3.3 Default correlation – binomial models

Desirable properties for a good model of portfolio credit risk modelling

- Default dependence – produce default correlations of a realistic magnitude.
- Estimation – number of parameters should be limited.
- Timing risk – producing "clusters" of defaults in time, several defaults that occur close to each other
- *Calibration* (i) Individual term structures of default probabilities

  (ii) Joint defaults and correlation information
- Implementation

Empirical evidence

There seems to be serial dependence in the default rates of subsequent years. A year with high default rates is more likely to be followed by another year with an above average default rate than to be followed by a low default rate.

Some definitions

Consider two obligors A and B and a fixed time horizon T.

$$
\begin{aligned}
p_A &= \text{prob of default of } A \text{ before } T\\
p_B &= \text{prob of default of } B \text{ before } T\\
p_{AB} &= \text{joint default probability that } A \text{ and } B \text{ default before } T\\
p_{A|B} &= \text{prob that } A \text{ defaults before } T\text{, given that } B \text{ has defaulted before } T
\end{aligned}
$$

$$
\begin{aligned}
p_{A|B} &= \frac{p_{AB}}{p_B}, \quad p_{B|A} = \frac{p_{AB}}{p_A}\\
\rho_{AB} &= \text{linear correlation coefficient}\\
&= \frac{p_{AB} - p_A p_B}{\sqrt{p_A(1-p_A)p_B(1-p_B)}}.
\end{aligned}
$$

Since default probabilities are very small, the correlation ρ_{AB} can have a much larger effect on the joint risk of a position

$$
\begin{aligned}
p_{AB} &= p_A p_B + \rho_{AB}\sqrt{p_A(1-p_A)p_B(1-p_B)}\\
p_{A|B} &= p_A + \rho_{AB}\sqrt{\frac{p_A}{p_B}(1-p_A)(1-p_B)} \quad \text{and}\\
p_{B|A} &= p_B + \rho_{AB}\sqrt{\frac{p_B}{p_A}(1-p_A)(1-p_B)}.
\end{aligned}
$$

For N obligors, we have $N(N-1)/2$ correlations, N individual default probabilities. Yet we have 2^N possible joint default events. The correlation matrix only gives the bivariate marginal distributions, while the full distribution remains undetermined.

Price bounds for first-to-default (FtD) swaps

fee on CDS on worst credit	$\leq$	fee on FtD swap	$\leq$	portfolio of CDSs on all credits
$\overline{s}_C$	$\leq$	$\overline{s}^{\text{FtD}}$	$\leq$	$\overline{s}_A + \overline{s}_B + \overline{s}_C$

With low default probabilities and low default correlation

$$\overline{s}^{\text{FtD}} \approx \overline{s}_A + \overline{s}_B + \overline{s}_C.$$

To see this, the probability of at least one default is

$$\begin{aligned} p &= 1-(1-p_A)(1-p_B)(1-p_C) \\ &= p_A + p_B + p_c - (p_A p_B + p_A p_C + p_B p_C) + p_A p_B p_C \end{aligned}$$

so that

$$p \lessapprox p_A + p_B + p_C \quad \text{for small} \quad p_A, p_B \text{ and } p_C.$$

Basic mixed binomial model

Mixture distribution randomizes the default probability of the binomial model to induce dependence, thus mimicking a situation where a common background variable affects a collection of firms. The default events of the firms are then *conditionally independent* given the mixture variable.

Binomial distribution

Suppose X is binomially distributed (n, p), then

$$E[X] = np \quad \text{and} \quad \text{var}(X) = np(1-p).$$

We randomize the default parameter p. Recall the following relationships for random variables X and Y defined on the same probability space

$$E[X] = E[E[X|Y]] \quad \text{and} \quad \text{var}(X) = \text{var}(E[X|Y]) + E[\text{var}(X|Y)].$$

Suppose we have a collection of n firms, $X_i = D_i(T)$ is the default indicator of firm i. Assume that $\widetilde{p}$ is a random variable which is independent of all the X_i. Assume that $\widetilde{p}$ takes on values in $[0, 1]$. Conditional on $\widetilde{p}, X_1, \cdots, X_n$ are independent and each has default probability $\widetilde{p}$.

$$\overline{p} = E[\widetilde{p}] = \int_0^1 pf(p)\,dp.$$

We have

$$E[X_i] = \overline{p} \quad \text{and} \quad \text{var}(X_i) = \overline{p}(1 - \overline{p})$$

and

$$\text{cov}(X_i, X_j) = E[\widetilde{p}^2] - \overline{p}^2, \quad i \neq j.$$

(i) When $\widetilde{p}$ is a constant, we have *zero* covariance.

