MATH 4512

Fundamentals of Mathematical Finance

Final Examination – Spring 2017

Time allowed: 135 minutes *Course instructor: Prof. Y. K. Kwok*

[points]

1. Consider a portfolio with N risky assets and a riskfree asset whose rate of return is r. The covariance matrix and expected rate of return vector of the N risky assets are Ω and $\boldsymbol{\mu}$, respectively.

   Recall that the weight vector of an efficient portfolio of the N risky assets on the tangent line from the riskfree point $(0, r)$ to the hyperbolic arc that bounds the feasible region of the N risky assets takes the form

   $$\boldsymbol{w} = \lambda\Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1}),$$

   where λ is the Lagrangian multiplier and $\mathbf{1} = (1 \ \ldots \ 1)^T$. Define $a = \mathbf{1}^T\Omega^{-1}\mathbf{1}$, $b = \boldsymbol{\mu}^T\Omega^{-1}\mathbf{1}$ and $c = \boldsymbol{\mu}^T\Omega^{-1}\boldsymbol{\mu}$. Let M be the tangency portfolio.

   (a) Let μ_M be the expected rate of return of M and σ_M^2 be the variance of rate of return of M. Write $\boldsymbol{w}_M$ as the portfolio weight vector of M. Provided that $r \neq \dfrac{b}{a}$, find $\boldsymbol{w}_M$ and show that

   $$\mu_M = \frac{c - br}{b - ar} \quad \text{and} \quad \sigma_M^2 = \frac{ar^2 - 2br + c}{(b - ar)^2}.$$

   Note that this tangency portfolio has sum of weights of the risky assets equal one. **[4]**

   (b) The tangent line through M is called the capital market line (CML). Find the equation of CML and explain why all portfolios on the CML are efficient portfolios. **[2]**

   (c) Assume $r \neq \dfrac{b}{a}$, let P be any portfolio and write $\boldsymbol{w}_P$ as its weight vector. Let r_P and r_M be the random rates of return of portfolios P and M, respectively. Recall that

   $$\mathrm{cov}(r_P, r_M) = \boldsymbol{w}_P^T\Omega\boldsymbol{w}_M.$$

   (i) Prove the following Capital Asset Pricing Model formula

   $$\beta_{PM} = \frac{\mathrm{cov}(r_P, r_M)}{\sigma_M^2} = \frac{\mu_P - r}{\mu_M - r},$$

   where μ_P is the expected rate of return of P. **[5]**

   (ii) Give the definition of the Sharpe ratio of any given portfolio P. Express the ratio of the Sharpe ratios of P and M in terms of ρ_{PM}, where $\rho_{PM} = \dfrac{\sigma_{PM}}{\sigma_P\sigma_M}$. **[3]**

   (d) Suppose $r = \dfrac{b}{a}$, then the tangency portfolio does not exist since μ_M and σ_M^2 in part(a) are not defined. Given the target expected rate of portfolio return μ_P, show that the corresponding efficient portfolio is to hold 100% on the riskfree asset and w_j on the j^{th} risky asset, $j = 1, 2, \ldots, N$, where w_j is given by the j^{th} component of $\dfrac{a(\mu_P - r)}{ac - b^2}\Omega^{-1}(\boldsymbol{\mu} - r\mathbf{1})$. **[6]**

2. Suppose the random rate of return r_P of portfolio P is formally expressed as

$$r_P = r + \beta_P(r_M - r) + \epsilon_P,$$

where ϵ_P is the residual risk, β_P is the portfolio beta, r_M and r are the random rate of return of the market portfolio M and riskfree interest rate, respectively.

(a) Show that the total risk as proxied by $\sigma_P^2 = \text{var}(r_P)$ is given by

$$\sigma_P^2 = \beta_P^2 \sigma_M^2 + \text{var}(\epsilon_P).$$ **[3]**

(b) Give the financial interpretation that the last term $\text{var}(\epsilon_P)$ is called the firm-specific risk. **[1]**

(c) Explain why $\text{var}(\epsilon_P)$ becomes zero when P is efficient. **[1]**

3. Suppose 3 risky assets whose random rates of return are governed by

$$\begin{aligned} r_1 &= 5 + 2f_1 + 3f_2 \\ r_2 &= 3 + f_1 + 2f_2 \\ r_3 &= 16 + 6f_1 + 10f_2, \end{aligned}$$

where f_1 and f_2 are risk factors observing $E[f_1] = E[f_2] = 0$.

(a) Suppose we would like to form a riskfree portfolio of these 3 risky assets, whose weights are w_1, w_2 and w_3. Explain why the solution of these 3 equations gives the required riskfree portfolio

$$\begin{aligned} &w_1 + w_2 + w_3 = 1 \\ &2w_1 + w_2 + 6w_3 = 0 \\ &3w_1 + 2w_2 + 10w_3 = 0. \end{aligned}$$ **[1]**

(b) The solution of the above system of equations is seen to be $w_1 = w_2 = \frac{2}{3}$ and $w_3 = -\frac{1}{3}$. Suppose the expected rates of return of the 3 assets are given by

$$\begin{aligned} \overline{r}_1 &= \lambda_0 + 2\lambda_1 + 3\lambda_2 \\ \overline{r}_2 &= \lambda_0 + \lambda_1 + 2\lambda_2 \\ \overline{r}_3 &= \lambda_0 + 6\lambda_1 + 10\lambda_2. \end{aligned}$$

