2.5 Debt negotiation models and optimal capital structure

- The option-based approach gives the prices of equity and bonds in a given capital structure. Instead of the static choice of capital structure, we would like to analyze some possible choices of dynamic capital structure. For example
  - maturity choice of debts
  - strategic models of negotiation between equity and debt holders.
- To analyze dynamic capital structure choice, we introduce bankruptcy costs and tax advantage to issuance of debts.
  - tax advantage favors higher debt levels
  - bankruptcy costs push the reverse (costs lost to third parties, like lawyers, accountants, courts, etc.)

Tax benefits and bankruptcy costs are the two competing factors.

Strategic debt negotiation

Merton's model: $\overline{B}(V,T) = \min(F,V)$, where $F =$ face value, $V =$ firm asset value.

Upon default of the debt contract, the debtholders use the bankruptcy courts to seize the assets of the firm instantly and costlessly which they then operate with no loss of value.

Real situations

1. Bankruptcies are costly because of the direct costs and the disruptions of the firm's activities.
2. Bankruptcy procedures give considerable scope for *opportunistic behaviors* by the various parties involved.
3. Deviations from the absolute priority of claims are common.

- Costly liquidations represent possible sources of inefficiency.
- Empirical studies on the Mertonian models show that the model systematically over-estimate the observed prices of corporate bonds. To reconcile the models, one has to assume implausibly higher levels for σ_V.
  - $\star$ Even with stochastic interest rates, yield spreads are found to be relatively insensitive to σ_r.
  - $\star$ *Bankruptcy is optional*: shareholder pays less than the contractual debt service but an amount which is sufficient to persuade the bondholders not to liquidate the firm.

Strategic default

"Risky debt, bond convenants and optimal capital structure," H. E. Leland, *Journal of Finance*, vol. 49 (1994), p. 1213-1252.

Endogenous bankruptcies – firm's management chooses the bankruptcy level V_B. This gives the management the option to postpone bankruptcy and pay the coupon from issuance of new equity if they consider the upside in the firm attractive enough.

Model setup

The debt is assumed to have an *infinite* maturity, with an aggregate coupon of C, and a market price of $\overline{B}$. Debt generates a *tax benefit* of $\tau^* C\,dt$, τ^* is the corporate tax rate.

- Default occurs at a barrier V_B of V. At default, debt recovers $(1-\alpha)V_B$ and there are bankruptcy costs of αV_B. This constitutes a disadvantage of debt which must balance the tax advantage of debt in the optimum.

- For every time step dt, a coupon of $C\,dt$ is paid to the debtholder. This is financed by issuance of new equity, while the dynamics of V remains unchanged:

$$dV = rV\,dt + \sigma V\,dZ.$$

Pricing model

$$\frac{\sigma^2}{2} V^2 \frac{\partial^2 \overline{B}}{\partial V^2} + rV \frac{\partial \overline{B}}{\partial V} - r\overline{B} + C = 0$$

with boundary conditions: $\overline{B} \to C/r$ as $V \to \infty$

$$\overline{B} = (1-\alpha)V_B \text{ at } V = V_B.$$

- Present value of receiving \$1 at default $= E[e^{-r\tau_B}] = \left(\frac{V}{V_B}\right)^{-2r/\sigma^2}$,

  where τ_B is the (random) first passage time that the firm value process reaches V_B. This solution satisfies the differential equation, together with the boundary value of 1 at $V = V_B$ and 0 at $V \to \infty$.

- Present value of flow of C received up until time $\tau_B = \frac{C}{r}\left[1 - \left(\frac{V}{V_B}\right)^{-2r/\sigma^2}\right]$

  This result is obvious since the sum of this flow plus the contingent claims of receiving $\frac{C}{r}$ upon reaching V_B is the same as the default free consol bond $\left(\text{whose value is } \frac{C}{r}\right)$.

- Value of tax benefit $= \frac{\tau^* C}{r}\left[1 - \left(\frac{V}{V_B}\right)^{-2r/\sigma^2}\right]$.

- Value of bankruptcy cost $= \alpha V_B \left(\frac{V}{V_B}\right)^{-2r/\sigma^2}$.

