3.4 Copula approach for modeling default dependency

Two aspects of modeling the default times of several obligors

1. Default dynamics of a single obligor.
2. Model the dependence structure of defaults between the obligors.

Question How to specify a joint distribution of survival times, with given marginal distributions?

- Knowing the joint distribution of random variables allows us to derive the marginal distributions and the correlation structure among the random variables but not vice versa.
- A copula function links univariate marginals to their full multivariate distribution.

Proposition

If a one-dimensional continuous random variable X has distribution function F, that is, $F(x) = P[X \leq x]$, then the distribution of the random variable $U = F(X)$ is a uniform distribution on $[0, 1]$.

Proof

$$P[U \leq u] \;=\; P[F(X) \leq u] = P[X \leq F^{-1}(u)] = \int_{-\infty}^{F^{-1}(u)} f(s)\,ds$$

where $f(x) = F'(x)$ is the density function of X.

Let $y = F(s)$, then $dy = f(s)\,ds$ and

$$P[U \leq u] = \int_{-\infty}^{F(F^{-1}(u))} dy = \int_{-\infty}^{u} dy.$$

Conversely, if U is a random variable with uniform distribution on $[0, 1]$, then $X = F^{-1}(U)$ has the distribution function F.

Remark To simulate an outcome of X, one may simulate an outcome u from a uniform distribution then let the outcome of X be $x = F^{-1}(u)$.

Multi-variate distribution function

$$F_{X_1,X_2,\cdots,X_n}(x_1,x_2,\cdots,x_n) = P[X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n]$$

- It is an increasing, right continuous function which maps a subset of the real numbers into the unit interval $[0,1]$.
- Monotonicity property for vector $\mathbf{a}$ and $\mathbf{b}$

$$\mathbf{a} < \mathbf{b} \quad \Rightarrow \quad F(\mathbf{b}) - F(\mathbf{a}) \geq 0$$

  $\mathbf{a} < \mathbf{b}$ means $\mathbf{b} - \mathbf{a}$ is a vector with non-negative entries and at least one strictly positive entry.

Cautious note

The probability assigned to $[x_1, y_1] \times [x_2, y_2]$ by F is

$$F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1).$$

For any distribution function, we require that it assigns positive mass to all rectangles.

Definition of a copula function

A function $C : [0,1]^n \rightarrow [0,1]$ is a copula if

(a) There are random variables $U_1, U_2, \cdots, U_n$ taking values in $[0,1]$ such that C is their distribution function.

(b) C has uniform marginal distributions; for all $i \leq n, u_i \in [0,1]$

$$C(1, \cdots, 1, u_i, 1, \cdots, 1) = u_i.$$

In the analysis of dependency with copula function, the joint distribution can be separated into two parts, namely, the marginal distribution functions of the random variables (marginals) and the dependence structure between the random variables which is described by the copula function.

Reference

Li, David (2000), "On default correlation: a copula function approach," *Journal of Fixed Income*, vol. 9(4) p.43-54.

Construction of multi-variate distribution function

Given univariate marginal distribution functions $F_1(x_1), F_2(x_2), \cdots, F_n(x_n)$, the function

$$C(F_1(x_1), F_2(x_2), \cdots, F_n(x_n)) = F(x_1, x_2, \cdots, x_n)$$

which is defined using a copula function C, results in a multivariate distribution function with univariate marginal distributions specified as $F_1(x_1), F_2(x_2), \cdots, F_n(x_n)$.

Proof

$$\begin{aligned} C(F_1(x_1), \cdots, F_n(x_n), \rho) &= P[U_1 \le F_1(x_1), \cdots, U_n \le F_n(x_n)] \\ &= P[F_1^{-1}(U_1) \le x_1, \cdots, F_n^{-1}(U_n) \le x_n] \\ &= P[X_1 \le x_1, \cdots, X_n \le x_n] \\ &= F(x_1, \cdots, x_n). \end{aligned}$$

The marginal distribution of X_i is

$$\begin{aligned} & C(F_1(\infty), \cdots, F_i(x_i), \cdots, F_n(\infty), \rho) \\ = \;& P[X_1 < \infty, \cdots, X_i \le x_i, \cdots, X_n < \infty] \\ = \;& P[X_i \le x_i] = F_i(x_i). \end{aligned}$$

Sklar's theorem

- Any multi-variate distribution function F can be written in the form of a copula function.

