Risk Management

Topic Three – Risk measures and economic capital

3.1 Types of financial risks and loss distributions

3.2 Global Correlation Model using factor models approach

3.3 VAR (value at risk), expected shortfall and coherent risk measures

3.4 Economic capital and risk-adjusted return on capital

3.1 Types of financial risks and loss distributions

Risk can be defined as loss or exposure to mischance, or more quantitatively as the volatility of unexpected outcomes, generally related to the value of assets or liabilities of concern. It is best measured in terms of probability distribution functions.

- While some firms may passively accept financial risks, others attempt to create a competitive advantage by judicious exposure to financial risk. In both cases, these risks must be monitored because of their potential for damage (or even ruin).

Risk management is the process by which various risk exposures are identified, measured, and controlled.

Major events / shocks in the financial markets

- On Black Monday, October 19, 1987, U.S. stocks collapsed by 23 percent, wiping out $1 trillion in capital.

- In the bond debacle (崩潰) of 1994, the Federal Reserve Bank, after having kept interest rates low for 3 years, started a series of six consecutive interest rate hikes that erased $1.5 trillion in global capital.

- The Japanese stock price bubble finally deflated at the end of 1989, sending the Nikkei index from $39,000$ to $17,000$ three years later. A total of $2.7 trillion in capital was lost, leading to an unprecedented financial crisis in Japan. Until October 2011, the Nikkei index remained to be below $9,000$.

- The Asian turmoil of 1997 wiped off about three-fourth of the dollar capitalization of equities in Indonesia, Korea, Malaysia, and Thailand.

- The Russian default in August 1998 sparked a global financial crisis that culminated in the near failure of a big hedge fund, Long Term Capital Management.

- The bankruptcy of Lehman Brothers and failure of other major financial institutions, like AIG, CitiGroup, Merill Lynch, triggered the financial tsunami in 2008.

- The European debt crisis in 2011 triggered crashes in the financial markets around the globe.

Wall Street fails Main Street

Market risk

Market risk arises from movements in the level or volatility of market prices.

- Absolute risk, measured in dollar terms, which focuses on the volatility of returns.
- Relative risk, measured relative to a benchmark index, can be expressed in terms of tracking errors (deviation from an index).

Directional risks

Exposures to the direction of movements in financial variables (linear exposure, like $\Delta B \approx \frac{\partial B}{\partial i}\Delta i$)

- Beta for exposure to stock market movement, $\beta_i = \frac{\sigma_{iM}}{\sigma_M^2} = \frac{\overline{r_i} - r_f}{\overline{r}_M - r_f}$, where $\sigma_{iM} = \mathrm{cov}(r_i, r_M)$ and $\sigma_M^2 = \mathrm{var}(r_M)$.

- Duration for exposure to interest rate, $D = \sum c_i^\star t_i / \sum c_i^\star$, where $c_i^\star$ is the discounted cash flow at time t_i.

- Delta for exposure of options to the underlying asset price, $\Delta = \frac{\partial V}{\partial S}$.

Non-linear exposures

Exposures to hedged positions or to volatilities. Second order statistics, like dispersion and volatility, come into play.

1. Second order or quadratic exposures are measured by
   - convexity when dealing with interest rates
   - gamma when dealing with options, $\sqcap = \frac{\partial^2 V}{\partial S^2} = \frac{\partial \Delta}{\partial S}$; higher gamma means more rapid change of the delta hedged position when the underlying prices move.
2. Volatility risk measures exposure to movements in the actual or implied volatility. Volatility risks are more significant in derivative instruments. Options and warrants become more expensive when volatility is high.

3. Basis risk is the risk arising when offsetting investments in a hedging strategy do not experience price changes in entirely opposite directions from each other. The imperfect correlation between the two investments creates the potential for excess gains or losses in a hedging strategy.

   - An example is the index futures arbitrage where the basket of stocks may not match exactly the stock index. This represents lack of exact (or sufficiently close) substitutes.
   - Another example is the failure of convergence of bond prices that caused the LTCM bankruptcy.

Systemic risk and systematic risk

- *Systemic risk* is the risk of loss from some catastrophic event that can trigger a collapse in a certain industry or economy.

- As an example of systemic risk, the collapse of Lehman Brothers in 2008 caused major reverberations throughout the financial system and the economy. Lehman Brother's size and integration in the economy caused its collapse to result in a domino effect.

- *Systematic risk* is the risk inherent in the aggregate market that cannot be solved by diversification. Though it cannot be fixed with diversification, it can be hedged, say, by entering into off-setting positions.

- It is the risk associated with aggregate market returns, so it is sometimes also called aggregate risk or undiversifiable risk.

- According to the CAPM, investors are compensated by excess expected returns above the riskfree rate for bearing the systematic risk.

- *Unsystematic risk*, also called specific risk, idiosyncratic risk, residual risk, or diversifiable risk, is the company-specific or industry-specific risk in a portfolio, which is uncorrelated with the aggregate market returns.

Credit risk

The credit risk associated with investment on a financial instrument can be quantified by the spread, which is the yield above the riskfree Treasury rate.

Credit risk consists of two components: default risk and spread risk.

1. Default risk (違約風險): any non-compliance with the exact specification of a contract.

2. Spread risk: reduction in market value of the contract / instrument due to changes in the credit quality of the debtor / counterparty.

Event of default

1. Arrival risk – timing of the event of default, modeled by a stopping time τ

   A stopping rule is defined such that one can determine whether a stochastic process stops or continues, given the information available at that time.

2. Magnitude risk – loss amount (exposure net of the recovery value)

   $$\begin{aligned}\text{Loss amount} &= \text{par value (possibly plus accrued interest)} - \\ &\quad\ \text{market value of a defaultable bond}\end{aligned}$$

Risk elements

1. Exposure at default and recovery rate, both are random variables.

2. Default probability (characterization of the default criteria and the associated random time of default).

3. Credit migration – the process of changing the creditworthiness of an obligor as characterized by the transition probabilities from one credit state to other credit states.

Credit event

Occurs when the calculation agent is aware of some publicly available information as to the existence of a credit condition.

- Credit condition means either a payment default or a bankruptcy (清盘) event.
- Legal definition of publicly available information means information that has been published in any two or more internationally recognized published or electronically displayed financial news sources.

