3.2 Pricing of defaultable claims using the intensity approach

To price any defaultable claim, the recovery mechanism must be modeled.

- **recovery of Treasury**
  \[ \phi(\tau) = \alpha P(\tau), \quad P(\tau) \text{ is the price of Treasury at } \tau, \, 0 < \alpha < 1 \]

- **multiple defaults**
  In the course of reorganization, the claim holders lose a fraction \( q \) of the face value of the claim, but the claim continues to live.

- **recovery of market value (fractional recovery)**
  \[ \phi(\tau) = (1 - q) \overline{P}(\tau_\infty), \quad \overline{P}(\tau_\infty) \text{ is the value of the defaultable claim right before } \tau. \]

- **recovery of par**
  \[ \phi(\tau) = \pi \hat{p}, \quad \hat{p} \text{ is the par} \]

- **zero recovery**
  \[ \phi(\tau) = 0 \]
Marked point process

To incorporate magnitude risk into the point process framework, we need to attach a “marker” to each event $\tau_i$. Thus we have a double sequence

$$\{(\tau_i, Y_i), i \in N\}$$

of points in time $\tau_i$ with marker $Y_i$. For default risk modeling, $Y_i$ may be a recovery rate or a new rating class.
Assumption 1

Defaults are triggered by the jumps of a Poisson process $N(t)$ with (possibly stochastic) intensity $\lambda(t)$. The Poisson arrivals are independent of all other modeling variables. Stochastic recovery parameters are markers to the Poisson process.

Assumption 2

Let $\overline{p}(t)$ be the price process of a defaultable asset, given that no default has occurred until time $t$. If a default occurs at time $\tau$, the asset has a recovery of $\phi(\tau)$ units of account at $\tau$. Here, $\phi(\tau)$ may be stochastic and it is known at the time of default ($\mathcal{F}_\tau$-measurable) but not necessarily before default.

Defaultable coupon bond

$$E \left[ \beta(\tau)\phi(\tau)1_{\{\tau \leq T\}} + \sum_{i=1}^{N} c\beta(t_i)1_{\{\tau > t_i\}} + \beta(t_N)1_{\{\tau > t_N\}} \right]$$
Recovery of market value

Consider the price process of a defaultable claim $V$ promising a payoff $f(X_T)$ at $T$. The recovery upon default is

$$\Phi(\tau) = \delta V(\tau_-) \quad \text{for} \quad \tau \leq T$$

where $\delta$ is a constant, and $\delta \in [0, 1)$. Given that there has not been a default at time $t$, we would like to show that

$$V(t) = E_t \left[ \exp \left( -\int_t^T [r + (1 - \delta)\lambda](X_s) ds \right) f(X_T) \right].$$
**Discrete-time argument**

In the discrete-time setting

- $\lambda_s$ is the probability of defaulting in $(s, s+1]$ given survival up to time $s$
- $r_s$ is the (continuously compounded) rate between $s$ and $s+1$
- $\delta$ is the fractional recovery received at time $s+1$ in the event of a default in $(s, s+1]$.

\[
V(t) = \lambda_t e^{-r_t} E_t[\delta V(t + 1)] + (1 - \lambda_t) e^{-r_t} E_t[V(t + 1)] \quad \text{for} \quad t < T
\]

\[
e^{-R_t} E_t[V(t + 1)]
\]

where $e^{-r_t} = \lambda_t e^{-r_t} \delta + (1 - \lambda_t) e^{-r_t}$. 
Iterating the expression, we obtain

\[ V(t) = E_t[e^{-(R_t+R_{t+1})}V(t + 2)] \]

and continuing on

\[ V(t) = E_t \left[ \exp \left( - \sum_{i=0}^{T-t-1} R_{t+i} \right) f(X_T) \right]. \]

Next, we move from a period of unit length to length of \( \Delta t \). Expanding \( e^{-R_t \Delta t} \) in powers of \( \Delta t \)

\[ e^{-R_t \Delta t} \approx 1 - R_t \Delta t \approx \lambda_t \delta \Delta t (1 - r_t \Delta t) + (1 - \lambda_t \Delta t)(1 - r_t \Delta t) \]

so that

\[ R_t \approx r_t + (1 - \delta) \lambda_t. \]
Repeated fractional recovery of terminal payoff

If there is a default, the promised payment is reduced from $f(X_T)$ to $\delta f(X_T)$.

$$V_t = \sum_{k=0}^{\infty} E_t \left[ \exp \left( - \int_t^T r(X_s) \, ds \right) \delta^k f(X_T) \mathbf{1}_{\{N_T = k\}} \right].$$