Theorem

If $F(x_1, x_2, \cdots, x_n)$ is a joint multi-variate distribution function with univariate marginal distribution functions $F_1(x_1), \cdots F_n(x_n)$, then there exists a copula function $C(u_1, u_2, \cdots, u_n)$ such that

$$F(x_1, x_2, \cdots, x_n) = C(F_1(x_1), F_2(x_2), \cdots, F_n(x_n)).$$

If each F_i is continuous, then C is unique.

Remark

Going through all copula functions gives us all the possible types of dependence structures that are compatible with the given one-dimensional marginal distributions.

CreditMetrics

- CreditMetrics uses the normal copula function in its default correlation formula even though it does not use the concept of copula function explicitly.
- CreditMetrics calculates joint default probability of two credits A and B using the following steps:

(i) Let q_A and q_B denote the one-year default probabilities for A and B, respectively. Obtain Z_A and Z_B such that

$$q_A = P[Z < Z_A] \quad \text{and} \quad q_B = P[Z < Z_B]$$

where Z is the standard normal random variable.

(ii) Let ρ denote the asset correlation, the joint default probability for credit A and B is given by

$$P[Z < Z_A, Z < Z_B] = \int_{-\infty}^{Z_A} \int_{-\infty}^{Z_B} n_2(x, y; \rho)\, dxdy = N_2(Z_A, Z_B; \rho). \tag{A}$$

Bivariate normal copula function

$$C(u,v) = N_2(N^{-1}(u), N^{-1}(v); \gamma), \quad -1 \leq \gamma \leq 1.$$

Suppose we use a bivariate normal copula function with a correlation parameter γ, and denote the survival times for A and B as T_A and T_B. The joint default probability is given by

$$P[T_A < 1, T_B < 1] = N_2(N^{-1}(F_A(1)), N^{-1}(F_B(1)), \gamma) \qquad (B)$$

where F_A and F_B are the distribution functions for the survival times T_A and T_B.

We observe that

$$q_i = P[T_i < 1] = F_i(1) \quad \text{and} \quad Z_i = N^{-1}(q_i) \quad \text{for} \quad i = A, B,$$

Eqs. (A) and (B) are equivalent if we have $\rho = \gamma$.

Note that this correlation parameter is not the correlation coefficient between the two survival times.

Simulation of survival times of a basket of obligors

Assume that for each credit i in the portfolio, we have constructed a credit curve or a hazard rate function for its survival time T_i. Let $F_i(t)$ denote the distribution function of T_i.

Using a copula function C, we obtain the joint distribution of the survival times

$$F(t_1, t_2, \cdots, t_n) = C(F(t_1), F_2(t_2), \cdots, F_n(t_n)).$$

For example, suppose we use the normal copula function, we have

$$F(t_1, t_2, \cdots, t_n) = N_n(N^{-1}(F_1(t_1)), N^{-1}(F_2(t_2)), \cdots, N^{-1}(F_n(t_n))).$$

To simulate correlated survival times, we introduce

$$Y_1 = N^{-1}(F_1(T_1)), Y_2 = N^{-1}(F_2(T_2)), \cdots, Y_n = N^{-1}(F_n(T_n)).$$

There is a one-to-one mapping between Y and T.

Simulation scheme

- Simulate $Y_1, Y_2, \cdots, Y_n$ from an n-dimensional normal distribution with correlation coefficient matrix Σ.
- Obtain $T_1, T_2, \cdots, T_n$ using $T_i = F_i^{-1}(N(Y_i)), \quad i = 1, 2, \cdots, n$.

With each simulation run, we generate the survival times for all the credits in the portfolio. With this information we can value any credit derivative structure written on the portfolio.

Exponential model for dependent defaults

Reference

Kay Giesecke, "A simple exponential model for dependent defaults," (2003) Working paper of Cornell University.

Model setup

- A firm's default is driven by idiosyncratic as well as other regional, sectoral or economy-wide shocks, whose arrivals are modeled by independent Poisson processes.
- Default times are assumed to be jointly exponentially distributed. In this case, the exponential copula arises naturally.

Advantages

1. All relevant results are given in closed form.
2. Efficient simulation of dependent default times is straightforward.
3. Parameter calibration relies on market data as well as data provided by rating agencies.

Bivariate version of the exponential models

Suppose there are Poisson processes N_1, N_2 and N with respective intensities λ_1, λ_2 and λ. Here, λ_i is the idiosyncratic shock intensity of firm i and λ is the intensity of a macro-economic shock affecting both firms simultaneously.

