2.2 Extensions of the structural approach to the pricing of risky debts

1. *Interest rate uncertainty*

   Debts are relatively long-term interest rate sensitive instruments. The assumption of constant interest rate is embarrassing.

2. Jump-diffusion process of the firm value.

   - Allows for jump component to shock the firm value process.
   - Remedy the unrealistic small short-maturity spreads in pure diffusion model. Default may occur by surprise.

3. *Bankruptcy–triggering mechanism*

   Black-Cox (1976) assume a cut-off level whereby intertemporal default can occur. The cut-off may be considered as a safety covenant which protects the bondholders. It is the liability level for the firm below which the firm bankrupts.

4. *Deviation from the strict priority rule*

   Empirical studies show that the absolute priority rule is enforced in only 25% of corporate bankruptcy cases. The write-down of creditor claims is usually the outcome of a bargaining process which results in shifts of gains and losses among corporate claimants relative to their contractual rights.

Quality spread differentials between fixed rate debt and floating rate debt

- In *fixed rate debts*, the par paid at maturity is fixed.

- A *floating rate debt* is similar to a *money market account*, where the par at maturity is the sum of principal and accrued interests. The amount of accrued interests depends on the realization of the stochastic interest rate over the life of bond.

  What is the appropriate proportion of debts put into either fixed rate or floating rate debts?

  1. Balance sheet duration
  2. Current interest rate environment
  3. Peer group practices.

Whether the default premiums demanded by investors are equal for both types of debts?

Related question: Does the swap rate in an interest rate swap depend on which party is serving as the fixed rate payer?

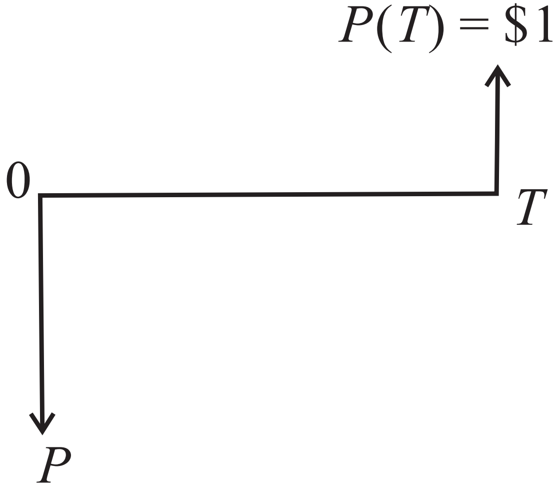
- Empirical studies reveal that the yield premiums for fixed rate debts are in general higher than those for floating rate debts. Why? On the other hand, when the yield curve is upward sloping, floating rate debt holders should demand a higher floating spread. Floating rate debts would appear to be less desirable to investors under such scenario. Consequently, supply and demand drives down the price. Can a risky debt model endogenously reflects this phenomenon?

Reference

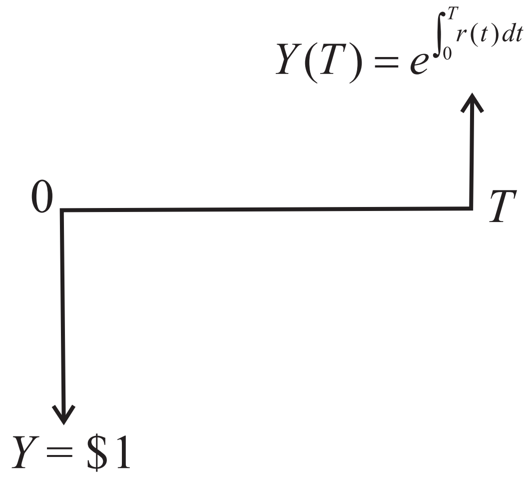
M. Ikeda, "Default premiums and quality spread differentials in a stochastic interest rate economy," *Advances in Futures and Options Research*, vol. 8 (1995) p.175-202.

