observable market price of a risky asset observed at the current time and $\widetilde{P}$ be the random price of the risky asset one period later. Let r be the riskless interest rate and r_M be the random rate of return of the market portfolio over the same investment period.

   (a) Show that the fair price of the risky asset under the equilibrium condition of the Capital Asset Pricing Model is given by

   $$P_{\text{fair}} = \frac{1}{1+r}\left\{E[\widetilde{P}] - \frac{\text{cov}(\widetilde{P}, r_M)(E[r_M] - r)}{\text{var}(r_M)}\right\}.$$ **[5]**

   (b) The quantity $\dfrac{\text{cov}(\widetilde{P}, r_M)(E[r_M] - r)}{\text{var}(r_M)}$ is usually called the dollar discount while the other quantity $E[\widetilde{P}] - \dfrac{\text{cov}(\widetilde{P}, r_M)(E[r_M] - r)}{\text{var}(r_M)}$ is called the certainty equivalent. Give your financial interpretation of these two terms. **[1]**

   (c) Suppose $P_{\text{fair}} < P_0$, what would be the expected reaction of investors' decisions on this risky asset? What would be the eventual price of the risky asset under market equilibrium as predicted by the Capital Asset Pricing Model? **[2]**

4. Consider the single-factor model with the risky factor f, the random rate of return of asset i is given by

   $$r_i = a_i + b_i f + e_i, \quad i = 1, 2, \ldots, n,$$

   where

   $$\text{cov}(f, e_i) = 0, \quad i = 1, 2, \ldots, n, \quad \text{and} \quad \text{cov}(e_i, e_j) = 0, \ i \neq j.$$

   Form a portfolio P, where

   $$r_P = \sum_{i=1}^{n} w_i a_i + \sum_{i=1}^{n} w_i b_i f + \sum_{i=1}^{n} w_i e_i.$$

   Here, w_i is the weight of asset i, $i = 1, 2, \ldots, n$.

   (a) Show that the overall variance of portfolio is given by

   $$\sigma_P^2 = b^2 \sigma_f^2 + \sigma_e^2$$

   where

   $$b = \sum_{i=1}^{n} w_i b_i, \quad \sigma_f^2 = \text{var}(f).$$

   Express σ_e^2 in terms of w_i and e_i, $i = 1, 2, \ldots, n$. **[4]**

   (b) Explain why the residual risk e_i is diversifiable when the number of assets in the portfolio tends to infinity. **[3]**

   (c) Find the covariance between r_i and r_j. **[1]**

5. Consider the three ratios that provide the risk adjusted performance measures:

   Treynor ratio: $T_P = \dfrac{\overline{r}_P - r}{\beta_P}$;

   Jensen alpha: $\alpha_P = \overline{r}_P - [r + \beta_P(\overline{r}_M - r)]$;

   Sharpe ratio: $S_P = \dfrac{\overline{r}_P - r}{\sigma_P}$.

   Here, $\overline{r}_P$ and $\overline{r}_M$ are the expected rate of return of portfolio P and market portfolio M, respectively, β_P is the beta of P, σ_P^2 is the variance of rate of return of P and r is the riskless interest rate.

(a) Explain why T_P gives the same ranking as the beta-adjusted Jensen alpha, $\dfrac{\alpha_P}{\beta_P}$. **[2]**

(b) For well diversified portfolios, explain why the Sharpe ratio and Treynor ratio give consistent rankings. **[4]**

Hint: When a portfolio is well diversified, it is almost efficient and so it lies on the capital market line.

6. Suppose the random return R of a risky asset after one period is given by

$$R = \begin{cases} 1.2 & \text{with probability } \dfrac{1}{2} \\ 0.8 & \text{with probability } \dfrac{1}{2} \end{cases}.$$

The return of the riskless asset is always one (neglecting the time value). Consider a mix of α portion of the risky asset and $1-\alpha$ of the riskless asset, $0 \leq \alpha \leq 1$. Find the optimal choice of α such that the investment achieves the highest long-term expected growth after an infinite number of investment periods. **[6]**

Hint: Maximization of the long-term expected growth is equivalent to maximization of single-period expected return under logarithm utility.

7. (a) Consider the power utility

$$u(x) = \frac{x^\alpha - 1}{\alpha}, \quad \alpha \leq 1.$$

(i) Show that the relative risk aversion coefficient of the power utility is a constant, independent of x. What is the economic interpretation of constant relative risk aversion? **[2]**

(ii) Explain why the logarithm utility is a special case of the class of power utilities. **[1]**

(b) Consider the quadratic concave utility

$$u(x) = ax - \frac{b}{2}x^2, \ a > 0, \ b > 0 \quad \text{and} \quad 0 \leq x \leq a/b.$$

Explain why

mean-variance analysis
$\Leftrightarrow$ maximum expected utility criterion based on quadratic concave utility. **[4]**

Hint: Consider the minimization of portfolio variance for a fixed expected return and maximization of expected return for a fixed portfolio variance.

— *End* —