Conditioning on the evolution of $X$ up to $T$

$$\sum_{k=0}^{\infty} E \left[ \delta^k f(X_T) \mathbf{1}_{\{N_T = k\}} | (X_t)_{0 \leq t \leq T} \right]$$

$$= \sum_{k=0}^{\infty} \delta^k \frac{\left( \int_t^T \lambda(X_s) \, ds \right)^k}{k!} \exp \left( - \int_t^T \lambda(X_s) \, ds \right) f(X_T)$$

$$= f(X_T) \exp \left( - \int_t^T \lambda(X_s) \, ds \right) \exp \left( \delta \int_t^T \lambda(X_s) \, ds \right)$$

$$= f(X_T) \exp \left( - \int_t^T [(1 - \delta) \lambda(X_s)] \, ds \right)$$

so that

$$V(t) = E_t \left[ \exp \left( - \int_t^T (r + (1 - \delta)\lambda(X_s)) \, ds \right) f(X_T) \right].$$
Tractable models of the spot intensity

- Stochastic intensity allows us to capture the risk of a change in the credit quality.
- Most empirical studies find a negative correlation of around 20% between default intensity and default-free interest rate.
- Analytic tractability and easy calibration of the model
  - Gaussian dynamics could mean negative default intensity.

Two-factor Gaussian model

\[
\begin{align*}
    dr(t) &= \left[ k(t) - ar \right] dt + \sigma(t) dZ(t) \\
    d\lambda(t) &= \left[ \bar{k}(t) - \bar{a}\lambda \right] dt + \bar{\sigma}(t) d\bar{Z}(t)
\end{align*}
\]

with \( dZ d\bar{Z} = \rho dt \). For default-free bonds, the discount bond price is

\[
\frac{dB(t, T)}{B(t, T)} = r(t) dt - \frac{\sigma(t)}{a} \left[ 1 - e^{-a(T-t)} \right] dZ(t)
\]

while the forward rate is

\[
df(t, T) = \frac{\sigma(t)^2}{a} e^{-a(T-t)} \left[ 1 - e^{-a(T-t)} \right] dt + \sigma(t) e^{-a(T-t)} dZ(t).
\]
Recall that

\[
E \left[ e^{-\int_t^T r(s) \, ds} \mid \mathcal{F}_t \right] = e^{\hat{A}(t,T)-r(t)\hat{B}(t,T)}
\]

where

\[
\hat{B}(t,T; a) = \frac{1}{a} \left[ 1 - e^{-a(T-t)} \right]
\]

\[
\hat{A}(t, T; a, k, \sigma) = \frac{1}{2} \int_t^T \sigma^2(s) \hat{B}(t, s; a)^2 \, ds - \int_t^T \hat{B}(t, s; a)k(s) \, ds.
\]

Recall

\[
\overline{B}(t, T) = E \left[ e^{-\int_t^T [r(s)+\lambda(s)] \, ds} \right] = B(t, T)E^{PT} \left[ e^{-\int_t^T \lambda(s) \, ds} \right].
\]

The dynamics of default intensity under the $T$-forward measure is

\[
d\lambda(t) = [\tilde{k}(t) - \tilde{\alpha}\lambda] \, dt + \tilde{\sigma}(t) \, d\tilde{Z}(t)
\]

where

\[
\tilde{k}(t) = \overline{k}(t) - \rho \overline{\sigma}(t) \sigma(t) \hat{B}(t, T).
\]
To price the defaultable claim of paying $1 at $T$ if a default happens at $T$,

$$e(t, T) = E \left[ \lambda(T) e^{-\int_t^T [r(s) + \lambda(s)] \, ds} \right].$$

We use $\overline{B}(t, T)$ as the numeraire so that

$$e(t, T) = \overline{B}(t, T) E^\overline{P} [\lambda(T)]$$

and the dynamics under the new measure

$$d\lambda(t) = \left[ \overline{k}(t) - \sigma(t)\overline{\sigma}(t) \rho \overline{B}(t, T; a) - \overline{\sigma}^2(t) \overline{B}(t, T; \overline{a}) - \overline{a}(t) \lambda(t) \right] \, dt$$

$$+ \overline{\sigma}(t) \, d\overline{Z}^\overline{P}(t).$$
The evaluation of the expectation gives

\[ e(t, T) = \overline{B}(t, T) \left[ \lambda(t)e^{\bar{a}(T-t)} + \int_t^T e^{-a(T-s)}\tilde{k}'(s) \, ds \right] \]

where

\[ \tilde{k}'(t) = \overline{k}(t) - \rho \overline{\sigma}(t) \sigma(t) \hat{B}(t, T; a) - \overline{\sigma}^2(t)\hat{B}(t, T; \bar{a}). \]