Riskfree debts

fixed rate debt

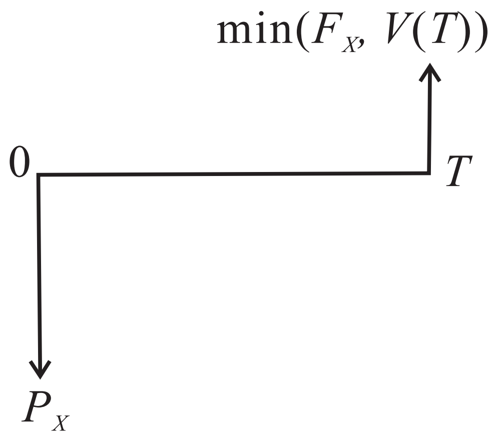


floating rate debt



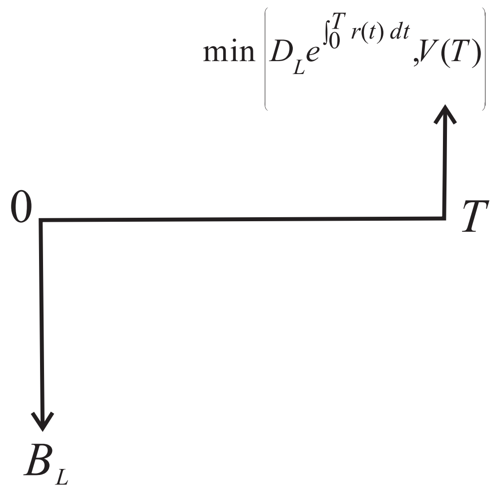
Risky debts

fixed rate debt



$B_X < PF_X$ since the debtholder bears the credit risk

floating rate debt



$B_L < D_L$ since the debtholder bears the credit risk

Interest rate dynamics

The short rate r_t follows the Vasicek model

$$dr_t = k(\theta - r_t)\,dt + \sigma_r\,dZ_r.$$

The price of the default free discount bond with unit par is given by

$$P = a(T)e^{-rb(T)}$$

where

$$\begin{aligned} b(T) &= \frac{1 - e^{-kT}}{k}, \\ a(T) &= \exp\left(-\left(\theta + \frac{\sigma\lambda}{k} - \frac{\sigma^2}{2k^2}\right)[T - b(T)]\right) - \frac{\sigma_r^2 b(T)^2}{4k}, \end{aligned}$$

λ is the market price of risk. The dynamics of P is given by

$$\frac{dP}{P} = r\,dt - \sigma_P\,dZ_r, \quad \text{where} \quad \sigma_P = b(T)\sigma_r.$$

Firm value process

Under the risk neutral measure Q

$$\frac{dV}{V} = r\,dt + \sigma_V\,dZ_V.$$

Fixed rate debt

The price of the fixed rate debt B_X of face amount F_X is the present value of the maturity payoff

$$\min(V(T), F_X) = V(T) - \max(V(T) - F_X, 0).$$

Now, $B_X = PE_{Q^T}[\min(V(T), F_X)]$, where Q^T is the forward measure. The forward measure is used since r_t appears only in the discount factor but not in the terminal payoff. Under Q^T, the variance of the firm value expressed in units of P is given by

$$\sigma_X^2 = \int_0^T [\sigma_P^2(t) + 2\rho\sigma_V\sigma_P(t) + \sigma_V^2]\, dt, \quad \text{where} \quad \rho\ dt = dZ_r\, dZ_V.$$

The variance of V/P can be obtained via the Ito lemma, using the information that V is lognormal with variance rate σ_V^2 and P is lognormal with variance rate $b(T)\sigma_r$.

Knowing that the debt value is the difference of firm value and equity value (call option with strike F_X), we obtain

$$B_X = VN(-h_X) + PF_X N(h_X - \sigma_X)$$

where

$$h_X = -\frac{\ln k_X}{\sigma_X} + \frac{\sigma_X}{2} \quad \text{and} \quad k_X = \frac{PF_X}{V}.$$

Note that B_X contains ρ and σ_X^2 is time-dependent. By making use of time-changed Brownian motion, σ_X is seen to play the same role as $\sigma\sqrt{\tau}$ in usual Black-Scholes price formula with constant σ.

Floating rate debt

A zero floating rate bond with a stochastic face amount

$$F_L(T) = D_L e^{\int_0^T r(t)\,dt} = D_L Y(T)$$

where D_L is a constant and $Y(t)$ is the money market account. The dynamics of $Y(t)$ is

$$dY(t) = r(t)Y(t)\,dt, \quad \text{and} \quad Y = 1.$$

We use $Y(t)$ as the numeraire and define $M(t) = V(t)/Y(t)$. The dynamics of M is given by

$$\frac{dM}{M} = \sigma_V\,dZ_V.$$

The terminal payoff normalized by Y is $M(T) - (M(T) - D_L)^+$. The price of the floating rate loan

$$\begin{aligned} B_L &= V - E_{Q^*}[(M(T) - D_L)^+] \\ &= VN(-h_L) + D_L N(h_L - \sigma_V\sqrt{T}) \end{aligned}$$

where Q^* is the measure with $Y(t)$ as the numeraire and

$$h_L = \frac{-\ln D_L/V}{\sigma_V\sqrt{T}} + \frac{\sigma_V\sqrt{T}}{2}.$$

Default premiums

Yield on the risky fixed rate debt

$$Y_X = \frac{1}{T} \ln \frac{F_X}{B_X}$$

and the default premium $\pi_X = Y_X - R_X$ where $R_X = -\dfrac{\ln P}{T}$, we have

$$\pi_X = -\frac{1}{T} \ln \left[N(h_X - \sigma_X) + \frac{N(-h_X)}{k_X} \right].$$

The default free yield of floating rate debt is

$$R_L = \frac{1}{T} \int_0^T r(t)\, dt.$$

The default premium $\pi_L = Y_L - R_L$, where $Y_L = \dfrac{1}{T} \ln \dfrac{F_L}{B_L}$, so that

$$\pi_L = -\frac{1}{T} \ln \left[N(h_L - \sigma_V \sqrt{T}) + \frac{N(-h_L)}{k_L} \right], \quad k_L = \frac{D_L}{V}.$$

Though R_L and F_L are stochastic but the default premium π_L is a deterministic function since the stochastic term $\dfrac{1}{T} \displaystyle\int_0^T r(t)\, dt$ is cancelled.