Chapter 11 Bankruptcy Code

- It is a chapter of the US Bankruptcy code.
- A company is protected from creditors while it restructures its business, usually by downsizing and narrowing focus.
- Keep the organization intact while seeking protection from creditors.

Liquidity risk

Asset liquidity risk

This arises when a transaction cannot be conducted at prevailing market prices due to the size of the position relative to normal trading lots.

– Some assets, like Treasury bonds, have deep markets where most positions can be liquidated easily with very little price impact.

– Other assets, like OTC (over-the-counter) derivatives or emerging market equities, any significant transaction can quickly affect prices.

Cash-flow risk

The inability to meet payment obligations, which may force early liquidation, thus transforming "paper" losses into realized losses. Examples include bankruptcy of Orange County and failure of Long Term Capital Management.

– This is a problem for portfolios that are leveraged and subject to margin calls from the lender.

Cash-flow risk interacts with asset liquidity risk if the portfolio contains illiquid assets that must be sold at less than the fair market value.

Operational risk

Arising from human and technical errors or accidents. Investors understand that the very function of trading is to take financial risk, few are willing to forgive losses due to lack of supervision.

- Frauds (traders intentionally falsify information), management failures, and inadequate procedures and controls.
- A combination of market or credit risk and failure of controls.

Model risk (mathematicians understand the limitations of the model while practitioners use them to advance their flavors)

The mathematical model used to value positions is misused. A good example is the rating of Constant Proportional Debts Obligations (CPDO) based on flawed mathematical models. Investors received LIBOR+200 bps coupon rate on CPDO, which had been rated AAA. Most investors on CPDO lost almost 100% of their investments during the financial tsunami in 2008.

Legal risk

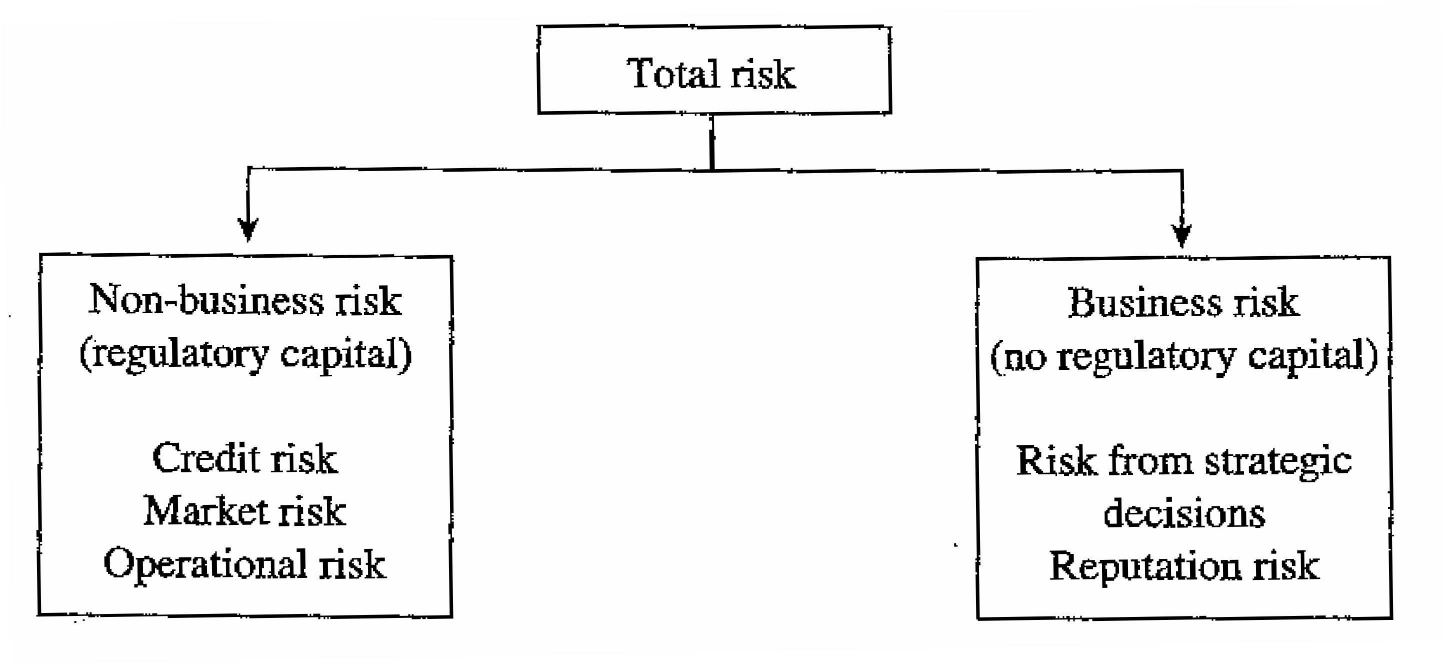
Arises when a transaction proves unenforceable in law.

- Legal risk is generally related to credit risk, since counterpaties that lose money on a transaction may try to find legal grounds for invalidating the transaction. It may take the form of shareholder law suits against corporations that suffer large losses.

Examples

Two municipalities in Britain had taken large positions in interest rate swaps that turned out to produce large losses. The swaps were later ruled invalid by the British High Court. The court decreed that the city councils did not have the authority to enter into these transactions and so the cities were not responsible for the losses. Their bank counterparties had to swallow the losses. A local example is the Lehman Brothers' mini-bonds.

Categorization of risks faced by a bank



Business risk: should we take on a new product or business line?

The relative importance of different risks depends on the business mix.

- Banks taking deposits and making loans – credit risk.
- Investment banks – both credit risk and market risk.
- Asset managers – operational risk.

The Barings Bank Disaster – Operational risk

- Nicholas Leeson, an employee of Barings Bank in the Singapore office in 1995, had a mandate to look for arbitrage opportunities between the Nikkei 225 futures prices on the Singapore exchange and the Osaka exchange. Over time Leeson *moved from being an arbitrageur to being a speculator* without anyone in the Barings London head office fully understanding that he had changed the way he was using derivatives.

- In 1994, Leeson is thought to have made \$20 million for Barings, one-fifth of the total firm's profit. He drew a $\$150,000$ salary with a \$1 million bonus.

- He began to make losses, which he was able to hide. He then began to take bigger speculative positions in an attempt to recover the losses, but only resulted in making the losses worse.