**Default digital put with maturity** $T$

Payoff of $1$ at default, if default occurs before $T$. Its price is

\[ \int_0^T e(0, t) \, dt = \int_0^T E \left[ \lambda(t)e^{-\int_0^t \lambda(s) \, ds}e^{-\int_0^t r(s) \, ds} \right] \, dt. \]
Partial differential equation formulation

Payoff structure

- Final payoff:  \( V(T) = F(r(T), \lambda(T)) \)
- Continuous payoffs: At all times \( t \leq T \), the security pays at the rate of \( f(t, r, \lambda) \), only paid before default, \( t \leq \tau \).
- Payoff at default: \( g(t, r, \lambda, \pi) \), where \( \pi \) is the stochastic recovery rate.

Let \( V = V(t, r(t), \lambda(t)) \) be the price of the default-sensitive security, \( t \leq \tau \). By Ito lemma,

\[
dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial r} dr + \frac{\sigma_r^2}{2} \frac{\partial^2 V}{\partial r^2} dt + \frac{\partial V}{\partial \lambda} d\lambda + \frac{\sigma_\lambda^2}{2} \frac{\partial^2 V}{\partial \lambda^2} dt + \rho \sigma_r \sigma_\lambda \frac{\partial^2 V}{\partial \lambda \partial r} dt + \int_0^1 [g(r, r, \lambda, \pi) - V(t, r, \lambda)] m(dt, d\pi)
\]

where \( m(dt, d\pi) \) is the indicator measure. The corresponding compensator measure is \( K(d\pi) \lambda dt \) and \( K(d\pi) \) is the conditional distribution of the recovery rate \( \pi \) at default.
The final integral represents the payoff of the credit derivative at default. At a default with recovery $\pi^*$, the increment in $V$ will be

$$g(r, T, \lambda, \pi^*) - V(t, r, \lambda).$$

The expected rate of return from holding any security under the martingale measure $Q$ must be the default-free short-term interest rate

$$E^Q[dV + f dt] = rV dt.$$ 

Suppose under $Q$, $dr = \mu_r dt + \sigma_r dZ_r$ and $d\lambda = \mu_\lambda dt + \sigma_\lambda dZ_\lambda$, then

$$rV dt = f dt + \frac{\partial V}{\partial t} dt + \mu_r \frac{\partial V}{\partial r} dt + \frac{\sigma_r^2}{2} \frac{\partial^2 V}{\partial r^2} dt + \mu_\lambda \frac{\partial V}{\partial \lambda} dt + \frac{\sigma_\lambda^2}{2} \frac{\partial^2 V}{\partial \lambda^2} dt + \rho \sigma_r \sigma_\lambda \frac{\partial^2 V}{\partial \lambda \partial r} dt + \int_0^1 [g(t, r, \lambda, \pi) - V(t, r, \lambda)] K(d\pi) \lambda dt.$$

Note that the jump measure is replaced by the compensator measure. We define the locally expected default payoff

$$g^e(t, r, \lambda) = \int_0^1 g(t, r, \lambda, \pi) K(d\pi).$$
The pricing pde becomes
\[
\frac{\partial V}{\partial t} + \mu_r \frac{\partial V}{\partial r} + \frac{\sigma_r^2}{2} \frac{\partial^2 V}{\partial r^2} + \mu_\lambda \frac{\partial V}{\partial \lambda} + \frac{\sigma_\lambda^2}{2} \frac{\partial^2 V}{\partial \lambda^2} + \rho \sigma_r \sigma_\lambda \frac{\partial^2 V}{\partial \lambda \partial r} - V(\lambda + r) + g^e \lambda + f = 0.
\]
Let \( L = \mu_r \frac{\partial}{\partial r} + \frac{\sigma_r^2}{2} \frac{\partial^2}{\partial r^2} + \mu_\lambda \frac{\partial}{\partial \lambda} + \frac{\sigma_\lambda^2}{2} \frac{\partial^2}{\partial \lambda^2} + \rho \sigma_r \sigma_\lambda \frac{\partial^2}{\partial \lambda \partial r} \) so that
\[
\frac{\partial V}{\partial t} + LV - (r + \lambda)V = -g^e \lambda - f.
\]

Remarks

1. Modified payout stream = \( f + g^e \lambda \)
2. Discount factor is modified from \( r \) to \( r + \lambda \).
3. We need to append boundary conditions at \( r = 0^+ , r \to \infty , \lambda = 0^+ \) and \( \lambda \to \infty \).
Valuation of basket credit swap


- Derive the joint survival probability of occurrence time of credit events in terms of stochastic intensity processes under the assumption of conditional independence.