Examination of fixed-floating differential

A firm is assumed to have the choice to issue between fixed rate and floating rate debt to raise the same dollar amount B.

$$\begin{aligned} B &= VN(-h_X) + PF_X N(h_X - \sigma_X) \\ B &= VN(-h_L) + D_L N(h_L - \sigma_V\sqrt{T}). \end{aligned}$$

Solve for F_X^* and D_L^* such that

$$\begin{aligned} F_X^* &= \frac{B - VN(-h_X^*)}{PN(h_X^* - \sigma_X)} \quad \text{where} \quad h_X^* = \frac{\ln \frac{V}{PF_X^*}}{\sigma_X} + \frac{\sigma_X}{2} \\ D_L^* &= \frac{B - VN(-h_L^*)}{N(h_L^* - \sigma_V\sqrt{T})} \quad \text{where} \quad h_L^* = \frac{\ln \frac{V}{D_L^*}}{\sigma_V\sqrt{T}} + \frac{\sigma_V\sqrt{T}}{2}. \end{aligned}$$

The default premiums evaluated at B are

$$\pi_X|_{B_X=B} = \frac{1}{T}\ln\frac{PF_X^*}{B} \quad \text{and} \quad \pi_L|_{B_L=B} = \frac{1}{T}\ln\frac{D_L^*}{B}.$$

Fixed-floating quality differential = DIF $= \pi_X|_{B_X=B} - \pi_L|_{B_L=B}$

$$= \frac{1}{T}\ln\frac{PF_X^*}{D_L^*}.$$

The quality differential is zero if and only if

$$PF_X^* = D_L^* \quad \text{or} \quad k_X^* = k_L^*$$

where

$$\begin{aligned} k_X^* &= \frac{PF_X^*}{V} = \text{ quasi-debt ratio for fixed rate debt} \\ k_L^* &= \frac{D_L^*}{V} = \text{ quasi-debt ratio for floating rate debt.} \end{aligned}$$

The sign of DIF is the same as the sign of $k_X^* - k_L^*$.

Lemma

$$k_X^* > k_L^* \Longleftrightarrow \sigma_X^2 > \sigma_V^2 T \quad \text{and} \quad k_X^* = k_L^* \Longleftrightarrow \sigma_X^2 = \sigma_V^2 T.$$

Proposition 1

When $\rho \geq 0$, DIF > 0 since $\sigma_X^2 - \sigma_V^2 T = \int_0^T [\sigma_P^2(t) + 2\rho\sigma_V\sigma_P(t)]\,dt$ which is positive whenever $\rho \geq 0$.

Proposition 2

When $\rho < 0$, DIF is generally positive.

Financial intuition

- An increase in ρ increases the default risk of the fixed rate debt through increasing the risk adjusted volatility measure σ_X while it leaves the default risk of floating rate debt $\sigma_V\sqrt{T}$ unchanged. Hence, DIF is an increasing function of ρ.

- When the firm value is decreased, the default risk of both fixed rate and floating rate debt is increased. In normal circumstance, $F_X^*/V > D_L^*/V$ (corresponding to positive DIF). A decrease in the firm value widens the difference in quasi-debt ratios, and the difference in default premiums is also widened. Therefore, when the firm value is decreased, the default risk of the fixed rate debt increases at a faster speed than that of the floating rate debt, and so DIF is increased.

Possible extensions

1. Inter-temporal default.

2. Incorporate the information of the current yield curve by using Hull-White model instead of Vasicek model with constant parameters.

Black-Cox model (1976)

Impact of various bond indenture provisions on risky debt valuation

1. *Inter-temporal default* (safety covenants)

   If the firm value falls to a specified level, the bondholders are entitled to force the firm into bankruptcy and obtain the ownership of the assets.