- In the end, Leeson's total loss was close to 1 billion dollars. As a result, Barings – a bank that had been in existence for 200 years – was wiped out.

- Lesson to be learnt: Both financial and nonfinancial corporations must set up controls to ensure that derivatives are being used for their intended purpose. Risk limits should be set and the activities of traders should be monitored daily to ensure that the risk limits are adhered to.

Similar cases

- Societe Generate lost $6.7 billion in January 2008 after trader Jerome Kerviel took unauthorized positions on European stock index futures.

- UBS lost $2 billion in September 2011 after Kweku Adoboli took unauthorized positions on currency swap trades.

Long Term Capital Management – Liquidity risk

- Long Term Capital Management (LTCM) is a hedge fund formed in the mid 1990s. The hedge fund's investment strategy was known as *convergence arbitrage*.

- It would find two bonds, X and Y, issued by the same company promising the same payoffs, with X being less liquid than Y. The market always places a value on liquidity. As a result the price of X would be less than the price of Y. LTCM would buy X, short Y and wait, expecting the prices of the two bonds to converge at some future time. Normally, with better liquidity on X provided by LTCM, price of X moves up while price of Y moves down.

- When interest rates increased (decreased), the company expected both bonds to move down (up) in price by about the same amount, so that the collateral it paid on bond X would be about the same as the collateral it received on bond Y. It therefore expected that there would be no significant outflow of funds as a result of its collateralization agreements.

- In August 1998, Russia defaulted on its debt and this led to what is termed a "flight to quality" in capital markets. One result was that investors valued liquid instruments more highly than usual and the spreads between the prices of the liquid and illiquid instruments in LTCM's portfolio increased dramatically (instead of convergence).

- The prices of the bonds LTCM had bought went down and the prices of those it had shorted increased. LTCM was required to post collateral on both.

- The company was highly leveraged and unable to make the payments required under the collateralization agreements. The result was that positions had to be closed out and there was a total loss of about $4 billion.

- If the company had been less highly leveraged, it would probably have been able to survive the flight to quality and could have waited for the prices of the liquid and illiquid bonds to become closer (eventual convergence of prices).

European Growth Trust (EGT) – Legal risk

- In 1996, Peter Young was a fund manager at Deutsche Morgan Grenfell, a subsidiary of Deutsche Bank. He was responsible for managing a fund called the European Growth Trust (EGT). It had grown to be a very large fund and Young had responsibilities for managing over £1 billion of investors' money.

- Certain rules were applied to EGT, one of these was that no more than 10% of the fund could be invested in unlisted securities. Peter Young violated this rule in a way that, it can be argued, benefited him personally.

- When the facts were uncovered, he was fired and Deutsche Bank had to compensate investors. The total cost to Deutsche Bank was over £200 million.

Balance sheet scorings based on drivers of firm's economic well being

- future earnings and cashflows
- debts, short-term and long-term liabilities, and financial obligations
- capital structure (leverage)
- liquidity of the firm's assets
- situation (political, social etc) of the firm's home country or industry
- management quality, company structure, etc.

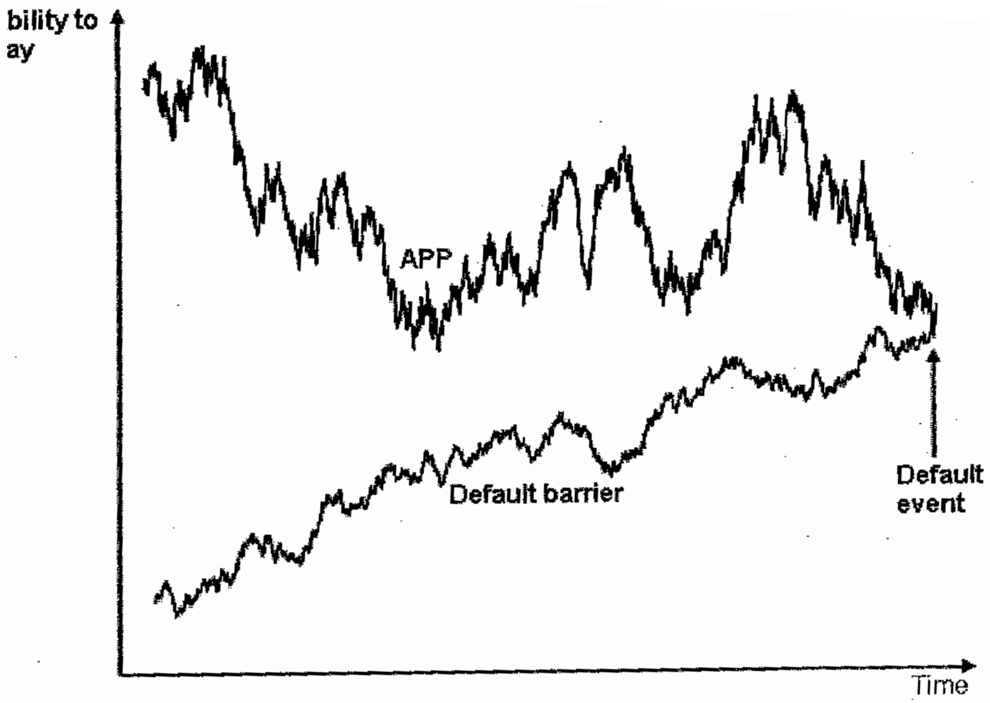
Rating on a company relies on the statistical analysis of financial variables plus soft factors.

S&P rating categories

AAA	*best credit quality* *extremely reliable with regard to financial obligations*
AA	*very good credit quality* *very reliable*
A	*more susceptible to economic conditions* *still good credit quality*
BBB	*lowest rating in investment grade*
BB	*caution is necessary* *best sub-investment credit quality*
B	*vulnerable to changes in economic conditions* *currently showing the ability to meet its financial obligations*
CCC	*currently vulnerable to non-payment* *dependent on favorable economic conditions*
CC	*highly vulnerable to a payment default*
C	*close to or already bankrupt* *payments on the obligation currently continued*
D	*payment default on some financial obligation has actually occurred*

- The assignment of a probability default to every rating grade in the given rating scale is called a *rating calibration*. However, calibration is not an easy task for rating methods that are based on financial statements (traditional methods adopted by the top three rating agencies). At best, these rating agencies provide data of historical corporate bond defaults.