Solution

$$\overline{B} = \left[1 - \left(\frac{V}{V_B}\right)^{-2r/\sigma^2}\right]\frac{C}{r} + \left(\frac{V}{V_B}\right)^{-2r/\sigma^2}(1-\alpha)V_B$$

$$\begin{aligned} & \text{value of equity} + \text{value of debt} \\ = \; & \text{value of assets} + \text{value of tax break} - \text{value of bankruptcy costs} \end{aligned}$$

so that

$$\begin{aligned} E &= \text{value of equity} \\ &= V - (1-\tau^*)\frac{C}{r} + \left[\frac{(1-\tau^*)C}{r} - V_B\right]\left(\frac{V}{V_B}\right)^{-2r/\sigma^2}. \end{aligned}$$

Note that the value of equity equals zero when $V = V_B$.

When V is sufficiently low, E may become negative. We find the default barrier V_B by using the optimality condition: $\frac{d}{dV}E|_{V=V_B} = 0$. This gives

$$V_B(C) = \frac{\frac{2r}{\sigma^2}(1-\tau^*)C}{r\left(1+\frac{2r}{\sigma^2}\right)}.$$

Note that $V_B(C)$ is linearly increasing in C and decreases as τ^* increases. Also, $V_B(C)$ is independent of α since equity equals zero upon bankruptcy.

Recent extension of this optimal capital structure model to stochastic interest rate:

"A model of optimal capital structure with stochastic interest rate," by Jing-Zhi Huang (Smeal School of Business), Nengjiu Ju (Smith School of Business) and Hui Ou-Yang (Fuqua School of Business) (Feb. 2003).

Assumption 1

Assume that bankruptcy occurs when the value of the firm falls below a constant level V_B. If $V_t > V_B$, the firm is solvent and pays the contractual coupon rate to its debt holders. In the event of bankruptcy, bond holders will receive ϕV_B with $\phi \in [0, 1)$ and equity holders get nothing.

Assumption 2

We consider a stationary debt structure where a firm continuously sells a constant (principal) amount of new debt with a maturity of m years to replace the same amount of principal of retiring debt. New bond principal and coupon are issued at rates $p = P/m$ and $c = C/m$ per year, where P and C are the total principal and total coupon rates of all outstanding bonds, respectively.

Define the first passage time τ as $\tau = \min\{t : V_t \leq V_B\}$, which is the first time that the firm value V_t hits V_B in some $\omega \in \Omega$ under Q^T. Denote by $F(T)$ the cumulative distribution function of τ under Q^T. Here, Q^T denotes the T-forward risk neutral measure. In their model, V_B is exogenously pre-specified.

Under Q^T, the firm value and the interest rate processes are given by

$$\begin{aligned} \frac{dV_t}{V_t} &= [r - \delta - \rho\sigma_v\sigma_r B(T-t)]\,dt + \sigma_v\,dw_{1t}^{Q^T}, \\ dr &= [\alpha - \beta r - \sigma_r^2 B(T-t)]\,dt + \sigma_r\,dw_{2t}^{Q^T}. \end{aligned}$$

Consider a bond that pays a coupon rate c, has a principal value p, and matures at time t. The payment rate $g(s)$ to the debt holders at any time s is equal to

$$g(s) = c\mathbf{1}_{\{s\le t\}}\mathbf{1}_{\{s\le\tau\}} + p\delta(s-t)\mathbf{1}_{\{s\le\tau\}} + \phi(t)V_B\delta(s-\tau)\mathbf{1}_{\{s\le t\}},$$

where $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function and $\delta(\cdot)$ is the Dirac delta function. Note that $g(s)$ is random because τ is random. $\phi(t)$ is the fraction of the asset value V_B, that the maturity-t bondholders receive in bankruptcy.

Under the risk-neutral measure Q, the value of the debt at time zero is given by

$$d(V; V_B, t) = \int_0^\infty E^Q \left[e^{-\int_0^s r(u)\,du} g(s) \right] ds = \int_0^\infty \Lambda(s) E^{Q^s}[g(s)]\, ds,$$

where

$$\Lambda(s) = E^Q \left[e^{-\int_0^s r(u)\,du} \right].$$

The term inside the square brackets represents the discounted cash flow received during time interval ds. Taking expectation under Q represents the present value of the cash flow, and integrating it gives rise to the present value of the debt. The last step results from the s-forward risk-neutral measure Q^s.