Suppose that we are given an approximated discrete-time model for the default processes \( h_i(k\Delta t) \), \( i = 1, 2, \ldots, n, k = 0, 1, \ldots \) and \( \Delta t \) is fixed. Now, \( h_i(t) \) are generated with a certain correlation structure until time \( t \). Under the assumption of conditional independence, given the realization of \( (h_1(t), \ldots, h_n(t)) \), the default event \( \{\tau_i \leq t + \Delta t\} \) occurs independently.
Conditional independence

Defaults are determined independently according to the probability
\[ P_t[t < \tau_i \leq t + \Delta t] = h_i(t)\Delta t, \quad i = 1, \ldots, n \]
while \( h_i(t) \) are generated non-independently.

Model setup

Let \( h_i(t), t \geq 0 \), be the default intensity process of defaultable discount bond \( i \), and assume that

\[ [R] \quad h_i(t) \text{ is continuous, bounded in any finite interval, and satisfies} \]
\[ h_i(t) \geq 0 \quad \text{and} \quad \int_0^\infty h_i(t) \, dt = \infty \text{ almost surely.} \]

Cumulative default intensity is defined by
\[ H_i(t, T) = \int_t^T h_i(u) \, du, t \leq T. \]

\( H_i(t, T) \) is non-decreasing in \( T \), continuous and bounded in any finite interval almost surely.
Since $e^{-H_i(t,T)}$ is non-increasing in $T$ and $\lim_{T \to \infty} e^{-H_i(t,T)} = 0$, there exists some random variable $\tau_i$ such that

$$P_T[\tau_i > t_i] = e^{-H_i(t,t_i)}, \quad t \leq t_i < T$$

given the realization $\mathcal{F}_T$. In other words, we can find a random variable $\tau_i$ whose distribution function equals $1 - e^{-H_i(t,t_i)}$.

Suppose $h_i(t)$ follows

$$dh_i(t) = [\phi_i(t) - a_i h_i(t)] dt + \sigma_i dZ_i(t), \quad 0 \leq t \leq T_i,$$

$$dZ_i(t) dZ_j(t) = \rho_{ij} dt$$

so that the default intensity processes $h_i(t)$ are not independent.

For conditional independence, we mean that given the realization $\mathcal{F}_T$ where $T \geq \max_i t_i$, we have

$$P_T[\tau_1 > t_1, \tau_2 > t_2, \cdots, \tau_n > t_n] = \prod_{i=1}^n P_T[\tau_i > t_i], \text{ for any } t \leq t_i \leq T.$$
Since $P_T[\tau_i > t_i] = e^{-H_i(t, t_i)}, t \leq t_i \leq T$, we have

$$P_t[\tau_1 > t_1, \tau_2 > t_2, \cdots, \tau_n > t_n] = E_t \left[ \exp \left( - \sum_{i=1}^{n} H_i(t, t_i) \right) \right].$$

Note that conditional independence does not imply the usual independence as given by

$$P_t[\tau_1 > t_1, \tau_2 > t_2, \cdots, \tau_n > t_n] = \prod_{i=1}^{n} P_t[\tau_i > t_i]$$

since

$$E_t \left[ \exp \left( - \sum_{i=1}^{n} H_i(t, t_i) \right) \right] \neq \prod_{i=1}^{n} P_t[\tau_i > t_i].$$

**Default-free short rate process, $h_0(t)$**

The time-$t$ price of the default-free discount bond maturing at time $T$

$$v_0(t, T) = E_t[e^{-H_0(t, T)}] = P_t[\tau_0 > T], \quad t \leq T.$$  

Here, $\tau_0$ is the pseudo default time (killing time).
Survival probability

\[ S_t(t_0, t_1, \cdots, t_n) = P_t[\tau_0 > t_0, \tau_1 > t_1, \cdots \tau_n > t_n]. \]

Under the Gaussian model,

\[ h_i(s) = h_i(t) e^{-a_i(s-t)} + \int_t^s \phi_i(u) e^{-a_i(s-u)} du \]
\[ + \sigma_i \int_t^s e^{-a_i(s-u)} dZ_i(u), \quad t \leq s \leq T_i. \]

As a defect, \( h_i(t) \) may become negative with positive probability.

