3.2 Pricing of defaultable claims using the intensity approach

To price any defaultable claim, the recovery mechanism must be modeled.

- *recovery of Treasury*

  $\phi(\tau) = \alpha P(\tau), P(\tau)$ is the price of Treasury at $\tau, 0 < \alpha < 1$
- *multiple defaults*

  In the course of reorganization, the claim holders lose a fraction q of the face value of the claim, but the claim continues to live.
- *recovery of market value (fractional recovery)*

  $\phi(\tau) = (1 - q)\overline{P}(\tau_-), \overline{P}(\tau_-)$ is the value of the defaultable claim right before τ.
- *recovery of par*

  $$\phi(\tau) = \pi\widehat{p}, \widehat{p} \text{ is the par}$$
- *zero recovery*

  $$\phi(\tau) = 0$$

Marked point process

To incorporate magnitude risk into the point process framework, we need to attach a "marker" to each event τ_i. Thus we have a double sequence

$$\{(\tau_i, Y_i), i \in N\}$$

of points in time τ_i with marker Y_i. For default risk modeling, Y_i may be a recovery rate or a new rating class.

Assumption 1

Defaults are triggered by the jumps of a Poisson process $N(t)$ with (possibly stochastic) intensity $\lambda(t)$. The Poisson arrivals are independent of all other modeling variables. Stochastic recovery parameters are *markers* to the Poisson process.

Assumption 2

Let $\overline{p}(t)$ be the price process of a defaultable asset, given that no default has occurred until time t. If a default occurs at time τ, the asset has a recovery of $\phi(\tau)$ units of account at τ. Here, $\phi(\tau)$ may be stochastic and it is known at the time of default ($\mathcal{F}_\tau$-measurable) but not necessarily before default.

Defaultable coupon bond

$$E\left[\beta(\tau)\phi(\tau)\mathbf{1}_{\{\tau\leq T\}}+\sum_{i=1}^{N}\overline{c}\beta(t_i)\mathbf{1}_{\{\tau>t_i\}}+\beta(t_N)\mathbf{1}_{\{\tau>t_N\}}\right]$$

Recovery of market value

Consider the price process of a defaultable claim V promising a payoff $f(X_T)$ at T. The recovery upon default is

$$\Phi(\tau) = \delta V(\tau_-) \quad \text{for} \quad \tau \leq T$$

where δ is a constant, and $\delta \in [0, 1)$. Given that there has not been a default at time t, we would like to show that

$$V(t) = E_t \left[\exp\left(- \int_t^T [r + (1-\delta)\lambda](X_s)\, ds \right) f(X_T) \right].$$

Discrete-time argument

In the discrete-time setting

- λ_s is the probability of defaulting in $(s, s+1]$ given survival up to time s
- r_s is the (continuously compounded) rate between s and $s+1$
- δ is the fractional recovery received at time $s+1$ in the event of a default in $(s, s+1]$.

$$\begin{aligned} V(t) &= \lambda_t e^{-r_t} E_t[\delta V(t+1)] + (1-\lambda_t) e^{-r_t} E_t[V(t+1)] \quad \text{for} \quad t < T \\ &= e^{-R_t} E_t[V(t+1)] \end{aligned}$$

where $e^{-r_t} = \lambda_t e^{-r_t} \delta + (1-\lambda_t) e^{-r_t}$.