Define the default time τ_i of firm i by

$$\tau_i = \inf\{t \geq 0 : N_i(t) + N(t) > 0\}.$$

That is, a default occurs completely unexpectedly if either an idiosyncratic or a systematic shock strikes the firm for the first time. Firm i defaults with intensity $\lambda_i + \lambda$ so that the survival function is

$$S_i(t) = P[\tau_i > t] = P[N_i(t) + N(t) = 0] = e^{-(\lambda_i+\lambda)t}.$$

The expected default time and variance are

$$E[\tau_i] = \frac{1}{\lambda_i + \lambda} \quad \text{and} \quad \text{var}(\tau_i = \frac{1}{(\lambda_i + \lambda)^2}.$$

The joint survival probability is found to be

$$\begin{aligned}
S(t,u) &= P[\tau_1 > t, \tau_2 > u] \\
&= P[N_1(t) = 0, N_2(u) = 0, N(t \vee u) = 0] \\
&= e^{-\lambda_1 t - \lambda_2 u - \lambda(t \vee u)} \\
&= e^{-(\lambda_1 + \lambda)t - (\lambda_2 + \lambda)u + \lambda(t \wedge u)} \\
&= S_1(t) S_2(u) \min(e^{\lambda t}, e^{\lambda u}).
\end{aligned}$$

Remark

All random variables are defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. Depending on the specific application, P is the physical probability (risk management setting) or some risk neutral probability (valuation setting).

Survival copula

There exists a unique solution $C^\tau : [0,1]^2 \to [0,1]$, called the survival copula of the default time vector (τ_1, τ_2) such that the joint survival probabilities can be represented by

$$S(t,u) = C^\tau(S_1(t), S_2(u)).$$

The copula C^τ describes the complete non-linear default time dependence structure.

Define $\theta_i = \dfrac{\lambda}{\lambda_i + \lambda}$, we obtain

$$C^\tau(u,v) = S(S_1^{-1}(u), S_2^{-1}(v)) = \min(vu^{1-\theta_1}, uv^{1-\theta_2}).$$

The parameter vector $\theta = (\theta_1, \theta_2)$ controls the degree of dependence between the default times.

1. Firms default independently of each other ($\lambda = 0$ or $\lambda_1, \lambda_2 \to \infty$)

$$\theta_1 = \theta_2 = 0, \quad C_\theta^\tau(u,v) = uv \text{ (product copula)}$$

2. Firms are perfectly correlated (firms default simultaneously, $\lambda \to \infty$ or $\lambda_1 = \lambda_2 = 0$)

$$\theta_1 = \theta_2 = 1 \quad \text{and} \quad C_\theta^\tau(u,v) = u \wedge v.$$

It can be shown that

$$uv \leq C^{\tau}_{\theta}(u,v) \leq u \wedge v, \theta \in [0,1]^2, u,v \in [0,1].$$

Also, the default can only be positively correlated.

Similarly, define K^{τ} by

$$K^{\tau}(P_1(t), P_2(t)) = P[\tau_1 \leq t, \tau_2 \leq u] = P(t,u)$$

where $P_i(t) = P[\tau_i \leq t] = 1 - S_i(t)$. Since

$$S(t,u) = 1 - P_1(t) - P_2(u) + P(t,u)$$

so that these copulas are related by

$$\begin{aligned} K^{\tau}(u,v) &= C^{\tau}(1-u, 1-v) + u + v - 1 \\ &= \min([1-v][1-u]^{1-\theta_1}; (1-u][1-v]^{1-\theta_2}) + u + v - 1. \end{aligned}$$

Correlation coefficients

1. *Spearman's rank correlation*

   It is simply the linear correlation ρ of the copula K^τ given by

$$\begin{aligned}\rho^S(\tau_1,\tau_2) &= \rho(P_1(\tau_1),P_2(\tau_2))\\ &= 12\int_0^1\int_0^1 K^\tau(u,v)\,dudv-3\\ &= \frac{3\lambda}{3\lambda+2\lambda_1+2\lambda_2}\end{aligned}$$