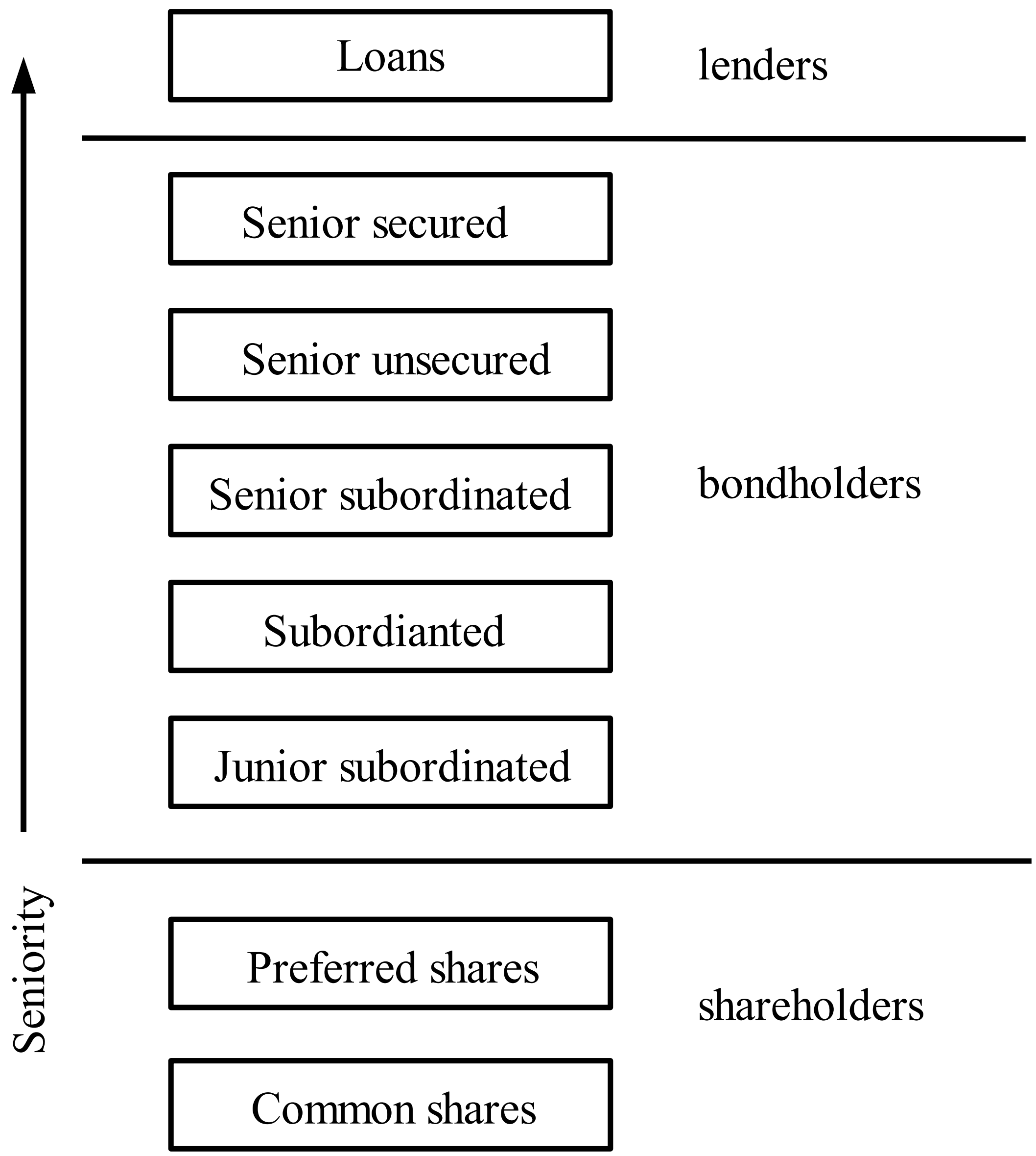
2. *Subordinated bonds*

   Payments can be made to the junior debt holders only if the full promised payment to the senior debt holders has been made.

claim	$V < P$	$P \leq V \leq P + Q$	$V > P + Q$
senior bond	V	P	P
junior bond	0	$V - P$	Q
equity	0	0	$V - P - Q$

$P =$ par value of senior bond

$Q =$ par value of junior bond



Typical capital structure of a firm arranged according to seniority

Recoveries on defaulted bonds

Recovery can refer to

- price of the bonds at the time of default
- their value at the end of the distressed-reogranization period

Average recovery rate on a sample of more than 700 defaulting bonds (1978-1995) was $41.70 per $100 face value.

Industry affiliation factor

The asset structure and regulatory environment of public utilities lead to better recovery rates than those of industries that operate in a highly competitive environment and have little tangible assets.

Seniority on recovery rates

Senior secured debt	58%
Senior unsecured debt	48%
Senior subordinate debt	34%
Junor subordinate debt	31%

Data from "Almost everything you wanted to know about recoveries on defaulted bonds," by E.I. Altman and V.M. Kishore, *Financial Analysts Journal* Nov-Dec issue (1996).

Across various industry sectors

	average	*median*	*standard deviation*
Public utilities	70.47	79.07	19.46
Casino, hotel and recreation	40.15	28.00	25.66
Lodging, hospitals and nursing facilities	26.49	16.00	22.65

Within the same industry

Chemicals, petroleum, rubber and plastic products	*no. of observations*	*average*	*weighted*
senior secured	6	75.04	89.17
senior unsecured	16	71.91	81.71
senior subordinated	7	63.07	77.81
subordinated	6	25.54	31.46

* weighted average > simple average means bond issues of larger size have higher recovery rate.

Longstaff-Schwartz model

F.A. Longstaff and E.S. Schwartz, "A simple approach to valuing risky fixed and floating rate debt," *Journal of Finance*, vol. 50(1) (1995) p.789-819.