Threshold model for default of a credit obligor



Ability to pay process (APP) as default trigger

- In the classical Merton structural model, the *ability to pay* or *distance to default* of an obligor is described as a function of *assets and liabilities*.

- If liabilities exceed the financial power of an obligor, either a firm or an individual, bankruptcy and payment failure will follow.

- Given the dynamics of the ability to pay process and default barrier, we can estimate the probability that the ability to pay process falls below the barrier. In this sense, the default probability can be quantified.

Characterization of the credit risk of loans

- Financial variables to be considered include
  - default probability (DP)
  - loss fraction called the loss given default (LGD)
  - exposure at default (EAD)

Goal: Derive the portfolio risk based on the information of individual risks and their correlations.

Loss variable

$$\tilde{L} = EAD \times SEV \times L$$

where $L = \mathbf{1}_D$, $E[\mathbf{1}_D] = DP$. Here, D is the default event that the obligor defaults within a certain period of time. We treat severity (SEV) of loss in case of default as a random variable with $E[SEV] = LGD$.

Based on the assumption that the exposure, severity and default event are independent, the expected loss (EL):

$$EL = E[\tilde{L}] = E[EAD] \times LGD \times DP.$$

Here, EAD is in general stochastic and $E[EAD]$ is the expectation of several relevant underlying random variables.

Independence assumption in SEV and $\mathbf{1}_D$ may be questionable since DP becomes high and LGD becomes low during a recession period.

Example of EAD calculation in a bank credit unit

Banks grant to obligors credit lines which is like a credit limit for the single-obligor exposure.

- Total credit line is \$20m. The borrower can draw \$12m as cash and can use the remaining \$8m for so-called contingent liabilities, say, guarantees or comparable credit constructs, but not for cash. Suppose \$10m has been drawn as cash with 2m remaining as open cash credit line.

- $EAD_{\text{cash}} = \mathbf{1}_{\text{D}raw} \times X \times 2\text{m}$, where $\mathbf{1}_{\text{D}raw}$ is the event that the obligor draws on the open cash credit line, $X \in [0,1]$ is the random fraction describing how much of the open 2m line is drawn. Suppose $P[\text{D}raw] = E[\mathbf{1}_{\text{D}raw}] = 80\%$ and $E[X] = 60\%$, then $EAD_{\text{cash}} = 48\% \times 2\text{m}$.

- For the contingent liability part of the credit line, we assume the existence of a rich database which allows for the calibration of a draw down factor (DDF), say, 40% for the contingent liability part. Also, another so-called cash equivalent exposure factor (CEEF) of 80% which is another conversion factor quantifying the conversion of the specific contingent liability, say, a guarantee into a cash exposure.

- Finally, we have

$$E[EAD] = 10\text{m} + 48\% \times 2\text{m} + 32\% \times 8\text{m} = 13.52\text{m}.$$

.

Loss given default

$$LGD = 1 - \text{recovery rate}$$

Driving factors

1. quality of collateral - how much the collateral can cover the loss
2. seniority of bank's claim on the borrower's assets

- How do banks share knowledge about their practical LGD experience?
- How can we derive better techniques for estimating LGD from historical data?

- Assume that a client has m credit products with the bank and pledged n collateral securities to the bank, called an m-to-n situation.

  Remark

  It would be difficult to get the interdependence and relation between products and collateral right (quite often, collateral may be dedicated to a particular credit product).

- $\$LGD = (EAD_1 + \cdots + EAD_m) - (\$REC_1 + \cdots + \$REC_n)$, where REC_i is the recovery proceed from the i^{th} collateral. We have

$$\text{percentage } LGD = \frac{\$LGD}{EAD_1 + \cdots + EAD_m}.$$

- Technical issues: require a rich database storing historical proceeds from collateral security categories; also, time value of money should be considered since recovery proceeds come later.

Unexpected loss – standard deviation of $\tilde{L}$

Holding capital as a cushion against expected losses is not enough. As a measure of the magnitude of the deviation of losses from the EL, a natural choice is the standard deviation of the loss variable $\tilde{L}$, which is termed unexpected loss.

$$\text{Unexpected loss } (UL) = \sqrt{\text{var}(\tilde{L})} = \sqrt{\text{var}(EAD \times SEV \times L)}.$$

Under the assumption that the severity and the default event D are independent, and also EDA is taken to be deterministic, we have

$$UL = EAD \times \sqrt{\text{var}(SEV) \times DP + LGD^2 \times DP(1 - DP)}.$$

In contrast to EL, UL is the "actual" uncertainty faced by the bank when investing in a portfolio since UL captures the deviation from the expectation.

Proof

We make use of var$(X) = E[X^2] - E[X]^2$, so that var$(\mathbf{1}_D) = DP(1-DP)$ since $E[\mathbf{1}_D^2] = E[\mathbf{1}_D] = DP$. Assuming SEV and $\mathbf{1}_D$ are independent, we have

$$
\begin{aligned}
\text{var}(SEV\,\mathbf{1}_D) &= E[SEV^2\mathbf{1}_D^2] - E[SEV\,\mathbf{1}_D]^2 \\
&= E[SEV^2]E[\mathbf{1}_D^2] - E[SEV]^2E[\mathbf{1}_D]^2 \\
&= \{\text{var}(SEV) + E[SEV]^2\}DP - E[SEV]^2DP^2 \\
&= \text{var}(SEV) \times DP + LGD^2 \times DP(1-DP).
\end{aligned}
$$

Remark

It is common to have the situation where the severity of losses and the default events are random variables driven by a common set of underlying factors. In this case we need to have the information of the joint distribution of SEV and $\mathbf{1}_D$ in order to perform the expectation calculations.

Portfolio losses

Consider a portfolio of m loans

$$\tilde{L}_i = EAD_i \times SEV_i \times \mathbf{1}_{D_i}, \quad i = 1, \ldots, m, \quad P[D_i] = E[\mathbf{1}_{D_i}] = DP_i.$$

The random portfolio loss $\tilde{L}_p$ is given by

$$\tilde{L}_p = \sum_{i=1}^{m} \tilde{L}_i = \sum_{i=1}^{m} EAD_i \times SEV_i \times L_i, \quad L_i = \mathbf{1}_{D_i}.$$

Using the additivity of expectation, we obtain

$$EL_p = \sum_{i=1}^{m} EL_i = \sum_{i=1}^{m} EAD_i \times LGD_i \times DP_i.$$

In the case UL, additivity holds if the loss variable $\tilde{L}_i$ are pairwise uncorrelated. Unfortunately, correlations are the "main part of the game" and a main driver of credit risk.

