2.6 Default correlations using structural approach

- ⋆ Estimation of probabilities of multiple defaults is an important task in credit analysis and risk management of bond portfolios.
- ⋆ The use of historical default data to estimate default correlations cannot capture *any firm specific information*. As defaults are rare events, historical statistics are generally very inaccurate due to lack of reliable time series data.
- ⋆ Default correlations are time varying, so past history may not reflect the current reality.

Reference

"An analysis of default correlations and multiple defaults," by Chun-sheng Zhou, *Review of Financial Studies*, vol. (14) (2001) pp. 555-576.

– develops an analytical formula for calculating default correlations using the first-passage-time approach.

First-passage-time model

Let V_1 and V_s denote the total asset values of Firm 1 and Firm 2. The dynamics of V_1 and V_s are given by

$$\begin{pmatrix} d\ln V_1 \\ d\ln V_2 \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} dt + \Omega \begin{pmatrix} dZ_1 \\ dZ_2 \end{pmatrix}, \qquad \begin{array}{l} dZ_1 \& dZ_2 \text{are independent} \\ \text{Wiener processes} \end{array}$$

where

$$\Omega = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \quad \text{such that} \quad \Omega\Omega^T = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}.$$

Here, Ω is the Cholesky decomposition of the covariance matrix, and ρ is the correlation coefficient between $d\ln V_1$ & $d\ln V_2$. To understand the above decomposition, we recall that

$$\Omega \begin{pmatrix} dZ_1 \\ dZ_2 \end{pmatrix} \left(\Omega \begin{pmatrix} dZ_1 \\ dZ_2 \end{pmatrix} \right)^T = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} dt$$

and

$$\begin{pmatrix} dZ_1 \\ dZ_2 \end{pmatrix} (dZ_1 \ dZ_2) = I\, dt.$$

Financial assumptions on default

The default of a firm is triggered by the decline in its firm value. If $V_i(t) \leq e^{\lambda_i t} K_i$ then Firm i defaults on all of its obligations.

Mathematical assumption: $\lambda_i = \mu_i$. As remarked by the author, this assumption makes possible to remove the drift term from $\ln[e^{-\lambda_i t} V_i(t)]$. Define $\tau_i = \inf_{t \geq 0}\{t | e^{-\lambda_i t} V_{i,t} \leq K_i\}$, then $D_i(t) = \{\tau_i \leq t\}$ is the event that Firm i defaults before some time $t > 0, i = 1, 2$.

$$P[D_i(t)] = P[\tau_i \leq t] = 2N\left(-\frac{\ln V_{i,0}/K_i}{\sigma_i \sqrt{t}}\right).$$

Write $Z_i = \dfrac{\ln V_{i,0}/K_i}{\sigma_i}, i = 1, 2$, then $P[D_i(t)] = 2N\left(-\dfrac{Z_i}{\sqrt{t}}\right)$. There is an one-to-one correspondence between Z_i and $P[D_i(t)]$. Recall that

$$D_i(t) = \{\tau_i \leq t\} = \{H_t^i = 1\} \text{ where } H_t^i = \mathbf{1}_{\{\tau_i < t\}}, i = 1, 2.$$

The default correlation is defined as Pearson's correlation between the random variables H_t^1 and H_t^2.

The default correlation between Firm 1 and Firm 2 over period $[0, t]$ is

$$\begin{aligned}\rho_D(t) &= \text{corr}(D_1(t), D_2(t)) \\ &= \frac{P[D_1(t)D_2(t)] - P[D_1(t)]P[D_2(t)]}{\sqrt{P[D_1(t)][1 - P[D_1(t)]]}\sqrt{P[D_2(t)][1 - P[D_2(t)]]}}.\end{aligned}$$

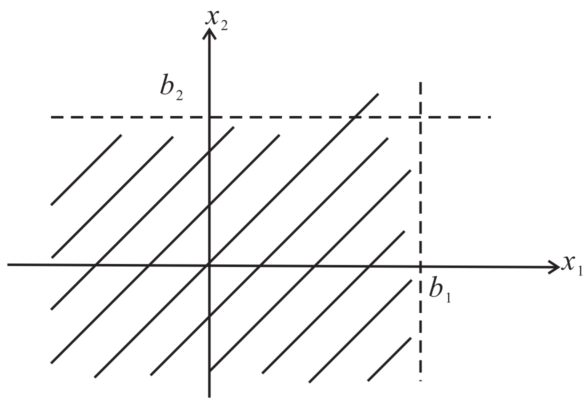
Since $P(P_1 \cap D_2) = P(D_1) + P(D_2) - P(D_1 \cup D_2)$, therefore we need to find $P(D_1 \cup D_2)$, that is, the probability that at least one default has occurred by time t.