Interest rate uncertainty

Vasicek interest rate process: $dr = (\zeta - \beta r)\, dt + \eta\, dZ_2$

Bankruptcy-triggering mechanism and payoff

Threshold value $\nu(t)$ for the firm value at which financial distress occurs; take $\nu(t) = K =$ constant. Upon default, the bondholder receives $1-w$ times the face value of the bond at maturity. Terminal payoff can be expressed as

$$1 - w\mathbf{1}_{\{\gamma \le T\}}$$

where γ is the first passage time of V hitting K. The firm value V follows

$$\frac{dV}{V} = \mu\, dt + \sigma\, dZ_1, \quad dZ_1\, dZ_2 = \rho\, dt.$$

The defaultable bond price $\overline{B}(r,t;T)$ and default free bond price $B(r,t;T)$ are related by

$$\overline{B}(r,t;T) = B(r,t;T)[1 - wQ(V,r,T)],$$

where $Q(V,r,t)$ is the probability of default of the risky debt over $[0,T]$. The solution to $B(r,t;T)$ takes the form $\exp(A(T) - B(T)r)$, where $B(T)$ is found to be $(1 - e^{-\beta T})/\beta$. Let $X = V/K$, then Q satisfies

$$\frac{\sigma^2}{2}X^2 Q_{XX} + \rho\sigma\eta X Q_{Xr} + \frac{\eta^2}{2}Q_{rr} + [r - \rho\sigma\eta B(T)]XQ_X$$
$$+[\alpha - \beta r - \eta^2 B(T)]Q_r - Q_T = 0,$$

subject to the initial condition: $Q(X,r,0) = \mathbf{1}_{\{\gamma \le T\}}$.

$Q(X,r,T)$ can be interpreted as the probability that the first passage time of $\ln X$ to zero is less than T. The initial condition can be changed to $Q(X,r,0) = 1$ after the imposition of the homogeneous Dirichlet condition (absorbing barrier condition) on $\ln X = 0$.

The joint processes of $\ln X$ and r are given by

$$\begin{aligned} d\ln X &= \left[r - \frac{\sigma^2}{2} - \rho\sigma\eta B(T-t)\right] dt + \sigma\, dZ_1 \\ dr &= [\alpha - \beta r - \eta^2 B(T-t)]\, dt + \eta\, dZ_2. \end{aligned}$$

Integrating the dynamics for r from time zero to time τ

$$\begin{aligned} r_\tau &= r\exp(-\beta\tau) + \left(\frac{\alpha}{\beta} - \frac{\eta^2}{\beta^2}\right)[1 - e^{-\beta\tau}] \\ &\quad + \frac{\eta^2}{2\beta^2}\exp(-\beta T)\left(e^{\beta\tau} - e^{-\beta\tau}\right) + \eta e^{-\beta\tau}\int_0^\tau e^{\beta s}\, dZ_2. \end{aligned}$$

Substituting in the value of r, we obtain

$$\ln X_T = \ln X + M(T,T) + \frac{\eta}{\beta}\int_0^T [1 - e^{-\beta(T-t)}]\, dZ_2 + \sigma\int_0^T dZ_1.$$

Thus, $\ln X_T$ is normally distributed with mean $\ln X + M(T,T)$ and variance $S(T)$. The last step is to determine the first passage time density $q(0,\tau|\ln X, 0)$ of $\ln X$ to zero at time τ starting from $\ln X$ at time zero. Finally

$$Q(X,r,T) = P[\gamma \le T] = \int_0^T q(0,\tau|\ln X, 0)\, d\tau.$$

First passage time for one-dimensional Markov processes

Consider a one-factor continuous Markov process ℓ_t. Define $\pi(\ell_t, t|\ell_s, s)$ as the free transition density. Further, define $g(\ell_s = \underline{\ell}, s|\ell_0, 0)$ as the probability density that the first passage time through a constant boundray $\underline{\ell}$ occurs at date-s. An implicit formula for $g(\cdot)$ in terms of $\pi(\cdot)$ is given by

$$\pi(\ell_t, t|\ell_0, 0) = \int_0^t g(\ell_s = \underline{\ell}, s|\ell_0, 0)\pi(\ell_t, t|\ell_s = \underline{\ell}, s)\, ds, \ell_t > \underline{\ell} > \ell_0. \quad (A1)$$

- In this integral equation for g, it is assumed that ℓ_t and ℓ_0 are on opposite sides of the boundary $\ell = \underline{\ell}$.
- When the process ℓ is one-factor Markov, the above equation has a very intuitive interpretation: The only way that the process can start below the boundary ($\ell_0 < \underline{\ell}$) and end up above the boundary ($\ell_t > \underline{\ell}$) is that the process, at some intermediate time s, must pass through the boundary for the first time.
- From the strong Markov property of Brownian processes, the Brownian path after s is independent of the path history before s. It only depends on the information that $\ell_s = \underline{\ell}$.

First passage time for two-dimensional Markov processes

Longstaff-Schwartz model is not one-factor Markov since the proposed firm value process is a function of the spot rate, which itself is stochastic.