Iterating the expression, we obtain

$$V(t) = E_t[e^{-(R_t+R_{t+1})}V(t+2)]$$

and continuing on

$$V(t) = E_t\left[\exp\left(-\sum_{i=0}^{T-t-1} R_{t+i}\right) f(X_T)\right].$$

Next, we move from a period of unit length to length of Δt. Expanding $e^{-R_t\Delta t}$ in powers of Δt

$$e^{-R_t\Delta t} \approx 1 - R_t\Delta t \approx \lambda_t\delta\Delta t(1 - r_t\Delta t) + (1 - \lambda_t\Delta t)(1 - r_t\Delta t)$$

so that

$$R_t \approx r_t + (1-\delta)\lambda_t.$$

Repeated fractional recovery of terminal payoff

If there is a default, the promised payment is reduced from $f(X_T)$ to $\delta f(X_T)$.

$$V_t = \sum_{k=0}^{\infty} E_t \left[\exp\left(- \int_t^T r(X_s)\, ds \right) \delta^k f(X_T) \mathbf{1}_{\{N_T = k\}} \right].$$

Conditioning on the evolution of X up to T

$$\begin{aligned}
& \sum_{k=0}^{\infty} E\left[\delta^k f(X_T) \mathbf{1}_{\{N_T=k\}} | (X_t)_{0 \le t \le T} \right] \\
= \ & \sum_{k=0}^{\infty} \delta^k \frac{\left(\int_t^T \lambda(X_s)\, ds \right)^k}{k!} \exp\left(- \int_t^T \lambda(X_s)\, ds \right) f(X_T) \\
= \ & f(X_T) \exp\left(- \int_t^T \lambda(X_s)\, ds \right) \exp\left(\delta \int_t^T \lambda(X_s)\, ds \right) \\
= \ & f(X_T) \exp\left(- \int_t^T [(1-\delta)\lambda(X_s)]\, ds \right)
\end{aligned}$$

so that

$$V(t) = E_t \left[\exp\left(- \int_t^T (r + (1-\delta)\lambda)(X_s)\, ds \right) f(X_T) \right].$$

Tractable models of the spot intensity

- Stochastic intensity allows us to capture the risk of a change in the credit quality.
- Most empirical studies find a negative correlation of around 20% between default intensity and default-free interest rate.
- Analytic tractability and easy calibration of the model
  - Gaussian dynamics could mean negative default intensity.

Two-factor Gaussian model

$$\begin{aligned} dr(t) &= [k(t) - ar]\,dt + \sigma(t)\,dZ(t) \\ d\lambda(t) &= [\overline{k}(t) - \overline{a}\lambda]\,dt + \overline{\sigma}(t)\,d\overline{Z}(t) \end{aligned}$$

with $dZ\,d\overline{Z} = \rho\,dt$. For default-free bonds, the discount bond price is

$$\frac{dB(t,T)}{B(t,T)} = r(t)\,dt - \frac{\sigma(t)}{a}\left[1 - e^{-a(T-t)}\right]\,dZ(t)$$

while the forward rate is

$$df(t,T) = \frac{\sigma(t)^2}{a}e^{-a(T-t)}\left[1 - e^{-a(T-t)}\right]\,dt + \sigma(t)e^{-a(T-t)}\,dZ(t).$$

Recall that

$$E\left[e^{-\int_t^T r(s)\,ds}|\mathcal{F}_t\right] = e^{\widehat{A}(t,T)-r(t)\widehat{B}(t,T)}$$

where

$$\widehat{B}(t,T;a) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]$$

$$\widehat{A}(t,T;a,k,\sigma) = \frac{1}{2}\int_t^T \sigma^2(s)\widehat{B}(t,s;a)^2\,ds - \int_t^T \widehat{B}(t,s;a)k(s)\,ds.$$

Recall

$$\overline{B}(t,T) = E\left[e^{-\int_t^T [r(s)+\lambda(s)]\,ds}\right] = B(t,T)E^{P_T}\left[e^{-\int_t^T \lambda(s)\,ds}\right].$$

The dynamics of default intensity under the T-forward measure is

$$d\lambda(t) = [\widetilde{k}(t) - \overline{a}\lambda]\,dt + \overline{\sigma}(t)\,d\overline{Z}(t)$$

where

$$\widetilde{k}(t) = \overline{k}(t) - \rho\overline{\sigma}(t)\sigma(t)\widehat{B}(t,T).$$