Evaluating $E^{Q^s}[g(s)]$, we have

$$\begin{aligned} E^{Q^s}[g(s)] &= E^{Q^s}[c\mathbf{1}_{\{s\le t\}}\mathbf{1}_{\{s\le\tau\}}] + E^{Q^s}[p\delta(s-t)\mathbf{1}_{\{s\le\tau\}}] \\ &\quad + E^{Q^s}[\phi(t)V_B\delta(s-\tau)\mathbf{1}_{\{s\le t\}}] \\ &= c\mathbf{1}_{\{s\le t\}}[1-F(s)] + p\delta(s-t)[1-F(s)] \\ &\quad + \phi(t)V_B\mathbf{1}_{\{s\le t\}}f(s). \end{aligned}$$

Note that $F(s) = E^{Q^s}[\mathbf{1}_{\{s\ge\tau\}}]$ and

$$f(s) = E^{Q^s}[\delta(s-\tau)] = \int_0^\infty f(\tau)\delta(s-\tau)\,d\tau$$

are the distribution function and density function of τ under the s-forward risk-neutral measure, respectively.

$$
\begin{aligned}
d(V;V_B,t) &= \int_0^\infty \Lambda(s) c \mathbf{1}_{\{s\le t\}}[1-F(s)]\,ds \\
&\quad + \int_0^\infty \Lambda(s) p\delta(s-t)[1-F(s)]\,ds \\
&\quad + \int_0^\infty \Lambda(s)\phi(t)V_B \mathbf{1}_{\{s\le t\}} f(s)\,ds \\
&= c\int_0^t \Lambda(s)[1-F(s)]\,ds \\
&\quad + \Lambda(t)p[1-F(t)] + \phi(t)V_B \int_0^t \Lambda(s) f(s)\,ds \\
&= \frac{C}{m}\int_0^t \Lambda(s)[1-F(s)]\,ds + \frac{P}{m}\{\Lambda(t)[1-F(t)]\} \\
&\quad + \frac{\phi V_B}{m}\left[\Lambda(t)F(t) - \int_0^t \Lambda'(s)F(s)\,ds\right],
\end{aligned}
$$

where $\phi(t) = \phi/m, c = C/m$ and $p = P/m$ have been substituted.

Assuming that the newly issued debt (at time 0) is priced at par, i.e., $d(V;V_B,m) = P/m$, the coupon rate C can then be solved in terms of P by

$$
C = \frac{P\{1-\Lambda(m)[1-F(m)]\} - \phi V_B[\Lambda(m)F(m) - \int_0^m \Lambda'(s)F(s)\,ds]}{\int_0^m \Lambda(s)[1-F(s)]\,ds}.
$$

Integrating $d(V, V_B, t)$ from 0 to m, we obtain the total value of all outstanding debts:

$$\begin{aligned} D(V) &= \frac{C}{m}\int_0^m \left(\int_0^t \Lambda(s)[1-F(s)]\,ds\right) dt + \frac{P}{m}\int_0^m \Lambda(t)[1-F(t)]\,dt \\ &+ \frac{\phi V_B}{m}\int_0^m \Lambda(t)F(t)\,dt - \frac{\phi V_B}{m}\int_0^m \left(\int_0^t \Lambda'(s)F(s)\,ds\right) dt, \end{aligned}$$

which can be simplified to

$$\begin{aligned} D(V) &= \frac{C}{m}\int_0^m \Lambda(t)[1-F(t)](m-t)\,dt + \frac{P}{m}\int_0^m \Lambda(t)[1-F(t)]\,dt \\ &+ \frac{\phi V_B}{m}\int_0^m [\Lambda(t)F(t) - \Lambda'(t)F(t)(m-t)]\,dt. \end{aligned}$$

The tax shield accumulation rate is given by $\theta C \mathbf{1}_{\{s<\tau\}}$, with θ being the corporate tax rate.