(ii) By Jensen's inequality, $\text{cov}(X_i, X_j) \geq 0$.

(iii) Default event correlation

$$\rho(X_i, X_j) = \frac{E[\widetilde{p}^2] - \overline{p}^2}{\overline{p}(1 - \overline{p})}.$$

Define $D_n = \sum_{i=1}^{n} X_i$, which is the total number of defaults; then

$$E[D_n] = n\overline{p} \quad \text{and} \quad \text{var}(D_n) = n\overline{p}(1-\overline{p}) + n(n-1)(E[\widetilde{p}^2] - E[\widetilde{p}]^2)$$

(i) When $\widetilde{p} = \overline{p}$, corresponding no randomness, $\text{var}(D_n) = n\overline{p}(1-\overline{p})$, like usual binomial distribution.

(ii) When $\widetilde{p} = 1$ with prob $\overline{p}$ and zero otherwise, then $\text{var}(D_n) = n^2\overline{p}(1-\overline{p})$, corresponding to perfect correlation between all default events.

(iii) One can obtain any default correlation in $[0, 1]$; correlation of default events depends only on the first and second moments of f. However, the distribution of D_n can be quite different.

(iv) $\text{var}\left(\frac{D_n}{n}\right) = \frac{\overline{p}(1-\overline{p})}{n} + \frac{n(n-1)}{n^2}\text{var}(\widetilde{p}) \longrightarrow \text{var}(\widetilde{p})$ as $n \to \infty$, that is, when considering the fractional loss for n large, the only remaining variance is that of the distribution of $\widetilde{p}$.

Large portfolio approximation

When n is large, the realized frequency of losses is close to the realized value of $\widetilde{p}$.

$$P\left[\frac{D_n}{n} < \theta\right] = \int_0^1 P\left[\frac{D_n}{n} < \theta \middle| \widetilde{p} = p\right] f(p)\,dp.$$

Note that $\dfrac{D_n}{n} \to p$ for $n \to \infty$ when $\widetilde{p} = p$, since var $\left(\dfrac{D_n}{n}\right) = \dfrac{p(1-p)}{n}$.
we have

$$P\left[\frac{D_n}{n} < \theta \middle| \widetilde{p} = p\right] \xrightarrow{n \to \infty} \begin{cases} 0 & \text{if } \theta < p \\ 1 & \text{if } \theta > p \end{cases}.$$

Furthermore,

$$P\left[\frac{D_n}{n} < \theta\right] \xrightarrow{n \to \infty} \int_0^1 \mathbf{1}_{\{\theta > p\}} f(p)\,dp = \int_0^\theta f(p)\,dp = F(\theta).$$

Summary

Firms share the same default probability and are mutually independent. The loss distribution is

$$P[D_n = k] = {}_nC_k \int_0^1 z^k (1-z)^{n-k}\,dF(z), \qquad k \leq n.$$

The above loss probability is considered as a *mixture* of binomial probabilities with the mixing distribution given by F.

Choosing the mixing distribution using Merton's model

Consider n firms whose asset values V_t^i follow

$$dV_t^i = rV_t^i\,dt + \sigma V_t^i\,dB_t^i$$

with

$$B_t^i = \rho\widetilde{B}_t^0 + \sqrt{1-\rho^2}\,\widetilde{B}_t^i.$$

The GBM driving V^i can be decomposed into a common factor $\widetilde{B}_t^0$ and a firm-specific factor $\widetilde{B}_t^i$. Also, $\widetilde{B}^0, \widetilde{B}^1, \widetilde{B}^2, \cdots$ are independent standard Brownian motions. Also, the firms are identical in terms of drift rate and volatility.