In general, we have

$$
\begin{aligned}
UL_p &= \sqrt{\mathrm{var}(\tilde{L}_p)} \\
&= \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m} EAD_i \times EAD_j \times \mathrm{cov}(SEV_i \times L_i, SEV_j \times L_j)}.
\end{aligned}
$$

For a portfolio with constant severities, we have the following simplified formula

$$
UL_p^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} EAD_i \times EAD_j \times LGD_i \times LGD_j \times \sqrt{DP_i(1-DP_i)DP_j(1-DP_j)}\rho_{ij},
$$

where

$$
\begin{aligned}
\rho_{ij} &= \text{correlation coefficient between default events} \\
&= \frac{\mathrm{cov}(\mathbf{1}_{D_i}, \mathbf{1}_{D_j})}{\sqrt{\mathrm{var}(L_i)\mathrm{var}(L_j)}},
\end{aligned}
$$

with $\mathrm{var}(L_i) = DP_i(1 - DP_i)$.

Example

Take $m = 2, LGD_i = EAD_i = 1, i = 1, 2$, then

$$UL_p^2 = p_1(1 - p_1) + p_2(1 - p_2) + 2\rho\sqrt{p_1(1 - p_1)p_2(1 - p_2)},$$

where p_i is the default probability of obligor i, $i = 1, 2$, and ρ is the correlation coefficient.

(i) When $\rho = 0$, the two default events are uncorrelated. Under full diversification with widely different assets of diversified classes, the correlation is viewed as being close to zero.

(ii) When $\rho > 0$, the default of one counterparty increases the likelihood that the other counterparty may also default. Consider

$$P[L_2 = 1|L_1 = 1] = \frac{P[L_2 = 1, L_1 = 1]}{P[L_1 = 1]} = \frac{E[L_1 L_2]}{p_1}$$
$$= \frac{p_1 p_2 + \text{cov}(L_1, L_2)}{p_1} = p_2 + \frac{\text{cov}(L_1, L_2)}{p_1}.$$

Positive correlation leads to a conditional default probability higher than the unconditional default probability p_2 of obligor 2.

- Under the case of perfect correlation and $p = p_1 = p_2$, we have

$$UL_p = 2\sqrt{p(1-p)}.$$

This means the portfolio contains the risk of only one obligor but with double intensity (concentration risk). The default of one obligor makes the other obligor defaulting almost surely.

Relations between portfolio variance UL_p^2 and individual unexpected loss UL_i

Recall

$$
\begin{aligned}
UL_p^2 &= \mathrm{var}(\tilde{L}_p) = \mathrm{var}\Big(\sum_{i=1}^m \tilde{L}_i\Big) \\
&= \sum_{i=1}^m \sum_{j=1}^m \mathrm{cov}(\tilde{L}_i, \tilde{L}_j) = \sum_{i=1}^m \sum_{j=1}^m UL_i UL_j \rho_{ij},
\end{aligned}
$$

where $UL_i^2 = \mathrm{var}(\tilde{L}_i)$, $i = 1, 2, \ldots, m$. Note that

$$
\frac{\partial UL_p^2}{\partial UL_i} = 2UL_p \frac{\partial UL_p}{\partial UL_i} = 2 \sum_{j=1}^m UL_j \rho_{ij},
$$

giving

$$
\frac{\partial UL_p}{\partial UL_i} = \sum_{j=1}^m \rho_{ij} \frac{UL_j}{UL_p}.
$$

Risk contribution

The risk contribution of a risky asset i to the portfolio unexpected loss is defined to be the *incremental risk* that the exposure of a single asset contributes to be the portfolio's total risk, namely,

$$RC_i = UL_i \frac{\partial UL_p}{\partial UL_i} = \frac{UL_i \sum_j UL_j \rho_{ij}}{UL_p}.$$

Using the unexpected losses UL_i and UL_p as the quantifiers of risk, we expect that the risk contributions from the risky assets is simply the total portfolio risk. As a verification, it is seen mathematically that

$$\sum_i RC_i = \frac{\sum_i UL_i \sum_j UL_j \rho_{ij}}{UL_p} = UL_p.$$

Calculation of EL, UL and RC for a two-asset portfolio

ρ	default correlation coefficient between the two exposures
EL_p	portfolio expected loss $EL_p = EL_1 + EL_2$
UL_p	portfolio unexpected loss $UL_p = \sqrt{UL_1^2 + UL_2^2 + 2\rho UL_1 UL_2}$
RC_1	risk contribution from Exposure 1 $RC_1 = UL_1(UL_1 + \rho UL_2)/UL_p$
RC_2	risk contribution from Exposure 2 $RC_2 = UL_2(UL_2 + \rho UL_1)/UL_p$

$$UL_p = RC_1 + RC_2$$

Features of portfolio risk

- The variability of default risk within a portfolio is substantial.
- The correlation between default risks is generally low.
- The default risk itself is dynamic and subject to large fluctuations.
- Default risks can be effectively managed through diversification.
- Within a well diversified portfolio, the loss behavior is characterized by lower than expected default credit losses for much of the time but very large losses which are incurred infrequently.

Monte Carlo simulation of losses

In a Monte Carlo simulation, losses are simulated and tabulated in the form of a histogram. Assume that we have simulated n runs of potential portfolio losses $\tilde{L}_p^{(1)}, \ldots, \tilde{L}_p^{(n)}$, hereby taking the driving distributions of the single loss variables and their correlations into account.