The cumulative default intensities are

\[ H_i(t, T) = h_i(t) B_i(t, T) + \hat{A}_i(t, T) + \sigma_i \int_t^T B_i(s, T) dZ_i(s), \]
\[ t \leq T, i = 0, 1, \cdots, n, \]

\[ B_i(t, T) = \frac{1 - e^{-a_i(T-t)}}{a_i} \quad \text{and} \quad \hat{A}_i(t, T) = \int_t^T \phi_i(s) \frac{1 - e^{-a_i(T-s)}}{a_i} ds. \]

Hence, \( H_i(t, T) \) are normally distributed.
The mean is given by

$$E[H_i(t, T)] = M_i(t, T) = h_i(t)B_i(t, T) + \tilde{A}_i(t, T)$$

and covariance between $H_i(t, u)$ and $H_j(t, v)$ is

$$C_{ij}(t_j, u, v) = \sigma_i\sigma_j \int_t^{\min(u,v)} B_i(s, u)B_j(s, v)\rho_{ij} \, ds$$

$$= \rho_{ij} \sigma_i\sigma_j \left[ s - \frac{e^{-a_i(u-s)}}{a_i} - \frac{e^{-a_j(v-s)}}{a_j} + \frac{e^{-a_i(u-s)-a_j(v-s)}}{a_i + a_j} \right]_{s=\min(u,v)}^{s=t}.$$ 

Lastly, the survival probability is given by

$$S_t(t_0, t_1, \cdots, t_n) = \exp \left( -\sum_{i=0}^{n} M_i(t, t_i) + \frac{1}{2} \sum_{i=0}^{n} \sum_{k=0}^{n} C_{ik}(t; t_i, t_k) \right).$$
Basket credit default swap

Let \( v_i(t, T_i) \) denote the time-\( t \) price of the \( i \)th discount bond maturing at time \( T_i \), \( \tau_i \) be the default time of discount bond \( i \), \( \tau = \min_{1 \leq i \leq n} \tau_i \) be the first-to-default time. Assume that the discount bonds are alive at time \( t \), \( \tau_i > t \), and that \( T_i \) are longer than the maturity \( T \) of the swap contract.

Contingent payment upon first-to-default

\[
Y(\tau) = v_i(\tau, T_i) - \phi_i(\tau), \quad \text{if } \tau = \tau_i \leq T,
\]

\( \phi_i(t) \) is the market value of discount bond \( i \) in the event of default at time \( t \). Using the recovery of market value assumption

\[
\phi_i(t) = [1 - L_i(t)]v_i(t, T_i), \quad t \leq T_i,
\]

\( L_i(t) \) is the (random) fractional loss of market value. Hence,

\[
Y(\tau) = L_i(\tau)v_i(\tau, T_i), \quad \text{if } \tau = \tau_i \leq T.
\]
First-to-default feature

Let $U$ denote the credit swap premium paid at $t_j, j = 1, 2, \cdots, m$, where

$$t \leq t_1 < t_2 < \cdots < t_m = T.$$ 

Money market account of the time interval $[t, T]$

$$B(t, T) = \exp \left( \int_t^T h_0(u) \, du \right), \quad t \leq T.$$ 

Present value of the annuity paid:

$$R_{ann} = E_t \left[ \sum_{k=1}^m \left( \sum_{j=1}^k \frac{U}{B(t, t_j)} \right) 1_{ \{ t_k < \tau \leq t_{k+1} \} } \right]$$ 

where $E_t$ is the time-$t$ conditional expectation operator under the risk neutral probability measure.
Present value of contingent payment upon first default:

\[
R_{\text{con}} = \sum_{i=1}^{n} E_t \left[ \frac{1}{B(t, \tau)} L_i(\tau) v_i(\tau, T_i) 1_{\{\tau=\tau_i \leq T\}} \right].
\]

To find the fair value of the premium, we set \( R_{\text{ann}} = R_{\text{con}} \), and obtain

\[
U = \frac{\sum_{i=1}^{n} E_t \left[ \frac{L_i(\tau) v_i(\tau, T_i)}{B(t, \tau)} 1_{\{\tau=\tau_i \leq T\}} \right]}{\sum_{j=1}^{m} E_t \left[ \frac{1_{\{\tau>t_j\}}}{B(t, t_j)} \right]}.
\]