To price the defaultable claim of paying \$1 at T if a default happens at T,

$$e(t,T) = E\left[\lambda(T)e^{-\int_t^T [r(s)+\lambda(s)]\,ds}\right].$$

We use $\overline{B}(t,T)$ as the numeraire so that

$$e(t,T) = \overline{B}(t,T)E^{\overline{P}}[\lambda(T)]$$

and the dynamics under the new measure

$$\begin{aligned} d\lambda(t) \;&=\; [\overline{k}(t) - \sigma(t)\overline{\sigma}(t)\rho\widehat{B}(t,T;a) - \overline{\sigma}^2(t)\widehat{B}(t,T;\overline{a}) - \overline{a}(t)\lambda(t)]\,dt \\ &\quad + \overline{\sigma}(t)\,dZ^{\overline{P}}(t). \end{aligned}$$

The evaluation of the expectation gives

$$e(t,T) = \overline{B}(t,T)\left[\lambda(t)e^{-\overline{a}(T-t)} + \int_t^T e^{-a(T-s)}\widetilde{k}'(s)\,ds\right]$$

where

$$\widetilde{k}'(t) = \overline{k}(t) - \rho\overline{\sigma}(t)\sigma(t)\widehat{B}(t,T;a) - \overline{\sigma}^2(t)\widehat{B}(t,T;\overline{a}).$$

Default digital put with maturity T

Payoff of \$1 at default, if default occurs before T. Its price is

$$\int_0^T e(0,t)\,dt = \int_0^T E\left[\lambda(t)e^{-\int_0^t \lambda(s)\,ds}e^{-\int_0^t r(s)\,ds}\right]dt.$$

Partial differential equation formulation

Payoff structure

- Final payoff: $V(T) = F(r(T), \lambda(T))$
- Continuous payoffs: At all times $t \leq T$, the security pays at the rate of $f(t, r, \lambda)$, only paid before default, $t \leq \tau$.
- Payoff at default: $g(t, r, \lambda, \pi)$, where π is the stochastic recovery rate.

Let $V = V(t, r(t), \lambda(t))$ be the price of the default-sensitive security, $t \leq \tau$. By Ito lemma,

$$\begin{aligned} dV &= \frac{\partial V}{\partial t}\,dt + \frac{\partial V}{\partial r}\,dr + \frac{\sigma_r^2}{2}\frac{\partial^2 V}{\partial r^2}\,dt + \frac{\partial V}{\partial \lambda}\,d\lambda + \frac{\sigma_\lambda^2}{2}\frac{\partial^2 V}{\partial \lambda^2}\,dt + \rho\sigma_r\sigma_\lambda\frac{\partial^2 V}{\partial \lambda \partial r}\,dt \\ &\quad + \int_0^1 [g(r, r, \lambda, \pi) - V(t, r, \lambda)]\, m(dt, d\pi) \end{aligned}$$

where $m(dt, d\pi)$ is the indicator measure. The corresponding compensator measure is $K(d\pi)\lambda\, dt$ and $K(d\pi)$ is the conditional distribution of the recovery rate π at default.

The final integral represents the payoff of the credit derivative at default. At a default with recovery π^*, the increment in V will be

$$g(r, T, \lambda, \pi^*) - V(t, r, \lambda).$$

The expected rate of return from holding any security under the martingale measure Q must be the default-free short-term interest rate

$$E^Q[dV + f\,dt] = rV\,dt.$$

Suppose under $Q, dr = \mu_r\,dt + \sigma_r\,dZ_r$ and $d\lambda = \mu_\lambda\,dt + \sigma_\lambda\,dZ_\lambda$, then

$$\begin{aligned} rV\,dt &= f\,dt + \frac{\partial V}{\partial t}\,dt + \mu_r \frac{\partial V}{\partial r}\,dt + \frac{\sigma_r^2}{2}\frac{\partial^2 V}{\partial r^2}\,dt + \mu_\lambda \frac{\partial V}{\partial \lambda}\,dt + \frac{\sigma_\lambda^2}{2}\frac{\partial^2 V}{\partial \lambda^2}\,dt \\ &\quad + \rho\sigma_r\sigma_\lambda \frac{\partial^2 V}{\partial \lambda \partial r}\,dt + \int_0^1 [g(t, r, \lambda, \pi) - V(t, r, \lambda)] K(d\pi)\lambda\,dt. \end{aligned}$$