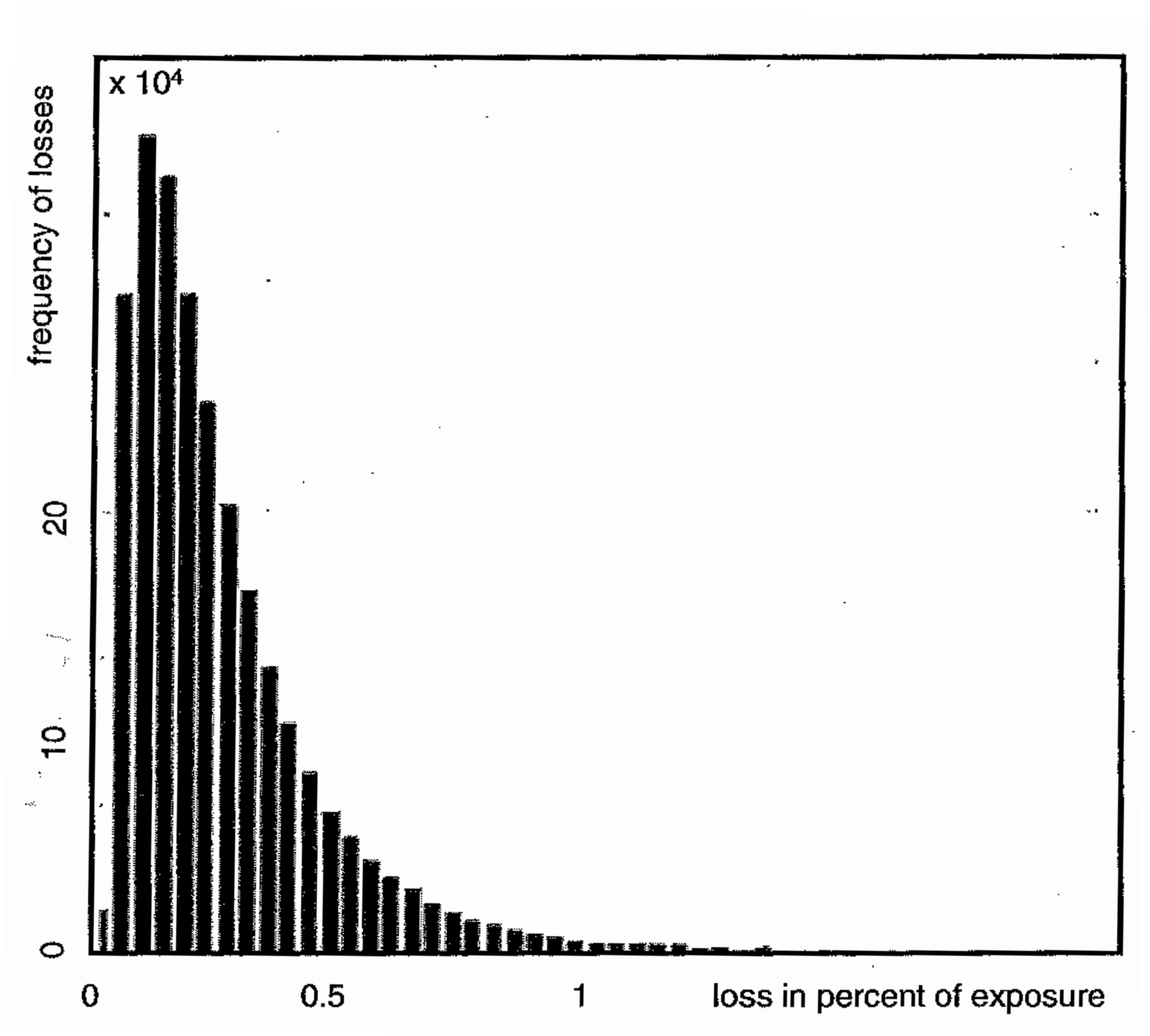
Define the indicator function, which is assigned the value one if y falls within $[0, x]$ and zero otherwise:

$$\mathbf{1}_{[0,x]}(y) = \begin{cases} 1 & y \leq x \\ 0 & y > x \end{cases}.$$

The empirical loss cumulative distribution function is given by

$$F(x) = \frac{1}{n} \sum_{j=1}^{n} \mathbf{1}_{[0,x]}(\tilde{L}_p^{(j)}).$$

That is, for a given value of x, we calculate the proportion of simulated portfolio losses out of n simulations where $\tilde{L}_p^{(j)}$ falls within $[0, x]$.



An empirical portfolio loss frequency distribution obtained by Monte Carlo simulation. The histogram is based on a portfolio of 2,000 corporate loans.

Monte Carlo simulation of loss distribution of a portfolio

1. *Estimate default probability and losses*
   Assign risk ratings to loss facilities and determine their default probability. Also, assign specified values of LGD_s and σ^s_{LGD}.

2. *Estimate asset correlation between obligors*
   Determine pairwise asset correlation whenever possible or assign obligors to industry groupings, then determine industry pair correlation. For example, every obligor in Industry A and every obligor in Industry B share the same correlation ρ_{AB}.

3. *Generate the random loss given default*

The LGD is a stochastic variable with an unknown distribution.

A typical example may be

	Recovery rate (%)	LGD (%)	σ_{LGD} (%)
secured	65	35	21
unsecured	50	50	28

$$LGD_i = LGD_s + f_i \times \sigma^s_{LGD}$$

where f_i is drawn from a uniform distribution whose range is selected so that the resulting LGD has a standard deviation that is consistent with historical observation.

1. Generate a set of random numbers drawn from a standard normal distribution.

2. Perform a cholesky decomposition on the asset correlation matrix to transform the independent set of random numbers (stored in the vector $\boldsymbol{e}$) into a set of correlated asset values (stored in the vector $\boldsymbol{e}'$). Recall that the covariance matrix Σ is symmetric and positive definite (neglect the unlikely event that Σ has the zero eigenvalue). By performing the usual LDU factorization, we have

$$\Sigma = LDU = L\sqrt{D}\sqrt{D}L^T = MM^T,$$

where $\sqrt{D}$ is the diagonal matrix whose diagonal entries are the positive square roots of those in D, and $M = L\sqrt{D}$. We compute $\boldsymbol{e}'$ via the transformation: $\boldsymbol{e}' = M\boldsymbol{e}$. As a check, we consider the covariance matrix of $\boldsymbol{e}'$:

$$E[\boldsymbol{e}'\boldsymbol{e}'^T] = E[M\boldsymbol{e}\boldsymbol{e}^T M^T] = ME[\boldsymbol{e}\boldsymbol{e}^T]M^T = MIM = \Sigma.$$

Calculation of the default point

We assume the credit indexes to be standard normal random variables, which are correlated with the covariance matrix Σ.