Firm i defaults when

$$V_0^i \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma B_T^i\right) < D^i$$

or

$$\ln V_0^i - \ln D^i + \left(r - \frac{\sigma^2}{2}\right)T + \sigma\left(\rho\widetilde{B}_T^0 + \sqrt{1-\rho^2}\,\widetilde{B}_T^i\right) < 0.$$

We write $\widetilde{B}_T^i = \epsilon_i\sqrt{T}$, where ϵ_i is a standard normal random variable. Then firm i defaults when

$$\frac{\ln V_0^i - \ln D^i + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} + \rho\epsilon_0 + \sqrt{1-\rho^2}\,\epsilon_i < 0.$$

Conditional on a realization of the common factor, say, $\epsilon_0 = u$ for some $u \in \mathcal{R}$, firm i defaults when

$$\epsilon_i < -\frac{c_i + \rho u}{\sqrt{1-\rho^2}}$$

where

$$c_i = \frac{\ln \frac{V_0^i}{D^i} + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}.$$

Assume that $c_i = c$ for all i, for given $\epsilon_0 = u$, the probability of default is

$$P(u) = N\left(-\frac{c + \rho u}{\sqrt{1-\rho^2}}\right).$$

Given $\epsilon_0 = u$, defaults of the firms are independent. The mixing distribution is that of the common factor ϵ_0, and N transforms ϵ_0 into a distribution on $[0, 1]$.

This distribution function $F(\theta)$ for the distribution of the mixing variable $\widetilde{p} = P(\epsilon_0)$ is

$$\begin{aligned} F(\theta) &= P[P(\epsilon_0) \leq \theta] = P\left[N\left(-\frac{c+\rho\epsilon_0}{\sqrt{1-\rho^2}}\right) \leq \theta\right] \\ &= P\left[-\epsilon_0 \leq \frac{1}{\rho}\left(\sqrt{1-\rho^2}N^{-1}(\theta) + c\right)\right] \\ &= N\left(\frac{1}{\rho}\left(\sqrt{1-\rho^2}N^{-1}(\theta) - N^{-1}(\overline{p})\right)\right) \quad \text{where} \quad \overline{p} = N(-c). \end{aligned}$$

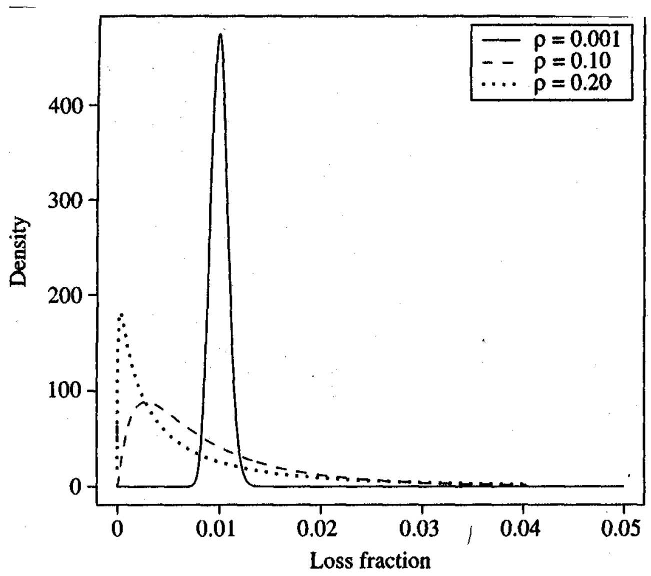
Note that $F(\theta)$ has the appealing feature that it has dependence on ρ and $\overline{p}$. The probability that no more than a fraction θ default is

$$P\left[\frac{D_n}{n} \leq \theta\right] = \int_0^1 \sum_{k=0}^{n\theta} {}_nC_k p(u)^k [1 - p(u)]^{n-k} f(u)\, du.$$

When $n \to \infty$,

$$P\left[\frac{D_n}{n} \leq \theta\right] \xrightarrow{n \to \infty} \int_0^\theta f(u)\, du = F(\theta).$$

$F(\theta)$ is the probability of having a fractional loss less than θ on a perfectly diversified portfolio with only factor risk.



The figure shows the loss distribution in an infinitely diversified loan portfolio consisting of loans of equal size and with one common factor of default risk. The default probability is fixed at 1% but the correlation in asset values varies from nearly 0 to 0.2.

Remarks

1. For a given default probability $\overline{p}$, increasing correlation increases the probability of seeing large losses and of seeing small losses compared with a situation with no correlation.
2. Recent reference

   "The valuation of correlation-dependent credit derivatives using a structural model," by John Hull, Mirela Predescu and Alan White, Working paper of University of Toronto (March 2005).

Randomizing the loss

Assume that the expected loss given $\widetilde{p}$ is $\ell(\widetilde{p})$ and it is strictly monotone. We expect the loss in default increases when systematic default risk is high, perhaps because of losses in the value of collateral.