Note that the jump measure is replaced by the compensator measure. We define the locally expected default payoff

$$g^e(t, r, \lambda) = \int_0^1 g(t, r, \lambda, \pi) K(d\pi).$$

The pricing pde becomes

$$\frac{\partial V}{\partial t} + \mu_r \frac{\partial V}{\partial r} + \frac{\sigma_r^2}{2}\frac{\partial^2 V}{\partial r^2} + \mu_\lambda \frac{\partial V}{\partial \lambda} + \frac{\sigma_\lambda^2}{2}\frac{\partial^2 V}{\partial \lambda^2} + \rho\sigma_r\sigma_\lambda \frac{\partial^2 V}{\partial \lambda \partial r}$$
$$- V(\lambda + r) + g^e\lambda + f = 0.$$

Let $\mathcal{L} = \mu_r \frac{\partial}{\partial r} + \frac{\sigma_r^2}{2}\frac{\partial^2}{\partial r^2} + \mu_\lambda \frac{\partial}{\partial \lambda} + \frac{\sigma_\lambda^2}{2}\frac{\partial^2}{\partial \lambda^2} + \rho\sigma_r\sigma_\lambda \frac{\partial^2}{\partial \lambda \partial r}$ so that

$$\frac{\partial V}{\partial t} + \mathcal{L}V - (r + \lambda)V = -g^e\lambda - f.$$

Remarks

1. Modified payout stream $= f + g^e\lambda$
2. Discount factor is modified from r to $r + \lambda$.
3. We need to append boundary conditions at $r = 0^+, r \to \infty, \lambda = 0^+$ and $\lambda \to \infty$.

Valuation of basket credit swap

Masaaki Kijima, "Valuation of a credit swap of the basket type," *Review of Derivatives Research*, vol. 4 p.81-97 (2000).

- Derive the joint survival probability of occurrence time of credit events in terms of *stochastic intensity processes* under the assumption of *conditional independence*.

  Suppose that we are given an approximated discrete-time model for the default processes $h_i(k\Delta t), i = 1, 2, \cdots, n, k = 0, 1, \cdots$ and Δt is fixed. Now, $h_i(t)$ are generated with a certain correlation structure until time t. Under the assumption of conditional independence, given the realization of $(h_1(t), \cdots, h_n(t))$, the default event $\{\tau_i \leq t + \Delta t\}$ occurs independently.

Conditional independence

Defaults are determined independently according to the probability

$$P_t[t < \tau_i \le t + \Delta t] = h_i(t)\Delta t, \quad i = 1, \cdots, n$$

while $h_i(t)$ are generated non-independently.

Model setup

Let $h_i(t), t \ge 0$, be the default intensity process of defaultable discount bond i, and assume that

$[R]$ $h_i(t)$ is continuous, bounded in any finite interval, and satisfies

$$h_i(t) \ge 0 \quad \text{and} \quad \int_0^\infty h_i(t)\,dt = \infty \text{ almost surely.}$$

Cumulative default intensity is defined by

$$H_i(t,T) = \int_t^T h_i(u)\,du, t \le T.$$

$H_i(t,T)$ is non-decreasing in T, continuous and bounded in any finite interval almost surely.

Since $e^{-H_i(t,T)}$ is non-increasing in T and $\lim_{T\to\infty} e^{-H_i(t,T)} = 0$, there exists some random variable τ_i such that

$$P_T[\tau_i > t_i] = e^{-H_i(t,t_i)}, \quad t \le t_i < T$$

given the realization $\mathcal{F}_T$. In other words, we can find a random variable τ_i whose distribution function equals $1 - e^{-H_i(t,t_i)}$.