2. *Linear default time correlation*

$$\rho(\tau_1,\tau_2)=\frac{\lambda}{\lambda+\lambda_1+\lambda_2}.$$

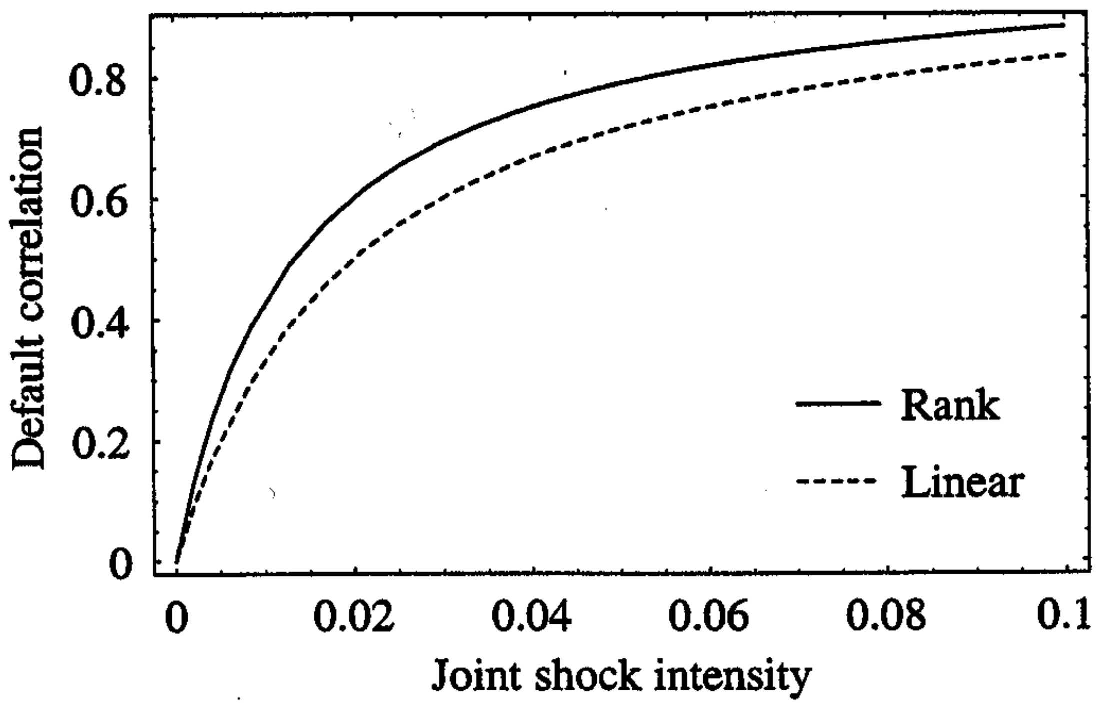
   Note that $\rho\le\rho^S$.

3. Linear correlation of the default indicator variables

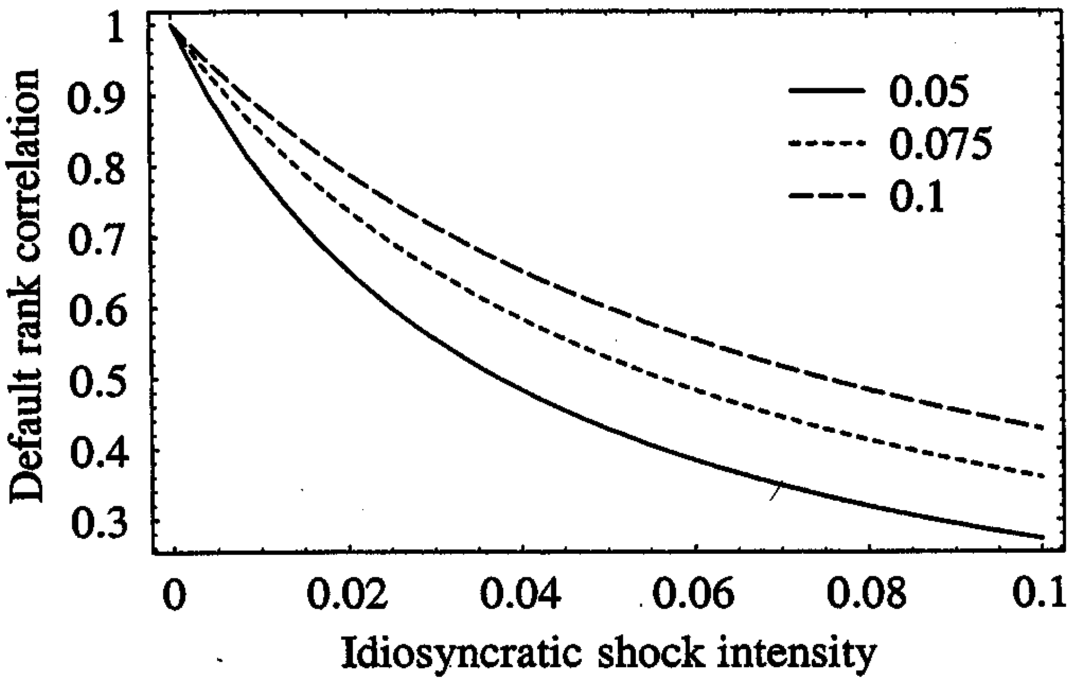
$$\rho\left(\mathbf{1}_{\{\tau_1\le t\}},\mathbf{1}_{\{\tau_2\le t\}}\right)=\frac{S(t,t)-S_1(t)S_2(t)}{\sqrt{P_1(t)S_1(t)P_2(t)S_2(t)}}.$$