Define $X_1(t) = -\ln[e^{-\lambda_1 t} V_1(t)/V_1(0)]$

$$X_2(t) = -\ln[e^{-\lambda_2 t} V_2(t)/V_2(0)]$$

$$b_1 = -\ln[K_1/V_1(0)] = \ln[V_1(0)/K_1]$$

$$b_2 = -\ln[K_2/V_2(0)] = \ln[V_2(0)/K_2].$$



- It is straightforward to verify that $[X_1(t), X_2(t)]$ follows a two dimensional Brownian motion:

$$\begin{pmatrix} dX_1 \\ dX_2 \end{pmatrix} = -\Omega \begin{pmatrix} dz_1 \\ dz_2 \end{pmatrix}.$$

  After this transformation, finding $P(\tau \leq t)$ is equivalent to finding the first passage time of the two dimensional Brownian motion $[X_1, X_2]'$ with initial condition $[X_1(0), X_2(0)]' = [0, 0]'$ to a boundary consisting of two intersecting lines $X_1 = b_1$ and $X_2 = b_2$.

  Let $f(x_1, x_2, t)$ be the transition probability density of the particle in the region $\{(x_1, x_1) | x_1 < b_1 \text{ and } x_2 < b_2\}$, i.e., the probability density that $[X_1(t), X_2(t)]' = [x_1, x_2]'$ and that the particle does not reach the barrier $\partial(b_1, b_2)$ in time interval $(0, t)$.

Consider

$$P\left[\max_{0\le s\le t} X_1(s) < b_1 \text{ and } \max_{0\le s\le t} X_2(s) < b_2, X_1(t) < y_1 \text{ and } X_2(t) < y_2\right]$$
$$= \int_{-\infty}^{y_1}\int_{-\infty}^{y_2} f(x_1, x_2, t)\, dx_1 dx_2 := F(y_1, y_2, t).$$

Thus $F(b_1, b_2, t)$ is the probability that absorption has not yet occurred by time t, i.e.,

$$F(b_1, b_2, t) = P(\tau > t) = 1 - P(\tau \le t).$$

According to $P(D_1 \cup D_2) = P(\tau \le t)$, to estimate the joint probability of Firms 1 and 2 not defaulting as well as the default correlation between the two firms, we only need to calculate $F(b_1, b_2, t)$.

Main Result 1 The probability that no firm has defaulted by time t is given by

$$\begin{aligned} F(b_1, b_2, t) &= 1 - P(D_1 \cup D_2) \\ &= \int_{-\infty}^{b_1} \int_{-\infty}^{b_2} f(x_1, x_2, t)\, dx_1 dx_2 \\ &= \frac{2r_0}{\sqrt{2\pi t}} e^{-\frac{r_0^2}{4t}} \sum_{n=1,3,5,\cdots} \frac{1}{n} \cdot \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \\ &\quad \left[I_{\frac{1}{2}(\frac{n\pi}{\alpha}+1)}\left(\frac{r_0^2}{4t}\right) + I_{\frac{1}{2}(\frac{n\pi}{\alpha}-1)}\left(\frac{r_0^2}{4t}\right) \right] \end{aligned}$$

where $I_\nu(z)$ is the modified Bessel function I with order ν and

$$\theta_0 = \begin{cases} \tan^{-1}\left(\frac{b_2\sigma_1\sqrt{1-\rho^2}}{b_1\sigma_2-\rho b_2\sigma_1}\right) & \text{if } (\cdot) > 0 \\ \pi + \tan^{-1}\left(\frac{b_2\sigma_1\sqrt{1-\rho^2}}{b_1\sigma_2-\rho b_2\sigma_1}\right) & \text{otherwise,} \end{cases} \qquad r_0 = \frac{b_2}{\sigma_2 \sin\theta_0},$$

$$\alpha = \begin{cases} \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{if } \rho < 0 \\ \pi + \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{otherwise,} \end{cases}$$

Remark Adjustment not required if $ATAN2(x, y)$ function is used in the Fortran code.