Based on conditional independence assumption

\[
R_{\text{ann}} = U \sum_{j=1}^{m} E_t \left[ e^{-H_0(t, t_j)} E_T \left[ 1_{\{\tau>t_j\}} \right] \right]
\]

\[
= U \sum_{j=1}^{m} E_t \left[ \exp \left( - \sum_{i=1}^{n} H_i(t, t_j) \right) \right] = U \sum_{j=1}^{m} S_t(t_j, t_j, \cdots, t_j).
\]
Present value of contingent payment

Suppose bondholder always receives $1 at $T_i$ if no default but $\delta_i$ dollars at $T_i$ if default occurs before $T_i$, then

$$v_i(t, T_i) = \delta_i v_0(t, T_i) + (1 - \delta_i) E_t \left[ e^{-H_0(t,T_i)} - H_i(t,T_i) \right]$$ on $\{\tau_i > t\}$.

Since $\tau_i$ are conditionally independent, given $\mathcal{F}_T$, we have

$$P_T \left[ s < \tau_i \leq s + ds, \tau_j > s \text{ for all } j \neq i \right] = h_i(s) \exp \left( - \sum_{i=1}^n H_i(t, s) \right) ds \text{ for } t < s \leq T.$$

Hence

$$R_{con} = \sum_{i=1}^n E_t \left[ \int_t^T e^{-H_0(t,s)} L_i(s) v_i(s, T_i) P_T[s < \tau_i \leq s + ds, \tau_j > s \text{ for all } j \neq i] \right]$$

$$= \sum_{i=1}^n E_t \left[ \int_t^T h_i(s) \exp \left( - \sum_{i=0}^n H_i(t, s) \right) L_i(s) v_i(s, T_i) ds \right]$$

$$= \sum_{i=1}^n \delta_i \int_t^T E_t \left[ h_i(s) L_i(s) \exp \left( - H_0(t, T_i) - \sum_{i=1}^n H_i(t, s) \right) \right] ds$$

$$+ \sum_{i=1}^n (1 - \delta_i) \int_t^T E_t \left[ h_i(s) L_i(s) \exp \left( - H_0(t, T_i) - H_i(t, T_i) - \sum_{k \neq 0, i} H_k(t, s) \right) \right] ds$$
Under the Gaussian model

1st term \[= \sum_{i=1}^{n} \delta_i \int_{0}^{T} L_i(s) \left\{ m_i(s) - \rho_i \frac{\sigma_i \sigma_0}{a_0} J_{i0}(s, T_i) - \sum_{k=1}^{n} \rho_{ik} \frac{\sigma_i \sigma_k}{a_k} J_{ik}(s, s) \right\} \]

2nd term \[= \sum_{i=1}^{n} \delta_i \int_{t}^{T} L_i(s) \left\{ m_i(s) - \sum_{k \in \{0, i\}} \rho_{ik} \frac{\sigma_i \sigma_k}{a_k} j_{ik}(s, T_i) - \sum_{k \neq 0, i} \rho_{ik} \frac{\sigma_i \sigma_k}{a_k} J_{ik}(s, s) \right\} k_i(s) \, ds \]
where \( K_i(s) = S_t(t_0, t_1, \ldots, t_0) \) with \( t_0 = t_i = T_i \) and \( t_j = s \) for \( j \neq 0, 1 \),

\[
m_i(s) = h_i(t) e^{-a_i(s-t)} + \int_t^s \phi_i(u) e^{-a_i(s-u)} \, du
\]

\[
J_{ik}(s, t_k) = \left[ \frac{e^{-a_i(s-u)}}{a_i} - \frac{e^{-a_i(s-u)-a_k(t_k-u)}}{a_i + a_k} \right]^{s \wedge t_k}_{u=t}.
\]

**Remark**

Given the default-free and defaultable bond prices, the parameter function \( \phi_j(t) \) can be determined by

\[
\phi_j(t) = a_j g_j(0, t) + \frac{\partial}{\partial t} g_j(0, t) + \frac{\sigma_j^2}{2a_j} (1 - e^{-2a_j t})
\]

\[
+ \rho_{0j} \sigma_0 \sigma_j \left( \frac{1 - e^{-a_0 t}}{a_0} + \frac{e^{-a_0 t} - e^{-(a_0 + a_j)t}}{a_j} \right), \quad j = 1, 2, \ldots, n
\]

where

\[
g_j(0, t) = -\frac{\partial}{\partial t} \ln \left( \frac{v_j(0, t)}{v_0(0, t) - \delta_j} \right).
\]