The value of the tax shields is given by

$$
\begin{aligned}
TB(V) &= \int_0^\infty E^Q\left[e^{-\int_0^s r(u)\,du}\theta C\mathbf{1}_{\{s\le\tau\}}\right] ds \\
&= \theta C\int_0^\infty \Lambda(s)E^{Q^s}[\mathbf{1}_{\{s\le\tau\}}]\,ds \\
&= \theta C\int_0^\infty \Lambda(s)[1-F(s)]\,ds.
\end{aligned}
$$

Similarly, the rate of bankruptcy costs is given by $(1-\phi)V_B\delta(s-\tau)$, and the value of bankruptcy costs is equal to

$$
\begin{aligned}
BC(V) &= \int_0^\infty E^Q\left[e^{-\int_0^s r(u)\,du}(1-\phi)V_B\delta(s-\tau)\right] ds \\
&= (1-\phi)V_B\int_0^\infty \Lambda(s)f(s)\,ds \\
&= -(1-\phi)V_B\int_0^\infty \Lambda'(s)F(s)\,ds.
\end{aligned}
$$

Finally, the total firm value consists of three terms: the firms unlevered asset value, plus the value of tax shields, less the value of bankruptcy costs:

$$\begin{aligned} v(V) &= V + TB(V) - BC(V) \\ &= V + \theta C \int_0^\infty \Lambda(s)[1 - F(s)]\, ds \\ &\quad + (1-\phi)V_B \int_0^\infty \Lambda'(s)F(s)\, ds. \end{aligned}$$

The total principal P shall be determined by maximizing $v(V)$. Substituting C in terms of P into the above expression for $v(V)$, we arrive at an unconstrained, univariate maximization problem.

Claimed contributions

- Combine two strands of the literature: debt valuation model with stochastic interest rate in the absence of optimal capital structure, and the optimal capital structure models in the absence of stochastic interest rate.

- Current level of the interest rate is critical in the pricing of risky bond.

- Long-run mean plays a key role in the determination of a firm's optimal structure, like optimal coupon rate and leverage ratio.

- ρ_{rV} has little impact on the firm's optimal capital structure.

"Design and valuation of debt contracts," by R.W. Anderson and S. Sundaresan, *Review of Financial Studies*, vol. 9, p.37–68 (1996).

- The paper combines a fully dynamic contingent claims analysis with more realistic assumptions about financial distress and contract renegotiation.

Set up of the model

- One owner-manager who has "access" to a technology and is endowed with a technology-specific human capital. His human capital is imbedded in a project which, if undertaken, will give rise to a stream of rents indefinitely into the future.

- The project requires financing in the amount of capital D (arranged through the issue of a debt contract). We assume there is a single homogeneous group of creditors. The terms of the contract call for a contracted payment of C_t up to maturity date T.

The ongoing project is represented as a stochastic process V_t, which is the present value of current and all future cash flows.

- Once underway, the control of the project can be transferred only at a cost.
  - direct form: legal costs
  - indirect form: loss of project-specific human capital

These costs are all summarized into a constant liquidation cost K so that the collateral value equals $V - K$.

Game

All cash flows from the project are paid out in the form of dividends, debt service or to cover bankruptcy cost.

$$\begin{aligned} & \text{value of financial claims (value of the levered firm)} \\ = \ & \text{equity value} + \text{debt value} \\ = \ & \text{asset value of the firm} \\ & - \text{present expected value of the future bankrupcy costs} \end{aligned}$$

- While the project is ongoing, there is a realization of the cash flow f_t at time t. Cash flows are assumed to be proportional to the value of the project; $f_t = \beta V_t$, where β is the payout ratio.