Suppose $h_i(t)$ follows

$$\begin{aligned} dh_i(t) &= [\phi_i(t) - a_i h_i(t)]\,dt + \sigma_i\,dZ_i(t), \quad 0 \le t \le T_i, \\ dZ_i(t)\,dZ_j(t) &= \rho_{ij}\,dt \end{aligned}$$

so that the default intensity processes $h_i(t)$ are not independent.

For conditional independence, we mean that given the realization $\mathcal{F}_T$ where $T \ge \max_i t_i$, we have

$$P_T[\tau_1 > t_1, \tau_2 > t_2, \cdots, \tau_n > t_n] = \prod_{i=1}^{n} P_T[\tau_i > t_i], \text{ for any } t \le t_i \le T.$$

Since $P_T[\tau_i > t_i] = e^{-H_i(t,t_i)}, t \leq t_i \leq T$, we have

$$P_t[\tau_1 > t_1, \tau_2 > t_2, \cdots, \tau_n > t_n] = E_t\left[\text{exp}\left(-\sum_{i=1}^{n} H_i(t, t_i)\right)\right].$$

Note that conditional independence does not imply the usual independence as given by

$$P_t[\tau_1 > t_1, \tau_2 > t_2, \cdots, \tau_n > t_n] = \prod_{i=1}^{n} P_t[\tau_i > t_i]$$

since

$$E_t\left[\text{exp}\left(-\sum_{i=1}^{n} H_i(t, t_i)\right)\right] \neq \prod_{i=1}^{n} P_t[\tau_i > t_i].$$

Default-free short rate process, $h_0(t)$

The time-t price of the default-free discount bond maturing at time T

$$v_0(t,T) = E_t[e^{-H_0(t,T)}] = P_t[\tau_0 > T], \quad t \leq T.$$

Here, τ_0 is the pseudo default time (killing time).

Survival probability

$$S_t(t_0, t_1, \cdots, t_n) = P_t[\tau_0 > t_0, \tau_1 > t_1, \cdots \tau_n > t_n].$$

Under the Gaussian model,

$$\begin{aligned} h_i(s) &= h_i(t)e^{-a_i(s-t)} + \int_t^s \phi_i(u)e^{-a_i(s-u)}\,du \\ &\quad + \sigma_i \int_t^s e^{-a_i(s-u)}\,dZ_i(u), \quad t \le s \le T_i. \end{aligned}$$

As a defect, $h_i(t)$ may become negative with positive probability.

The cumulative default intensities are

$$H_i(t,T) = h_i(t)B_i(t,T) + \widehat{A}_i(t,T) + \sigma_i \int_t^T B_i(s,T)\,dZ_i(s),$$
$$t \le T, i = 0, 1, \cdots, n,$$

$$B_i(t,T) = \frac{1 - e^{-a_i(T-t)}}{a_i} \quad \text{and} \quad \widehat{A}_i(t,T) = \int_t^T \phi_i(s)\frac{1 - e^{-a_i(T-s)}}{a_i}\,ds.$$

Hence, $H_i(t,T)$ are normally distributed.

The mean is given by

$$E[H_i(t,T)] = M_i(t,T) = h_i(t)B_i(t,T) + \widehat{A}_i(t,T)$$

and covariance between $H_i(t,u)$ and $H_j(t,v)$ is

$$\begin{aligned} C_{ij}(t_j,u,v) &= \sigma_i\sigma_j \int_t^{\min(u,v)} B_i(s,u)B_j(s,v)\rho_{ij}\,ds \\ &= \rho_{ij}\frac{\sigma_i\sigma_j}{a_i a_j}\left[s - \frac{e^{-a_i(u-s)}}{a_i} - \frac{e^{-a_j(v-s)}}{a_j} + \frac{e^{-a_i(u-s)-a_j(v-s)}}{a_i+a_j}\right]_{s=t}^{s=\min(u,v)}. \end{aligned}$$