- The default point threshold, DP, of the i^{th} obligor can be defined as default probability $= N(DP)$.
- For example, $N(-2.5) = 0.0062 = 0.62\%$; that is, the default point threshold equals -2.5 when the default probability is 0.62%. For obligor i, we take

$$\text{default if } \boldsymbol{e}_i' < DP_i$$
$$\text{no default if } \boldsymbol{e}_i' \geq DP_i.$$

Calculation of loss

Summing all the simulated losses from one single scenario

$$\text{Loss} = \sum_{\substack{\text{Obligors}\\ \text{in default}}} \text{exposure}_i \times \text{LGD}_i$$

Simulated loss distribution

The simulated loss distribution is obtained by repeating the above process sufficiently number of times.

Fitting of loss distribution

The two statistical measures about the credit portfolio are

1. mean, or called the portfolio expected loss;

2. standard deviation, or called the portfolio unexpected loss.

At the simplest level, we approximate the loss distribution of the original portfolio by a *beta distribution* through matching the first and second moments of the portfolio loss distribution.

The risk quantiles of the original portfolio can be approximated by the respective quantities of the approximating random variable X. The price for such convenience of fitting is *model risk*.

Reservation

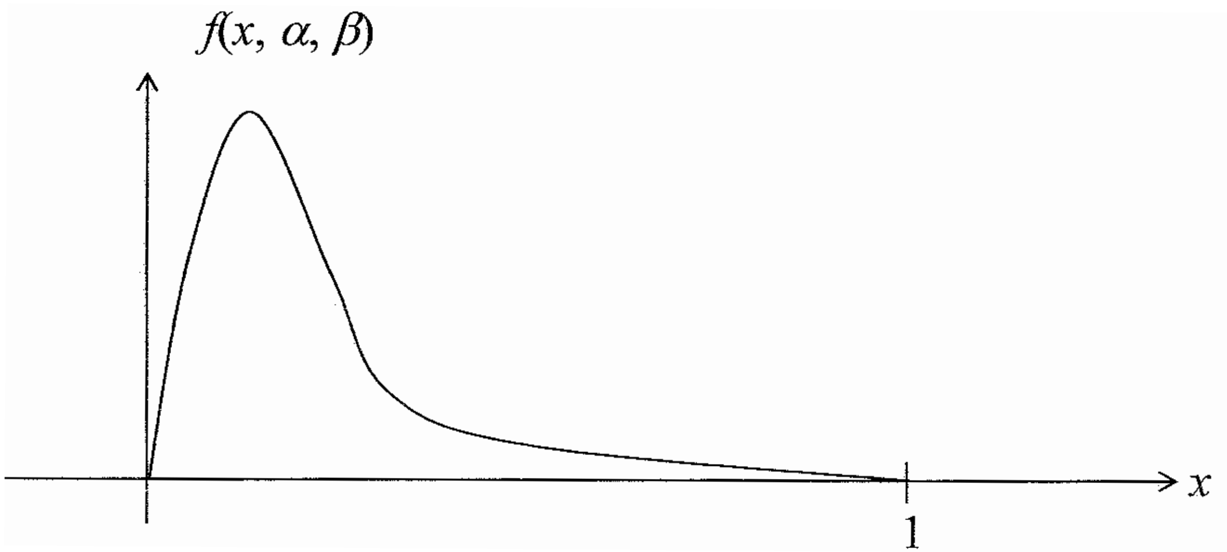
A beta distribution with only two degrees of freedom is perhaps insufficient to give an adequate description of the tail events in the loss distribution.

Beta distribution

The density function of a beta distribution is

$$f(x,\alpha,\beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \alpha > 0, \beta > 0,$$

where $\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1}\, dx$.
Mean $\mu = \frac{\alpha}{\alpha+\beta}$ and variance $\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.



Third and fourth order moments of a distribution

- Skewness describes the departure from symmetry:

$$\gamma = \frac{\int_{-\infty}^{\infty}(x - E[X])^3 f(x)\ dx}{\sigma^3}.$$

The skewness of a normal distribution is zero. Positive skewness indicates that the distribution has a long right tail and so entails large positive values.

- Kurtosis describes the degree of flatness of a distribution:

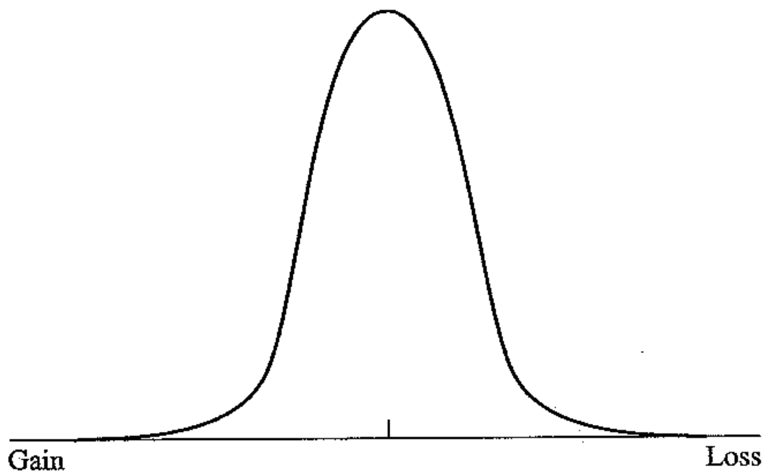
$$\delta = \frac{\int_{-\infty}^{\infty}(x - E[X])^4 f(x)\ dx}{\sigma^4}.$$

The kurtosis of a normal distribution is 3. A distribution with kurtosis greater than 3 has the tails decay less quickly than that of the normal distribution, implying a greater likelihood of large value in both tails.