It is obvious that the probability $F(b_1, b_2, t)$ at any given time horizon t is solely determined by *standardized distances to default* $Z_i = b_i/\sigma_i$ *and* ρ. One can easily verify that

$$\theta_0 = \begin{cases} \tan^{-1}\left(\frac{Z_2\sqrt{1-\rho^2}}{Z_1-\rho Z_2}\right) & \text{if } (\cdot) > 0 \\ \pi + \tan^{-1}\left(\frac{Z_2\sqrt{1-\rho^2}}{Z_1-\rho Z_2}\right) & \text{otherwise,} \end{cases}$$

$$r_0 = Z_2/\sin\theta_0.$$

Remark One should distinguish between the default correlation coefficient $\rho_{ij}^D(t)$ defined above, and the (linear) correlation coefficient ρ^{τ_i,τ_j} between *default times*, defined as follows:

$$\rho^{\tau_i,\tau_j} = \frac{E_P(\tau_i\tau_j) - E_P(\tau_i)E_P(\tau_j)}{\sqrt{\text{Var}_P(\tau_i)}\sqrt{\text{Var}_P(\tau_j)}},$$

where $\text{Var}_P(\tau_i)$ is the variance of τ_i under P.

Two incorrect results

1. Observe that $\widetilde{P}\{V_t^i \leq \widehat{v}^i\} = N(-d_1)$, and

$$\widetilde{P}\{V_t^i \leq \widehat{v}^i, V_t^j \leq \widehat{v}^j\} = N_2(-d_i, -d_j; \rho_{ij}),$$

where $d_i = d_i(t)$ and $d_j = d_j(t)$. CreditMetrics of J.P. Morgan computes the value of the default correlation $\rho_{ij}^D(t)$ as

$$\rho_{ij}^D(t) = \frac{N_2(-d_i, -d_j; \rho_{ij}) - N(-d_i)N(-d_j)}{\sqrt{N(-d_i)(1 - N(-d_i))}\sqrt{N(-d_j)(1 - N(-d_j))}}.$$

This result does not seem to be correct, though. The reason being that, in general, the inclusion $\{V_t^i \leq \widetilde{v}^i\} \subset D_i(t)$ and

$$\{V_t^i \leq \widehat{v}^i, V_t^j \leq \widetilde{v}^j\} \subset D_i(t) \cap D_j(t)$$

are strict, so that the inclusions may not be replaced with equalities.

2. For every $t > 0$ and arbitrary $b_i, b_j > 0$ we have

$$P[D_i(t) \cap D_j(t)] = 4 \int_{b_i}^{\infty} \int_{b_j}^{\infty} f(x, y; \sigma_i \sqrt{t}, \sigma_j \sqrt{t}, \rho_{ij})\, dx dy,$$

and for every $(x, y) \in R_+^2$,

$$f(x, y; \sigma_i \sqrt{t}, \sigma_j \sqrt{t}, \rho_{ij}) = \frac{1}{2\pi t \sigma_i \sigma_j \sqrt{1 - \rho_{ij}^2}} \exp\left(-\frac{x^2 - 2\rho_{ij}\sigma_i\sigma_j txy + y^2}{2t^2\sigma_i^2\sigma_j^2(1 - \rho_{ij}^2)} \right).$$