Define $g[\ell_s = \underline{\ell}, r_s, s|\ell_0, r_0, 0]$ as the probability density that the first passage time is at time s, and that the random process r takes on the value r_s at that time. We claim that the two-dimensional generalization is for $\ell_0 < \underline{\ell} < \ell_t$

$$\begin{aligned}&\pi(\ell_t, r_t, t|\ell_0, r_0, 0)\\ =& \int_0^t \int_{-\infty}^{\infty} g[\ell_s = \underline{\ell}, r_s, s|\ell_0, r_0, 0]\pi(\ell_t, r_t, t|\ell_s = \underline{\ell}, r_s, s)\, dr_s ds \qquad (B1)\end{aligned}$$

Remark

At time s, the r-process takes on some value r_s. We need to integrate over dr_s from $-\infty$ to ∞.

Numerical procedure

Define

$$\begin{aligned}\psi(r_t,t) &= \int_{\underline{\ell}}^{\infty} \pi(\ell_t,r_t,t|\ell_0,r_0,0)\,d\ell_t \\ \phi(r_t,t;r_s,s) &= \int_{\underline{\ell}}^{\infty} \pi(\ell_t,r_t,t|\ell_s=\underline{\ell},r_s,s)\,d\ell_t \\ g(r_s,s) &= g(\ell_s=\underline{\ell},r_s,s|\ell_0,r_0,0).\end{aligned}$$

Integrating (B1) by $\int_{\underline{\ell}}^{\infty} d\ell_t$, we obtain

$$\psi(r_t,t) = \int_0^t \int_{-\infty}^{\infty} g(r_s,s)\phi(r_t,t;r_s,s)\,dr_s\,ds.$$

Discretized version of the above equation takes the form

$$\psi(r_t,t) = \sum_{v=1}^{j}\sum_{u=1}^{n_r} q(r_u,t_v)\phi(r_i,t_j|r_u,t_v)$$

where $q(r_u,t_v) = \Delta t\Delta r\, g(r_u,t_v)$. The probability that the first passage time is less than T is given by

$$Q^T(r_0,\ell_0,T) = \sum_{j=1}^{n_T}\sum_{i=1}^{n_r} q(r_i,t_j).$$

Briys and de Varenne model

"Valuing risky fixed debt: an extension," *Journal of Financial and Quantitative Analysis*, vol. 32, p. 239-248 (1997).

Assume the existence of a unique probability measure Q (risk neutral measure) under which the continuously discounted price of any security is a Q-martingale.

Under Q, the short rate r_t follows

$$dr_t = a(t)[b(t) - r_t]\,dt + \sigma_r(t)\,dW_t.$$

For Gaussian type interest rate models, the dynamics of the default free bond price $B(t, T)$ under Q is

$$\frac{dB}{B} = r_t\,dt - \sigma_P(t, T)\,dW_t$$

where

$$\sigma_P(t, T) = \sigma_r(t) \int_t^T \exp\left(-\int_t^u a(s)\,ds\right)\,du.$$

Under Q, the firm value process V_t follows

$$\frac{dV_t}{V_t} = r_t\,dt + \sigma_V\left[\rho\,dW_t + \sqrt{1-\rho^2}\,d\widetilde{W}_t\right],$$

where W_t and $\widetilde{W}_t$ are uncorrelated Wiener processes.

The default-trigger barrier $\overline{K}(t) = \alpha F B(t,T), 0 \le \alpha \le 1, F$ is the face value.

Define the first passage time of the process V_u through the barrier $\overline{K}(u), t \le u \le T$.

$$T_{V,\overline{K}} = \text{inf}\left\{u \ge t : V_u = \overline{K}(u) = \alpha F B(t,T)\right\}.$$

The price as of time t of the risky zero coupon bond is

$$\begin{aligned}\overline{B}(r,t;T) \;=\; & E_Q\left[\exp\left(-\int_t^T r_u\,du\right)\right.\\ & \left.\left\{f_1 \alpha F \mathbf{1}_{\{T_{V,\overline{K}}<T\}} + F\mathbf{1}_{\{T_{V,\overline{K}}\ge T, V_T \ge F\}} + f_2 V_T \mathbf{1}_{\{T_{V,\overline{K}}\ge T, V_T < F\}}\right\}\right]\end{aligned}$$

where f_1 and f_2 are recovery rates.

Advantages of the model

The term structure of corporate spreads is affected by the presence of safety covenant and the violations of the absolute priority rule.

- Larger corporate spreads than those derived by Merton's model.
- Corporate spreads will exhibit more complex structures since there are more parameters in the model.

Good analytic tractability since the default-trigger barrier normalized by $B(t,T)$ becomes the constant barrier αF. The bond pricing model becomes a barrier option model. Note that $V_t/B(r,t;T)$ follows the Geometric Brownian motion with zero drift rate.