Remark

Correlation is an increasing function of the joint shock λ and a decreasing function of idiosyncratic intensities.



Rank and linear default correlation



Rank default correlation as a function of idiosyncratic shock intensity, varying joint shock intensity.

Multi-variate extension

Assume that there are $n \geq 2$ firms. The default of an individual firm is driven by some idiosyncratic shock as well as other sectoral, industry, country-specific or economy-wide shocks.

Define a matrix $(a_{ij})_{n\times m}$, when $a_{ij} = 1$ if shock $j \in \{1, 2, \cdots, m\}$ modeled through the Poisson process N_j with intensity λ_j, leads to a default of firm $i \in \{1, 2, \cdots, n\}$ and $a_{ij} = 0$ otherwise. For example, when $n = 3$

$$(a_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Note that $m = \sum_{k=1}^{n} {}_nC_k = 2^n - 1$. Suppose economy-wide shock events are excluded, one then set $a_{ij} = 0$ for $i = 1, 2, 3$. This corresponds to bivariate dependence only.

Joint survival function

$$T_i = \inf\left\{t \geq 0 : \sum_{k=1}^{m} a_{ik} N_k(t) > 0\right\}$$

meaning that firm i defaults with intensity $\sum_{k=1}^{m} a_{ik}\lambda_k$ and

$$S_i(t) = \exp\left(-\sum_{k=1}^{m} a_{ik}\lambda_k t\right).$$

The joint survival function

$$\begin{aligned} S(t_1, t_2, \cdots, t_n) &= P[\tau_1 > t_1, \cdots \tau_n > t_n] \\ &= \exp\left(-\sum_{k=1}^{m} \lambda_k \max(a_{1k}t_1, \cdots, a_{nk}t_n)\right). \end{aligned}$$

Joint default probabilities p is given by

$$p(t_1, \cdots, t_n) = \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} S(v_{vi_1}, \cdots, v_{ni_n}),$$

where $v_{j1} = t_j$ and $v_{j2} = 0$.

Survival copula function

The exponential survival copula associated with S can be found via $C^\tau(u_1, \cdots, u_n) = S(S_1^{-1}(u_1), \cdots, S_n^{-1}(u_n))$. Fixing some $i, j \in \{1, 2, \cdots, n\}$ with $i \neq j$, the two-dimensional marginal copula is given by

$$\begin{aligned} C^\tau(u_i, u_j) &= C^\tau(1, \cdots, 1, u_i, 1, \cdots, 1, u_j, 1, \cdots, 1) \\ &= \min(u_j u_i^{1-\theta_i}, u_i u_j^{1-\theta_j}) \end{aligned}$$

where we define, analogously to the bivariate case,

$$\theta_i = \frac{\sum_{k=1}^m a_{ik} a_{jk} \lambda_k}{\sum_{k=1}^m a_{ik} \lambda_k}, \quad \theta_j = \frac{\sum_{k=1}^m a_{ik} a_{jk} \lambda_k}{\sum_{k=1}^m a_{jk} \lambda_k}$$

as the ratio of joint default intensity of firms i and j to default intensity of firm i or j, respectively.