Lastly, the survival probability is given by

$$S_t(t_0,t_1,\cdots,t_n) = \exp\left(-\sum_{i=0}^{n} M_i(t,t_i) + \frac{1}{2}\sum_{i=0}^{n}\sum_{k=0}^{n} C_{ik}(t;t_i,t_k)\right).$$

Basket credit default swap

Let $v_i(t, T_i)$ denote the time-t price of the i^{th} discount bond maturing at time T_i, τ_i be the default time of discount bond $i, \tau = \min_{1 \leq i \leq n} \tau_i$ be the first-to-default time. Assume that the discount bonds are alive at time t, $\tau_i > t$, and that T_i are longer than the maturity T of the swap contract.

Contingent payment upon first-to-default

$$Y(\tau) = v_i(\tau, T_i) - \phi_i(\tau), \quad \text{if } \tau = \tau_i \leq T,$$

$\phi_i(t)$ is the market value of discount bond i in the event of default at time t. Using the recovery of market value assumption

$$\phi_i(t) = [1 - L_i(t)]v_i(t, T_i), \quad t \leq T_i,$$

$L_i(t)$ is the (random) fractional loss of market value. Hence,

$$Y(\tau) = L_i(\tau)v_i(\tau, T_i), \quad \text{if } \tau = \tau_i \leq T.$$

First-to-default feature

Let U denote the credit swap premium paid at $t_j, j = 1, 2, \cdots, m$, where

$$t \leq t_1 < t_2 < \cdots < t_m = T.$$

Money market account of the time interval $[t, T]$

$$B(t,T) = \exp\left(\int_t^T h_0(u)\,du\right), \quad t \leq T.$$

Present value of the annuity paid:

$$R_{ann} = E_t\left[\sum_{k=1}^{m}\left(\sum_{j=1}^{k}\frac{U}{B(t,t_j)}\right)\mathbf{1}_{\{t_k<\tau\leq t_{k+1}\}}\right]$$

where E_t is the time-t conditional expectation operator under the risk neutral probability measure.

Present value of contingent payment upon first default:

$$R_{con} = \sum_{i=1}^{n} E_t\left[\frac{1}{B(t,\tau)} L_i(\tau) v_i(\tau, T_i) \mathbf{1}_{\{\tau=\tau_i \le T\}}\right].$$

To find the fair value of the premium, we set $R_{ann} = R_{con}$, and obtain

$$U = \frac{\sum_{i=1}^{n} E_t\left[\frac{L_i(\tau)v_i(\tau,T_i)}{B(t,\tau)} \mathbf{1}_{\{\tau=\tau_i \le T\}}\right]}{\sum_{j=1}^{m} E_t\left[\frac{\mathbf{1}_{\{\tau>t_j\}}}{B(t,t_j)}\right]}.$$

Based on conditional independence assumption

$$\begin{aligned} R_{ann} &= U \sum_{j=1}^{m} E_t\left[e^{-H_0(t,t_j)} E_T\left[\mathbf{1}_{\{\tau>t_j\}}\right]\right] \\ &= U \sum_{j=1}^{m} E_t\left[\exp\left(-\sum_{i=0}^{n} H_i(t,t_j)\right)\right] = U \sum_{j=1}^{m} S_t(t_j, t_j, \cdots, t_j). \end{aligned}$$

Present value of contingent payment

Suppose bondholder always receives \$1 at T_i if no default but δ_i dollars at T_i if default occurs before T_i, then

$$v_i(t, T_i) = \delta_i v_0(t, T_i) + (1 - \delta_i) E_t \left[e^{-H_0(t,T_i) - H_i(t,T_i)} \right] \text{ on } \{\tau_i > t\}.$$