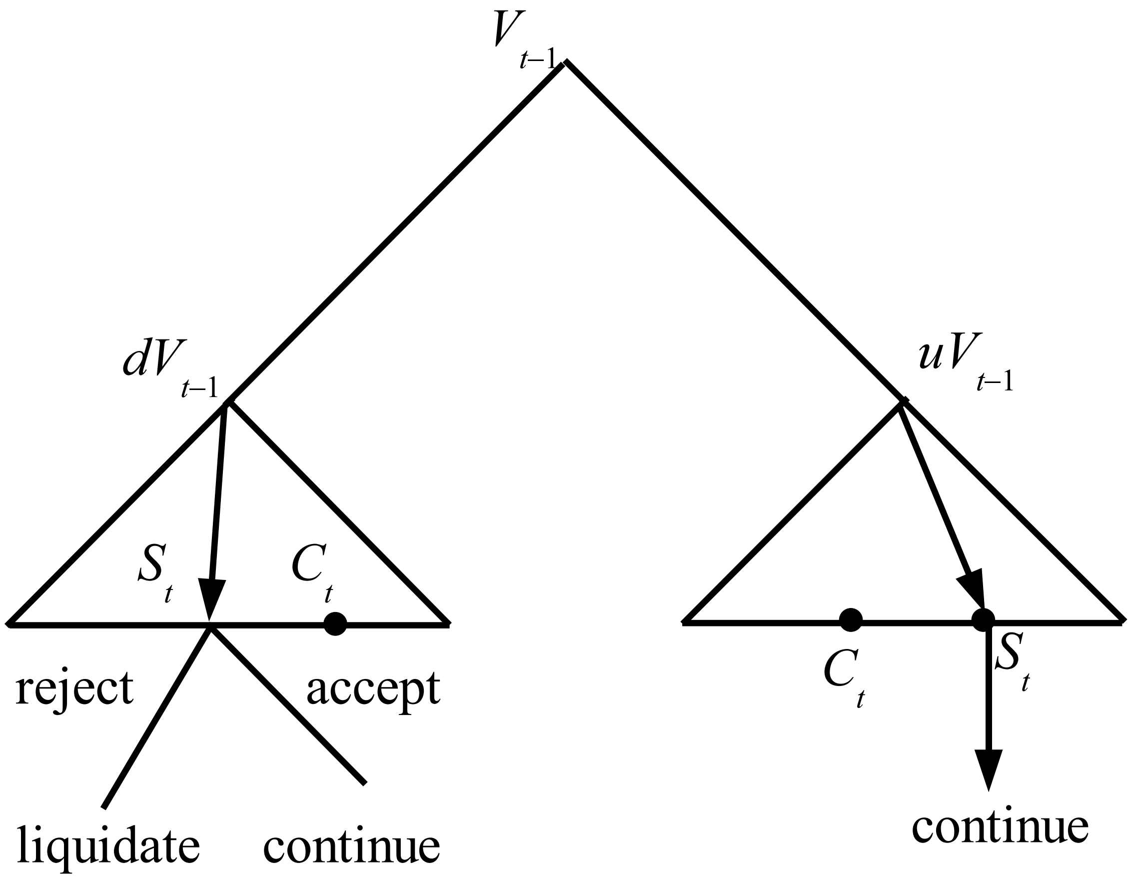
- The owner chooses a level of debt service

$$S_t \in [0, f_t] \quad (S_t \text{ is bounded by available cash flow})$$

  ⋆ If $S_t \geq C_t$ (contracted amount), the game continues to the next period.

  ⋆ If $S_t < C_t$, the creditor either

  (i) accepting the service (not initiating legal action)
    – the project continues
  (ii) rejecting the service
    – the project is liquidated.



- The owner has the right to underperform his debt contract, even when the health of the project would enable him to fully meet his obligations.

- The creditor needs not force bankruptcy when it is not in his interest to do so.

Assume that no new securities are issued to service existing debt or sale of asset.

- $\star$ Loan indentures do forbid the issue of additional debt with priority equal or superior.
- $\star$ Restrict asset liquidation since this might be exploited by management as a means of extracting value at the expense of undermining collateral values.

Terminal payoff

S_T: debt service selected by owner at time T.