Spearman's rank default time correlation matrix $(\rho_{ij}^S)_{n \times n}$ is given by

$$\rho_{ij}^S = \frac{3\theta_i \theta_j}{2\theta_i + 2\theta_j - \theta_i \theta_j}.$$

Extensions

1. *Shocks are not necessarily fatal*

   In the bivariate case, suppose an idiosyncratic shock leads to a default of firm i only with a pre-specified probability q_i. An economy-wide shock leads to a default of both firms with probability q_{11}, to a default of firm 1 only with probability q_{10}, and to a default of firm 2 only with probability q_{01}. These lead us to the exponential default time distribution

   $$S(t,u) = e^{-\gamma_1 t - \gamma_2 u - \gamma(t \vee u)},$$

   where $\gamma_1 = \lambda_1 q_1 + \lambda q_{10}, \gamma_2 = q_2 + \lambda q_{01}$, and $\gamma = \lambda q_{11}$. Of course, with $q_1 = q_2 = q_{11} = 1$ we obtain the earliest model.

   In the non-fatal model interpretation, the number of model parameters is quite high, which makes the model calibration very challenging.

2. *Variability of intensities over time*

- In practice credit spreads vary substantially over time. To capture these effects, in a first step, the Poisson framework can be generalized to deterministically varying intensities i.e. to inhomogeneous Poisson shock arrivals.
- The intensity function may be assumed to be piece-wise constant, which is a reasonably flexible approximation in certain application. In the bivariate case,

$$S_i(t) = \exp\left(-\int_0^t [\lambda_i(r) + \lambda(r)]\,dr\right)$$

and

$$S(t,u) = e^{-\int_0^t \lambda_1(r)\,dr - \int_0^u \lambda_2(r)\,dr - \int_0^{t\vee u} \lambda(r)\,dr}.$$

- We can extend to general stochastic intensities. Such models would capture, in a realistic way, the stochastic variation in the term structure of credit spreads. One needs, however, a large and reliable data base to calibrate the parameters of such a stochastic intensity model. Now

$$S_i(t) = E\left[\exp\left(-\int_0^t [\lambda_i(u) + \lambda(u)]\,du\right)\right]$$

and

$$S(t,u) = E\left[e^{-\int_0^t \lambda_1(r)\,dr - \int_0^u \lambda_2(r)\,dr - \int_0^{t\vee u} \lambda(r)\,dr}\right].$$

Simulation of correlated default arrival times

Four-step algorithm which generates default arrival times with an exponential dependence structure C^τ while allowing for arbitrary marginal default time distributions.

1. Simulate an m-vector $(t_1, \cdots, t_m)$ of independent exponential shock arrival times with given parameter vector $(\lambda_1, \cdots, \lambda_m)$ where $\lambda_k > 0$. This is done by drawing, for $k \in \{1, 2, \cdots, m\}$, an independent standard uniform random variate U_k and setting

$$t_k = -\frac{1}{\lambda_k} \ln U_k.$$

   Indeed, $P[t_k > T] = P[-\ln U_k/\lambda_k > T] = P[U_k \leq e^{-\lambda_k T}] = e^{-\lambda_k T}$.

2. Simulate an n-vector $(T_1, \cdots, T_n)$ of joint exponential default times by considering, for each firm $i \in \{1, 2, \cdots, n\}$, the minimum of the relevant shock arrival times:

$$T_i = \min\{t_k : 1 \leq k \leq m, a_{ik} = 1\}.$$

3. Generate a sample $(v_1, \cdots, v_n)$ from the (survival) default time copula $C^\mathcal{T}$ by setting, for $i \in \{1, 2, \cdots, n\}$,

$$v_i = S_i(T_i) = \exp\left(-T_i \sum_{k=1}^{m} a_{ik}\lambda_k\right).$$

4. In order to generate an n-vector $Z = (Z_1, \cdots, Z_n)$ of correlated default arrival times with given marginal survival function q_i and exponential default dependence structure $C^\mathcal{T}$, set, for $i \in \{1, s, \cdots, n\}$,

$$Z_i = q_i^{-1}(v_i)$$

provided that the inverse q_i^{-1} exists.

Remark

One may use a general stochastic intensity-based model or a structural model for q_i, or use some estimated arrival function. To account for different types of idiosyncratic default risk, we can also choose different survival marginals q_i for different firms. The above algorithm then generates correlated default times Z_i with these given q_i and exponential dependence structure, i.e.

$$P[Z_1 > t_1, \cdots, Z_n > t_n] = C^\mathcal{T}(q_1(t_1), \cdots, q_n(t_n)).$$

Default distribution

Assume zero recovery in case of default. The default loss L_t at some fixed horizon t is then equal to the number of the defaulted firm $L_t = n - M_t$, where

$$M_t = \sum_{i=1}^{n} 1_{\{\tau_i > t\}}$$

is the number of firms which still operate at t. The distribution of M_t can be computed directly from the joint survival probabilities. By standard arguments we find