By using the methodology of the change of numeraires and time change, it can be shown that

$$\begin{aligned}\overline{B}(r,t;T) &= FB(t,T)\left[1-P_E(\ell_t)+\frac{q_t}{\ell_t}P_E\left(\frac{q_t^2}{\ell_t}\right)-(1-f_1)\frac{1}{\ell_t}[N(-d_3)\right.\\ &\quad \left.+\ q_tN(-d_4)]-(1-f_2)\frac{1}{\ell_t}\left\{N(d_3)-N(d_5)+q_t\left[N(d_4)-N(d_6)\right]\right\}\right]\end{aligned}$$

where

$$\ell_t = \frac{FB(t,T)}{A_t}$$

$$q_t = \frac{\overline{K}(t)}{V_t} = \frac{\alpha FB(t,T)}{V_t}$$

and

$$\begin{aligned} d_1 &= \frac{-\ln \ell_t + \Sigma(t,T)^2/2}{\Sigma(t,T)} = d_2 + \Sigma(t,T) \\ d_3 &= \frac{-\ln q_t + \Sigma(t,T)^2/2}{\Sigma(t,T)} = d_4 + \Sigma(t,T) \\ d_5 &= \frac{-\ln(q_t^2/\ell_t) + \Sigma(t,T)^2/2}{\Sigma(t,T)} = d_6 + \Sigma(t,T) \\ \Sigma(t,T)^2 &= \int_t^T \left\{[\rho\sigma_V + \sigma_B(u,T)]^2 + (1-\rho^2)\sigma_V^2\right\} du \end{aligned}$$

$P_E(\ell_t)$ and $P_E(q_t^2/\ell_t)$ denote the price as of time t of two European put options of maturity T:

$$\begin{aligned} P_E(\ell_t) &= -\frac{1}{\ell_t}N(-d_1) + N(-d_2) \\ P_E\left(\frac{q_t^2}{\ell_t}\right) &= -\frac{\ell_t}{q_t^2}N(-d_5) + N(-d_6) \end{aligned}$$

Financial interpretation

The defaultable bond price formula consists of two ratios, ℓ_t and q_t.

- ℓ_t is the classical Merton's quasi-debt ratio ℓ_t. It is not equal to the true debt to asset ratio because the numerator (i.e. the face value of corporate debt) is discounted by the riskless rate. As a result, it is an upward-biased estimate of the real debt to asset ratio.

- q_t can be defined as the bankruptcy or *early default ratio*. It is simply the ratio of the current default threshold to the current value of the firm. As soon as q_t is equal to 1, bankruptcy is forced.

Bond price formula has 5 *basic components*

1. The first term corresponds to an otherwise identical riskless zero-coupon bond.

2. The second term is the usual put-to-default at maturity.

3. The thrid term, a long position on a European put, appears because of the possibility of an early default triggered by the safety covenant. As such it contributes to mitigating the effect of the previous traditional put-to-default.

4. The last two terms represent the effect of the deviations from the strict priority rule. This effect is negative to the bondholders due to the non-enforecement of the strict priority rule.

Some polar cases

- When the absolute priority rule is strictly enforced ($f_1 = f_2 = 1$). The last two terms disappear.
- In the polar case $\alpha = 1$, then $q_t = \ell_t$ and the two put options cancel out. The bondholder's situation has become riskless. As soon as the value of corporate assets reaches the present value of liabilities discounted at the risk-free rate, early bankruptcy is forced. The bondholder is then sure to receive the face value F at maturity T.

Towards dynamic capital structure: Stationary leverage ratios

"Do credit spreads reflect stationary leverage ratio?" P. Collin-Dufresne and R.S. Goldstein, *Journal of Finance*, vol. 56 (5), p.1929-1957 (2001).

Background

- Firms adjust outstanding debt levels in response to changes in firm value, thus generating mean-reverting leverage ratios.
- Develop a structural model of default with stochastic interest rates that generates stationary leverage ratios (exogenous assumption on future leverage).
- Empirical studies show the support for the existence of target leverage ratios within an industry. Theoretical dynamic models of optimal capital structure find that firm value is maximized when a firm acts to keep its leverage ratio within a certain band.

Assume that under the risk neutral measure Q,

$$\frac{dV_t}{V_t} = (r - \delta)\,dt + \sigma\,dZ_t \tag{1}$$

where δ is the payout rate. The default threshold changes dynamically over time. Let k_t denote the log-default threshold,

$$dk_t = \lambda(y_t - \nu - k_t)\,dt, \quad \text{where } y_t = \ln V_t. \tag{2}$$

- When $k_t < y_t - \nu$, the firm acts to increase k_t. That is, firms tend to issue debt when their leverage ratio falls below some target, and are most hesitant to replace maturing debt when their leverage ratio is above that target.

Define the log-leverage $\ell_t = k_t - y_t$, then

$$d\ell_t = \lambda(\bar{\ell} - \ell_t)\,dt - \sigma\,dZ_t, \tag{3}$$

where the stationary target leverage $\bar{\ell}$ is given by

$$\bar{\ell} = \frac{-r + \delta + \frac{\sigma^2}{2}}{\lambda} - \nu.$$

Define $\widetilde{\tau}$ as the random time at which $\ell(t)$ reaches zero for the first time triggering default. The risky discount bond with maturity T receives \$1 at T if $\widetilde{\tau} > T$ or $1 - w$ at time T if $\widetilde{\tau} \leq T$.

$$P^T(\ell_0) = e^{-rT} E_Q \left[\mathbf{1}_{\{\widetilde{\tau} > T\}} + (1 - w)\mathbf{1}_{\{\widetilde{\tau} < T\}} \right] = e^{-rT}[1 - wQ(\ell_0, T)],$$

where $Q(\ell_0, T)$ is the risk neutral probability that default occurs before time T given that the leverage ratio is ℓ_0 at time 0.