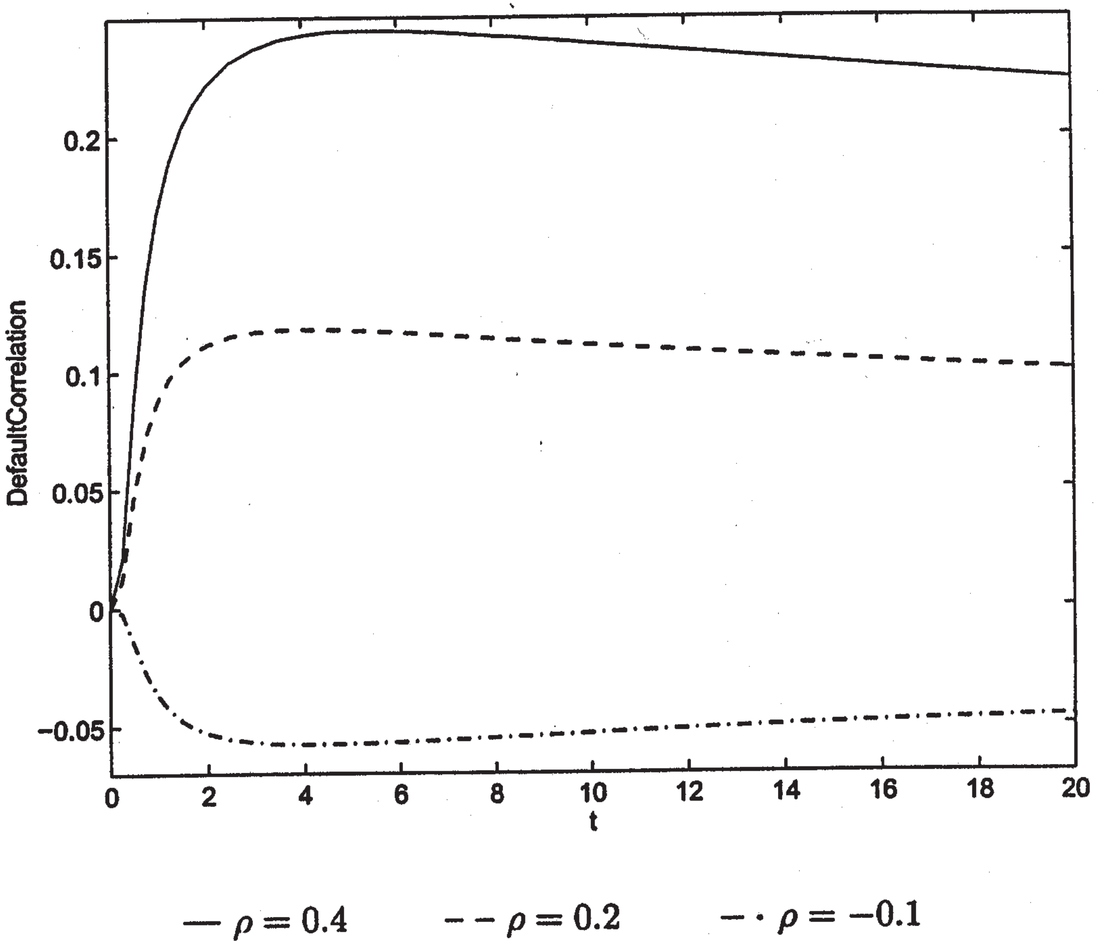


Figure 1 The relation between default correlation and time for given asset return correlations

Here ρ represents the asset level correlation between firms and t is the length of time horizon. The parameter values used here are $V_1/K_1 = V_2/K_2 = .18$ and $\sigma_1 = \sigma_2 = 0.4$.

1. Default correlation and the underlying asset return correlation have the *same sign*.

   ⋆ Firms in the same industry (region) often have higher default correlations than do the firms in different industries (regions).

2. Default correlations are generally very small over short investment horizons. They increase and then slowly decrease with time.

   - Over a short investment horizon, default correlations are low because quick defaults are rare and are almost idiosyncratic (peculiar to that firm).
   - Over a long time horizon, the default of a firm is virtually inevitable [non-default events become rare and idiosyncratic].