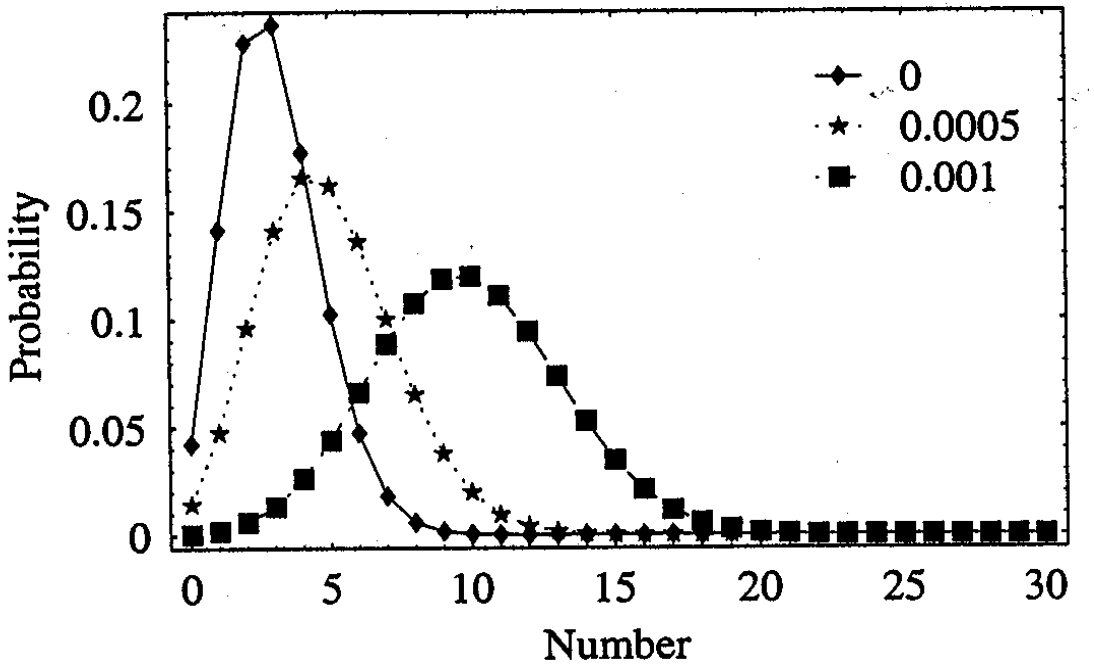
$$P[M_t = k] = \sum_{i=k}^{n} {}_iC_k(-1)^{i-k} \sum_{J\subset\{1,\cdots,n\},|J|=i} P\left[\bigcap_{j\in J}\{\tau_j > t\}\right].$$

where the $|J|$-dimensional marginal joint survival probability $P\left[\bigcap_{j\in J}\{\tau_j > t\}\right]$ is directly available from the joint survival function for all $J \subset \{1, \cdots, n\}$.

This can be simplified if the firm in the portfolio are homogeneous and symmetric, i.e. if the default time vector is exchangeable:

$$(\tau_1, \cdots, \tau_n) \stackrel{d}{=} (\tau_{z(1)}, \cdots, \tau_{z(n)})$$

for any permutation $z(1), \cdots, z(n)$ of indices $(1, \cdots, n)$ ($\stackrel{d}{=}$ we mean equality in distribution).



Distribution of the number of defaults, varying joint shock intensity, for a 10-year time horizon with $n = 30$ firms.

First-to-default basket

Consider a binary first-to-default swap, which involves the payment of one unit of account upon the *first* default in the reference portfolio in exchange for a periodic payment (the swap spread). The swap spread is paid up to the maturity T of the swap or the first default, whichever is first. The index set of the reference portfolio is $\{1, 2, \cdots, n\}$.

Let us denote by $\tau = \min_i(\tau_i)$ the first-to-default time. We have

$$P[\tau > t] = \exp\left(-t \sum_{k=1}^{m} \lambda_k \max(a_{1k}, \cdots, a_{nk})\right).$$

Assuming that investors are risk-neutral (i.e. P is some risk-neutral probability), the value c of the contingent leg of the swap at time zero is given by

$$c = E\left[e^{-\int_0^\tau r_s\, ds} \mathbf{1}_{\{\tau \leq T\}}\right]$$

where $(r_t)_{t \geq 0}$ is the riskless short rate.