Constant interest rate

Define $\pi(\ell_t, t|\ell_s, s)$ as the unrestricted transition density of ℓ_t and $g(\ell_s = \underline{\ell}, \ell_0, 0)$ as the probability density that the first passage time through a constant boundary $\underline{\ell}$ occurs at date-s. Recall that

$$\pi(\ell_t, t|\ell_0, 0) = \int_0^t g(\ell_s = \underline{\ell}, s|\ell_0 = 0)\pi(\ell_t, t|\ell_s, s)\, ds, \quad \ell_t > \underline{\ell} > \ell_0.$$

From Eq. (3), ℓ_t is a Gaussian process with

$$M_t = E_Q[\ell_t|\ell_0] = \ell_0 e^{-\lambda t} + \overline{\ell}(1 - e^{-\lambda t})$$

$$E_Q[\ell_t|\ell_s = 0] = L(t-s) = \overline{\ell}[1 - e^{-\lambda(t-s)}]$$

$$\mathrm{var}_s^Q[\ell_t] = s^2(t-s) = \left(\frac{\sigma^2}{2\lambda}\right)\left[1 - e^{-\lambda(t-s)}\right].$$

With the default boundary at $\underline{\ell} = 0$, we integrate both sides with respect to ℓ_t from 0 to ∞ to obtain

$$N\left(\frac{M(t)}{s(t)}\right) = \int_0^t g(\ell_s = \underline{\ell}, s|\ell_0, 0) N\left(\frac{L(t-s)}{s(t-s)}\right) ds.$$

Discretize time into n equal intervals, and define date $t_j = jT/n \equiv j\Delta t$ for $j \in (1, 2, \cdots, n)$. The price of a risky discount bond is given by

$$P^T(\ell_0) = e^{-rT}[1 - wQ(\ell_0, T)]$$

where

$$\begin{aligned}
Q(\ell_0, t_j) &= \sum_{i=1}^{j} q_i \quad j = 2, 3, \cdots, n \\
q_1 &= \frac{N(a_1)}{N(b_{(1/2)})} \\
q_i &= \left(\frac{1}{N(b_{(1/2)})}\right)\left[N(a_i) - \sum_{j=1}^{i-1} q_j N(b_{i-j+\frac{1}{2}})\right] \quad i = 2, 3, \cdots, n \\
a_i &= \frac{M(i\Delta t)}{s(i\Delta t)} \\
b_i &= \frac{L(i\Delta t)}{s(i\Delta t)}.
\end{aligned}$$

Credit Spreads

Consider a coupon bond with promised coupon payments C at dates $t_j, j \in (1, N)$, where $t_N \equiv T$. Treating a risky coupon bond as the sum of discount risky bonds, the price of this coupon bond can be written as

$$
\begin{aligned}
P^T(\ell_0) &= \sum_{j=1}^{N} Ce^{-rt_j} E^Q \left[\mathbf{1}_{\{\widetilde{\tau} > t_j\}} + (1 - \omega_{coup}) \mathbf{1}_{\{\widetilde{\tau} < t_j\}} \right] \\
&\quad + e^{-rT} E^Q \left[\mathbf{1}_{\{\widetilde{\tau} > T\}} + (1 - \omega) \mathbf{1}_{\{\widetilde{\tau} < T\}} \right] \\
&\equiv \sum_{j=1}^{N} Ce^{-rt_j} [1 - \omega_{coup} Q(\ell_0, t_j)] + e^{-rT} [1 - \omega Q(\ell_0, T)].
\end{aligned}
$$

1. Most theoretical models of risky debt limit their investigation to discount bonds. However, the term structure of credit spreads generated by discount bonds is qualitatively different than those generated by coupon bonds.

2. In practice, claims to future coupon payments are of the lowest priority, and rarely receive any compensation in bankruptcy. We thus set $\omega_{coup} = 1$, that is, only future principal payments receive compensation in bankruptcy.

3. The yield to maturity for this coupon bond Y^T is defined implicitly through the equation

$$P_c^T(\ell_0) = e^{-Y^T T} + C \sum_{j=1}^{N} e^{-Y^T t_j}.$$

Finally, the credit spread $CS(T)$ is defined via

$$CS(T) = Y^T - r.$$

Performance of the model

The traditional model predicts for all maturities counter-intuitive low credit spreads for low leverage firms and high credit spreads for speculative grade debt.

In contrast, modeling the leverage ratio dynamics as mean reverting improves the predictions of structural models.

The intuition is straightforward.

1. With constant default boundary, a low leverage firm almost never defaults and a high leverage firm almost certainly defaults over a short period of time.

2. With mean reversion in leverage ratios, default depends on both the initial leverage ratio and the long-term mean. If the latter is assumed to take on "moderate" values for all firms, then default probabilities will be less variable across firms.