Since τ_i are conditionally independent, given $\mathcal{F}_T$, we have

$$\begin{aligned} & P_T \left[s < \tau_i \leq s + ds, \tau_j > s \quad \text{for all} \quad j \neq i \right] \\ = \; & h_i(s) \exp \left(- \sum_{i=1}^{n} H_i(t, s) \right) ds \quad \text{for} \quad t < s \leq T. \end{aligned}$$

Hence

$$\begin{aligned} R_{con} \; = \; & \sum_{i=1}^{n} E_t \left[\int_t^T e^{-H_0(t,s)} L_i(s) v_i(s, T_i) P_T[s < \tau_i \leq s + ds, \tau_j > s \quad \text{for all} \quad j \neq i] \right] \\ = \; & \sum_{i=1}^{n} E_t \left[\int_t^T h_i(s) \exp \left(- \sum_{i=0}^{n} H_i(t, s) \right) L_i(s) v_i(s, T_i) \, ds \right] \\ = \; & \sum_{i=1}^{n} \delta_i \int_t^T E_t \left[h_i(s) L_i(s) \exp \left(-H_0(t, T_i) - \sum_{i=1}^{n} H_i(t, s) \right) \right] ds \\ & + \sum_{i=1}^{n} (1 - \delta_i) \int_t^T E_t \left[h_i(s) L_i(s) \exp \left(-H_0(t, T_i) - H_i(t, T_i) - \sum_{k \neq 0, i} H_k(t, s) \right) \right] ds \end{aligned}$$

Under the Gaussian model

$$\text{1st term} = \sum_{i=1}^{n} \delta_i \int_0^T L_i(s) \left\{ m_i(s) - \rho_{i0} \frac{\sigma_i \sigma_0}{a_0} J_{i0}(s, T_i) - \sum_{k=1}^{n} \rho_{ik} \frac{\sigma_i \sigma_k}{a_k} J_{ik}(s, s) \right\} S_t(T_i, s, \cdots, s)\, ds$$

$$\text{2nd term} = \sum_{i=1}^{n} \delta_i \int_t^T L_i(s) \left\{ m_i(s) - \sum_{k \in \{0,i\}} \rho_{ik} \frac{\sigma_i \sigma_k}{a_k} j_{ik}(s, T_i) - \sum_{k \neq 0,i} \rho_{ik} \frac{\sigma_i \sigma_k}{a_k} J_{ik}(s, s) \right\} k_i(s)\, ds$$

where $K_i(s) = S_t(t_0, t_1, \cdots, t_0)$ with $t_0 = t_i = T_i$ and $t_j = s$ for $j \neq 0, 1$,

$$m_i(s) = h_i(t)e^{-a_i(s-t)} + \int_t^s \phi_i(u)e^{-a_i(s-u)}\,du$$

$$J_{ik}(s, t_k) = \left[\frac{e^{-a_i(s-u)}}{a_i} - \frac{e^{-a_i(s-u)-a_k(t_k-u)}}{a_i + a_k}\right]_{u=t}^{s\wedge t_k}.$$

Remark

Given the default-free and defaultable bond prices, the parameter function $\phi_j(t)$ can be determined by

$$\begin{aligned}\phi_j(t) \;&=\; a_j g_j(0,t) + \frac{\partial}{\partial t} g_j(0,t) + \frac{\sigma_j^2}{2a_j}(1 - e^{-2a_j t}) \\ &\quad + \rho_{0j}\sigma_0\sigma_j\left(\frac{1 - e^{-a_0 t}}{a_0} + \frac{e^{-a_0 t} - e^{-(a_0+a_j)t}}{a_j}\right), \qquad j = 1, 2, \cdots, n\end{aligned}$$

where

$$g_j(0,t) = -\frac{\partial}{\partial t} \ln\left(\frac{v_j(0,t)}{v_0(0,t)} - \delta_j\right).$$