- Supposing for simplicity that $r_t = r > 0$ for all t, we get

$$c = \int_0^\infty e^{-ru} \mathbf{1}_{\{u \le T\}} P[\tau \in du] = \Lambda \int_0^T e^{-(r+\Lambda)u}\, du$$

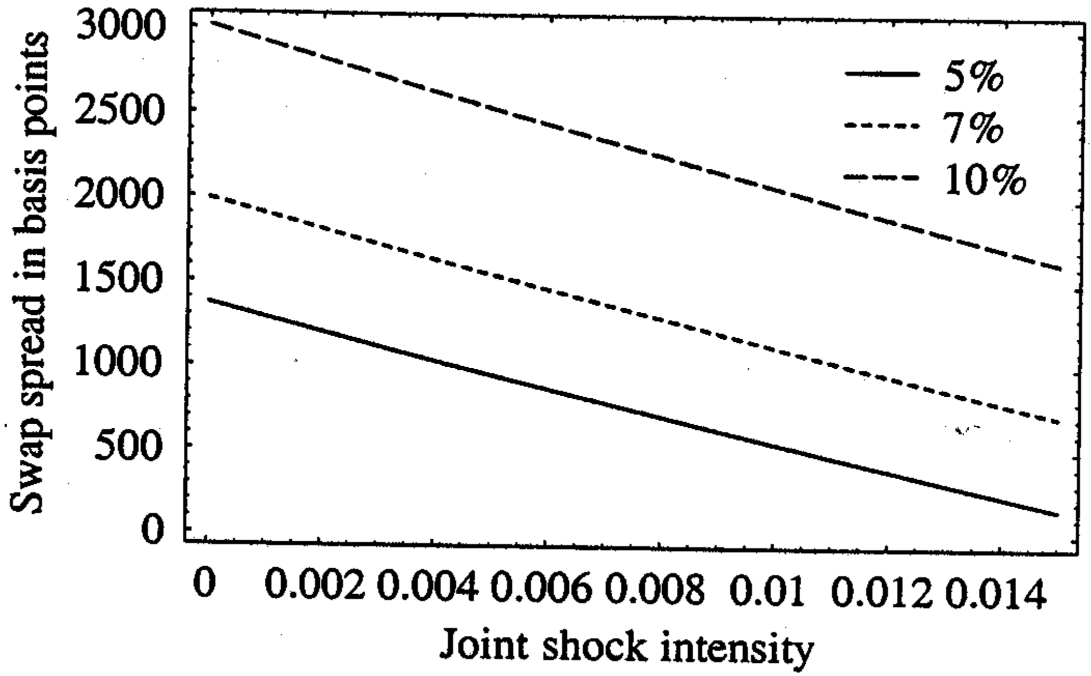
  where $\Lambda = \sum_{k=1}^{m} \lambda_k \max(a_{1k}, \cdots, a_{nk})$.

- If the (constant) swap spread R is paid at dates $t_1 < t_2 < \cdots < t_j = T$, then the fee leg has a value of

$$\begin{aligned} f &= \sum_{i: t_i \le T} E\left[e^{-\int_0^{t_i} r_s\, ds} R \mathbf{1}_{\{\tau > t_i\}} \right] \\ &= R \sum_{i: t_i \le T} e^{-(r+\Lambda)t_i} \end{aligned}$$

  where we invoke the assumption of constant short rates.

- We neglect any accrued swap spread here. The value of the fee leg paid by the protection buyer compensates the protection seller for paying one unit of account upon the first default in the reference portfolio. The swap spread R is that $c = f$ at inception of the contract ($t = 0$).



Swap spread as a function of the joint shock intensity, varying individual default probability, assuming symmetry and homogeneity.

- Supposing that $n = 5, r = 0, T = 1, t_1 = 0.5$, and $t_2 = 1$ (i.e. semi-annual coupon payments), we plot the swap spread R as a function of the joint shock intensity λ for varying one-year default probabilities (while increasing $\overline{\lambda}$, we decrease λ such that the default probability remains constant).

- Since the likelihood of a payment by the protection seller is increasing in individual firms' default probabilities, the spread is increasing in these default probabilities.

- The spread of decreasing in $\overline{\lambda}$, which is also intuitively clear: for increasing (positive) default correlation the probability of multiple defaults increases and the degree of default protection provided by a first-to-default swap is diminished.

- With zero correlation the premium is at its maximum, because the likelihood of multiple defaults is at its minimum (given that negative correlation is excluded).