Solve for λ_0, λ_1 and λ_2. Give the financial interpretation of the factor risk premiums: λ_1 and λ_2. **[5]**

4. Suppose you are asked to analyze the following two portfolios with the characteristics listed below:

	observed expected rate of return, $\overline{r}_P$	beta, β_P	residual variance
Portfolio A	18%	1.5	0.07
Portfolio B	10%	0.5	0

The riskfree rate of return r is assumed to be 6%. The expected rate of return and variance of the market portfolio M are 12% and 4%, respectively. Recall the following definitions of the performance indexes:

$$\begin{aligned} &\text{Jensen alpha:} \quad && \alpha_P = \overline{r}_P - [r + \beta_P(\overline{r}_M - r)] \\ &\text{Treynor:} && T_P = \frac{\overline{r}_P - r}{\beta_P} \\ &\text{Sharpe:} && S_P = \frac{\overline{r}_P - r}{\sigma_P}. \end{aligned}$$

(a) Compute the values of the above indexes for both Portfolios A and B. Desk calculators are not required since the square roots and divisions can be computed in exact numeration. **[3]**

(b) Explain why the three indexes give different assessments of the performance of the two portfolios. If you seek for both depth and breadth of the fund managers among these two portfolios, which fund is more preferred and why. **[3]**

5. Suppose the financial market provides one riskfree asset and one risky asset. The risky asset either doubles with probability α or halves with probability $1-\alpha$ in each investment period. The random return vector is

$$\boldsymbol{X} = \begin{cases} (1 \quad 2) & \text{with probability } \alpha \\ (1 \quad \frac{1}{2}) & \text{with probability } 1-\alpha \end{cases}.$$

We assume $\frac{1}{3} < \alpha < \frac{2}{3}$. Let the weight vector be

$$\boldsymbol{w} = (w \quad 1-w),$$

where w is the weight assigned to the riskfree asset. Using the log-utility criterion, find the optimal growth function in terms of α. **[4]**

6. Consider the expansion of the utility function $U(x)$ in powers of x. A second-order expansion of $U(x)$ near the mean $\overline{x} = E(x)$ gives

$$U(x) \approx U(\overline{x}) + U'(\overline{x})(x - \overline{x}) + \frac{1}{2}U''(\overline{x})(x-\overline{x})^2.$$

(a) Explain why

$$E[U(x)] \approx U(\overline{x}) + \frac{1}{2}U''(\overline{x})\text{var}(x)?$$ **[1]**

(b) If we let c denote the certainty equivalent and assume it is close to $\overline{x}$, we can use the first-order expansion

$$U(c) \approx U(\overline{x}) + U'(\overline{x})(c - \overline{x}).$$

Using these approximations, find an approximation to the certainty equivalent in terms of $\overline{x}$, $U'(\overline{x})$, $U''(\overline{x})$ and $\text{var}(x)$. **[3]**

7. (a) Suppose the random wealth of a portfolio is a normal random variable so that for an utility function U, the corresponding expected utility value $E[U(y)]$ is a function

of M and σ. Here, M and σ are the mean and standard deviation of the random wealth variable y. We write

$$E[U(y)] = \int_{-\infty}^{\infty} U(y)\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(y-M)^2/2\sigma^2}\ \mathrm{d}y = f(M,\sigma).$$

Here, $U(y)$ is concave and an increasing function of y. Explain why $\dfrac{\partial f}{\partial M} > 0$ and $\dfrac{\partial f}{\partial \sigma} < 0$. **[3]**

(b) Suppose we choose the quadratic concave utility function

$$u(x) = ax - \frac{b}{2}x^2, \quad 0 \le x \le a/b, \quad a > 0 \text{ and } b > 0.$$

Show that

$$E[u(z)] = aE[z] - \frac{b}{2}(E[z])^2 - \frac{b}{2}\mathrm{var}(z),$$

where z is the random wealth value. **[1]**

Explain why

(i) For a given value of $E[Z]$, **[1]**

$$\text{maximizing } E[u(z)] \Leftrightarrow \text{ minimizing var}(z);$$

(ii) For a given value of var(z), **[1]**

$$\text{maximizing } E[u(z)] \Leftrightarrow \text{ maximizing } E[z].$$

8. (a) Consider the following two investment choices. What can be said about their desirability using (i) first-order stochastic dominance, (ii) second-order stochastic dominance?

A		B	
Probability	Outcome	Probability	Outcome
0.2	4%	0.4	5%
0.3	7%	0.3	6%
0.4	8%	0.2	8%
0.1	10%	0.1	10%

[5]

(b) Given a risky project with the distribution (x_i, p_i), $i = 1, 2, \ldots, n$, where x_i is the outcome and p_i is the probability, the geometric mean $\overline{X}_{geo}$ is defined as

$$\overline{X}_{geo} = \prod_{i=1}^{n} x_i^{p_i}, \quad x_i > 0.$$

Show that $\overline{X}_{geo}(F) \ge \overline{X}_{geo}(G)$ is a necessary condition for dominance of F over G by the second order stochastic dominance. **[4]**

— *End* —