Define the loss on individual loan as

$$L_i(\widetilde{p}) = \ell(\widetilde{p})\mathbf{1}_{\{D_i=1\}},$$

then

$$E[L_i|\widetilde{p} = p] = p\ell(p) = \wedge(p).$$

Define

$$L = \frac{1}{n}\sum_{i=1}^{n} L_i \xrightarrow{n \to \infty} \widetilde{p}\ell(\widetilde{p})$$

so that the loss-weighted loss probability is

$$P[L \leq \theta] \xrightarrow{n \to \infty} \int_0^1 \mathbf{1}_{\{p\ell(p)\leq\theta\}} f(p)\, dp = F(\wedge^{-1}(\theta))$$

where F is the distribution function of $\widetilde{p}$ and $\wedge$.

Contagion model

Reference

Davis, M. and V. Lo (2001), "Infectious defaults," *Quantitative Finance*, vol. 1, p. 382-387.

Drawback in earlier model

It is the common dependence on the background variable $\widetilde{p}$ that induces the correlation in the default events. It requires assumptions of large fluctuations in $\widetilde{p}$ to obtain significant correlation.

Contagion means that once a firm defaults, it may bring down other firms with it. Define Y_{ij} to be an "infection" variable. Both X_i and Y_{ij} are Bernuolli variables

$$P[X_i] = p \quad \text{and} \quad P[Y_{ij}] = q.$$

The default indicator of firm i is

$$Z_i = X_i + (1 - X_i)\left[1 - \prod_{j \neq i}(1 - X_j Y_{ji})\right].$$

Note that Z_i equals one either when there is a direct default of firm i or if there is no direct default and $\prod_{j \neq i}(1 - X_j Y_{ji}) = 0$. The latter case occurs when at least one of the factor $X_j Y_{ji}$ is 1, which happens when firm j defaults and infects firm i.

Define $D_n = Z_1 + \cdots + Z_n$, Davis and Lo (2001) find that

$$\begin{aligned} E[D_n] &= n[1 - (1-p)(1-pq)^{n-1}] \\ \text{var}(D_n) &= n(n-1)\beta_n^{pq} - (E[D_n])^2 \end{aligned}$$

where

$$\begin{aligned} \beta_n^{pq} &= p^2 + 2p(1-p)[1 - (1-q)(1-pq)^{n-2}] \\ &\quad + (1-p)^2[1 - 2(1-pq)^{n-2} + (1 - 2pq + pq^2)^{n-2}]. \end{aligned}$$

$$\text{cov}(Z_i, Z_j) = \beta_n^{pq} - \text{var}(D_n/n)^2.$$

Binomial approximation using diversity scores

Seek reduction of problem of multiple defaults to binomial distributions.

If n loans each with equal face value are independent, have the same default probability, then the distribution of the loss is a binomial distribution with n as the number of trials.

Let F_i be the face value of each bond, p_i be the probability of default within the relevant time horizon and ρ_{ij} between the correlation of default events. With n bonds, the total principal is $\sum_{i=1}^{n} F_i$ and the mean and variance of the loss of principal $\widehat{P}$ is

$$\begin{aligned} E[\widehat{P}] &= \sum_{i=1}^{n} p_i F_i \\ \text{var}(\widehat{P}) &= \sum_{i=1}^{n} \sum_{j=1}^{n} F_i F_j \rho_{ij} \sqrt{p_i(1-p_i)p_j(1-p_j)}. \end{aligned}$$

We construct an approximating portfolio consisting D independent loans, each with the same face value F and the same default probability p.

$$\sum_{i=1}^{n} F_i = DF$$
$$\sum_{i=1}^{n} p_i F_i = DFp$$
$$\mathrm{var}(\widehat{P}) = F^2 Dp(1-p).$$

Solving the equations

$$\begin{aligned}
p &= \frac{\sum_{i=1}^{n} p_i F_i}{\sum_{i=1}^{n} F_i} \\
D &= \frac{\sum_{i=1}^{n} p_i F_i \sum_{i=1}^{n} (1-p_i) F_i}{\sum_{i=1}^{n} \sum_{j=1}^{n} F_i F_j \rho_{ij} \sqrt{\rho_i (1-p_i) \rho_j (1-p_j)}} \\
F &= \sum_{i=1}^{n} F_i \Big/ D.
\end{aligned}$$

Here, D is called the *diversity score*.