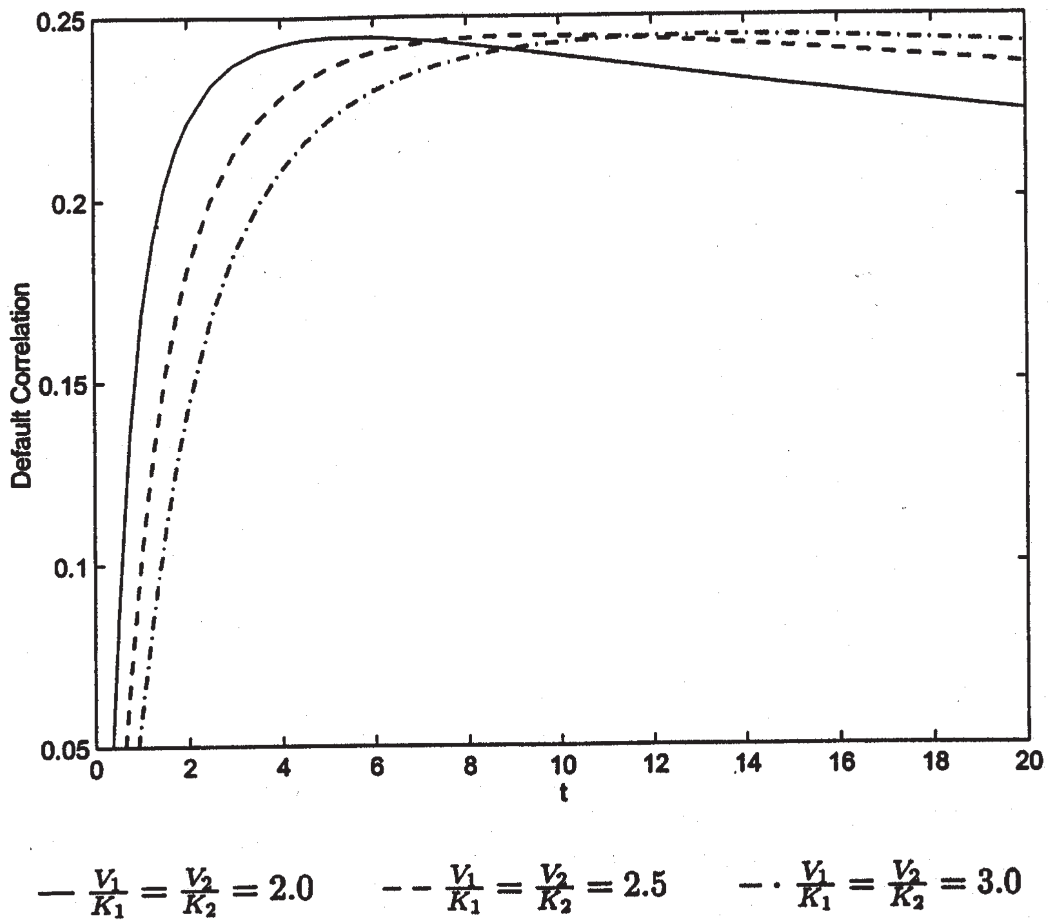


Figure 2 The relation between default correlation and time for various credit qualities

Here, ρ represents the asset level correlation between firms and t is the length of time horizon. The parameter values used here are $\rho = 0.4$ and $\sigma_1 = \sigma_2 = 0.4$.

1. *High credit quality implies a low default correlation over typical horizons*. For the higher credit quality firms, the conditional default probability $P[D_2(t) = 1|D_1(t) = 1]$ is small. Although the default of firm 1 signals that the value of firm 2, V_2, may have declined, because the original ratio V_2/K_2 is high, the probability that V_2 falls below K_2 is still very small. This result is consistent with the well-known empirical feature regarding the relation between default correlation and credit ratings.

2. *The time of peak default correlation depends on the credit quality of the underlying firms*. The higher quality firms take a longer time to peak. The short-term defaults of the higher credit quality firms are idiosyncratic events and the joint defaults of the higher quality firms are rare. It takes a long time for the default correlation of high quality firms to reach a high level.

3. *As the credit quality of firms is time-varying, the default correlation is dynamic*. An increase in the credit quality leads to a substantial drop in the default correlation, and a decline in the credit quality leads to a rise in the default correlation. This dynamic behavior arises even when the underlying assets and liabilities of the firm have constant expected returns and risks.

Risk management implications

1. Since the default correlation over short horizons is small, portfolio diversification should substantially reduce default risk over short horizons.

2. For long-term investment (e.g., 5 to 10 years), the default correlation can be quite a significant factor if the underlying firm values are highly correlated. In this case, concentration in one industry or one region, where defaults are highly correlated, could be very risky.

3. The dynamic nature of default correlations requires active management of the portfolio. This is true even if the expected returns and risks of the underlying assets and liabilities are constant over time.

4. The dynamic nature of default correlations has implications for capital requirements. Since the change in individual default risks may substantially affect the credit risk of a portfolio, the capital requirements must be adjusted accordingly. For example, suppose the default probability of each loan doubles, the probability of multiple defaults in a portfolio may be significantly more than doubled.