V_T: asset value at time T

Payoffs to the debtholder and the owner are, respectively,

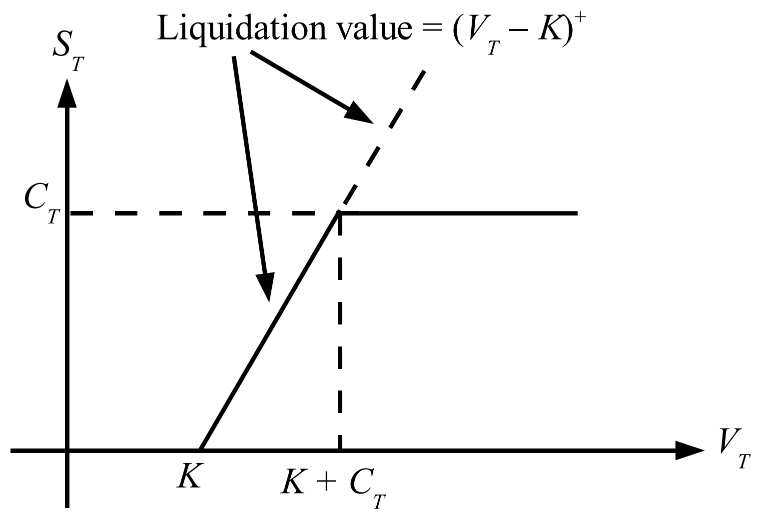
(i) $(S_T, V_T - S_T)$ if debt service fulfils the contract or the creditors accept

(ii) $(\max(V_T - K, 0), 0)$ if the debt service is rejected by creditor

Given under performance, $S_T < C_T$ [C_T is the contracted amount], the creditor should accept if $S_T \geq \max(V_T - K, 0)$ and to reject otherwise.

For the owner, he sets

$$\begin{cases} S_T = C_T & \text{if } V_T - K > C_T, \\ S_T = \max(V_T - K, 0) & \text{if otherwise.} \end{cases}$$



debt: $B(V_T) = \min(C_T, \max(V_T - K, 0))$

equity: $E(V_T) = V_T - B(V_T)$

Summary of strategies for the owner:

1. If the firm value is relatively high, liquidation value exceeds the contracted debt service, the owner is better off by simply honoring the contract.
2. For relatively low values of the firm, the owner is best off by making the minimum debt service which leaves the creditor indifferent between accepting or liquidating the firm.

Remark

When the contracted payment is less than the firm value but greater than its liquidation value, the owner underperforms the debt contract, but by an amount that is insufficient to provoke the creditor to take legal action.

(i) $S_t \geq C_t$, the game continues

(ii) $S_t < C_t$ (under performance) creditor accepts either liquidation value $= \max(V_t - K, 0)$

or the sum of debt service and continuation value

$$S_t + \overbrace{\frac{pB(uV_t) + (1-p)B(dV_t)}{R}}^{\text{continuation value}}; R = 1 + r$$

If liquidation does not occur, then the level of debt service is

$$\begin{aligned} & S(V_t) \\ = \ & \min\left(C_t, \left(0, \max(V_t - K, 0) - \frac{pB(uV_t) + (1-p)B(dV_t)}{R}\right)^+\right) \end{aligned}$$

Summary

If liq value $>$ cont value, then choose S_t such that $S_t =$ liq value $-$ cont value; else if liq value $<$ cont value, $S_t = 0$.

The value of debt

$$B(V_t) = S(V_t) + \frac{pB(uV_t) + (1-p)B(dV_t)}{R},$$

and the corresponding value of equity is

$$E(V_t) = f_t - S(V_t) + \frac{pE(uV_t) + (1-p)E(dV_t)}{R}.$$

Stopping rule

In some states, cash flows will be insufficient to pay an amount acceptable to the creditor. Forced liquidation occurs if $S(V_t) > f_t$; the value of debt then becomes

$$B(V_t) = \max(0, \min(V_t - K, C_t + P_t))$$

where P_t is the principal of the loan outstanding at t. The value of equity is

$$E(V_t) = V_t - K - B(V_t).$$

Note that costly forced liquidation can occur since asset sales are not allowed in the model.

Remark

1. The states in which the contract default occurs have been determined endogenously from the primitives of the model, unlike traditional contingent claim models [prescribed barrier versus free boundary].

2. Control of the project is transferred to the creditors only in a subset of default stated where the firm is illiquid.

3. Outcome of a negotiation process – a deviation from absolute priority in *favor of equity*. The size of the deviation depends on the health of the asset in place and costliness of liquidation.

4. Strategic considerations in the bankruptcy decision only enter into play for strictly positive bankruptcy costs ($K > 0$). The present model reduces to the Mertonian mode (in discrete version) when $K = 0$ and coupon rate $= 0$. In this case, $S_t = 0$ for $t < T$ and $B(V_T) = \text{min}(\text{par}, V_T)$.

Continuous model [“Strategic analysis of contingent claims,” by R.W. Anderson, S. Sundaresan and P. Tychon, *European Economic Review*, vol. 40 (1996) p.871-881.]

Let $S^*(V,t)$ denote the service flow rate

$B(V,\tau)$ be the bond value

Governing equation:

$$\frac{\sigma^2}{2}V^2\frac{\partial^2 B}{\partial V^2}+(r-\beta)\frac{\partial B}{\partial V}-\frac{\partial B}{\partial \tau}-rB+S^*(V,t)=0$$

$\beta =$ dividend yield of the firm

How to choose the appropriate function for the service flow S^* and specify the boundary conditions? How to determine the free boundary that separates the liquidation region and continuation region?

Discount debt

Time to maturity = 2.00 years, $V = 1, \beta = 0.00, c = 0.00, d = Pe^{-rT}/V_0$

σ^2	d	Risk, premium, $R - r(\%)$		
		$K = 0$	$K = 1$	$K = 0.2$
0.030	0.200	0.000	0.000	0.000
0.030	0.500	0.012	0.106	0.513
0.030	1.000	5.129	8.396	12.734
0.030	1.500	20.578	25.556	31.429
0.100	0.200	0.002	0.050	0.378
0.100	0.500	0.817	2.031	4.138
0.100	1.000	9.738	13.365	17.779
0.100	1.500	23.022	27.581	32.845
0.200	0.200	0.118	0.796	2.702
0.200	0.500	3.087	5.682	9.358
0.200	1.000	14.266	18.404	23.310
0.200	1.500	26.601	31.232	36.345

The effect of **positive bankruptcy costs** is indicated in the table by the yield spreads calculated under the assumption that $K = 0.1, 0.2$. For example, for a firm with $\sigma^2 = 0.03$ and $d = 0.5$, 2-year debt requires an insignificant spread (about 1 basis point) if liquidation is costless, whereas it carries spreads of 10 basis points and 51 basis points, respectively, when **liquidation costs are 10 percent and 20 percent of value**.

The amount that bankruptcy costs increase the yield spreads depends systematically on the degree of leverage and the volatility of the underlying asset.

Optimal debt design – choose the contractual features so as to minimize the inefficiencies that emerge in the associated equilibrium.

Let T be debt maturity, P be the par and τ be the tax rate. We solve the following problem:

$$\max_{C,T,P} E(V_0; \sigma^2, \beta, r, K, \tau) \text{ such that}$$
$$D \leq B(V_0, \sigma, \beta, r, K, \tau).$$

Maximize the value of equity subject to the constraint that the value of the debt be at least equal to the funding requirement.

Optimal design of debt contracts

$V = 1, \tau = 0.10, r = 1.05$

Debt ratio	β	*Coupon (%)*	*Equity*
Time to maturity = 5 years			
Liquidation cost = 0.1			
0.250	0.040	3.69	0.66470
0.250	0.060	5.25	0.65648
0.250	0.080	7.69	0.64992
0.250	0.100	7.80	0.64380
0.500	0.040	2.58	0.42695
0.500	0.060	4.44	0.41670
0.500	0.080	5.44	0.41174
0.500	0.100	5.60	0.40583
Liquidation cost = 0.2			
0.250	0.040	3.71	0.61797
0.250	0.060	5.67	0.60799
0.250	0.080	7.71	0.60086
0.250	0.100	9.67	0.59649
0.500	0.040	3.00	0.39859
0.500	0.060	0.43	0.38694
0.500	0.080	5.43	0.38262
0.500	0.100	7.36	0.38149

Summary of results

1. Tax shield and probability of liquidation
   - raising coupon interest brings tax benefits but its effect on default is ambiguous.
2. Firm with the higher leverage selects a lower level of contractual interest. Higher leverage tends to make forced liquidation in advance of maturity more likely.
3. Some of the burden of realized liquidation costs falls on equity.
4. Equity is higher for $T = 10$ than for $T = 5$ in most cases. Firms prefer longer terms debts.