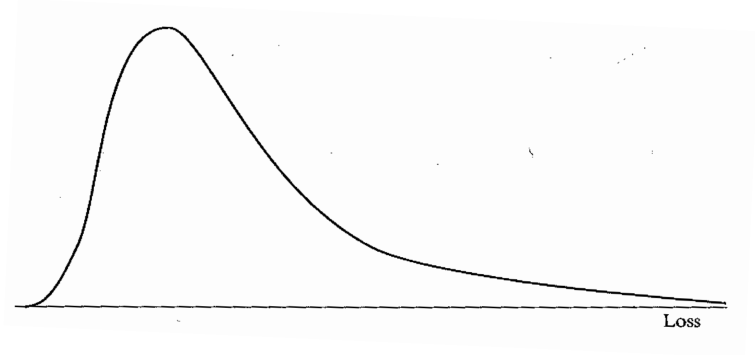
Characteristics of loss distributions for different risk types

Type of risk	*Second moment (standard deviation)*	*Third moment (skewness)*	*Fourth moment (kurtosis)*
Market risk	High	Zero	Low
Credit risk	Moderate	Moderate	Moderate
Operational risk	Low	High	High

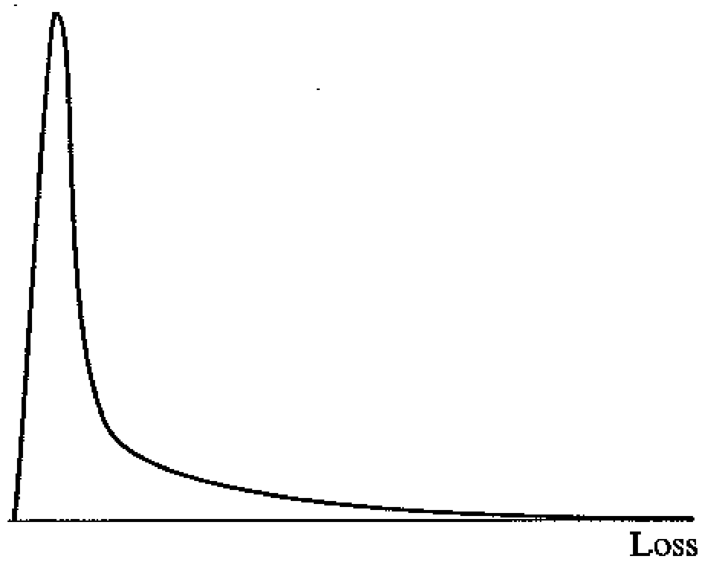
- The market risk loss distribution is symmetrical but not perfectly normally distributed.



- The credit risk loss distribution is quite skewed.

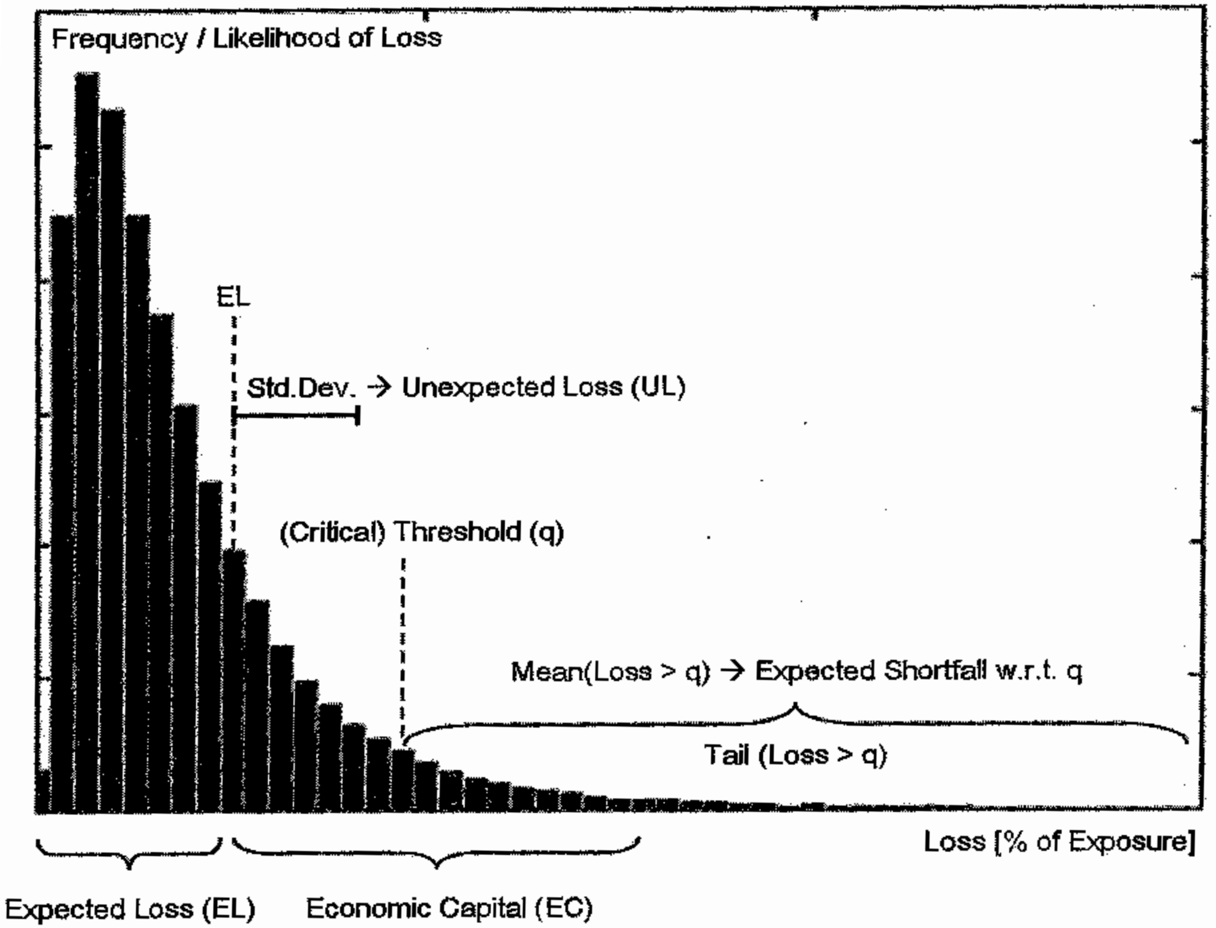


- The operational risk distribution has a quite extreme shape. Most of the time, losses are modest, but occasionally they are very large.



Loss distribution of a credit portfolio

- All risk quantities on a portfolio level are based on the portfolio loss variable $\tilde{L}_p$. Once the loss distribution is generated, all risk measures can be calculated (details to be discussed in Topic 2.3).

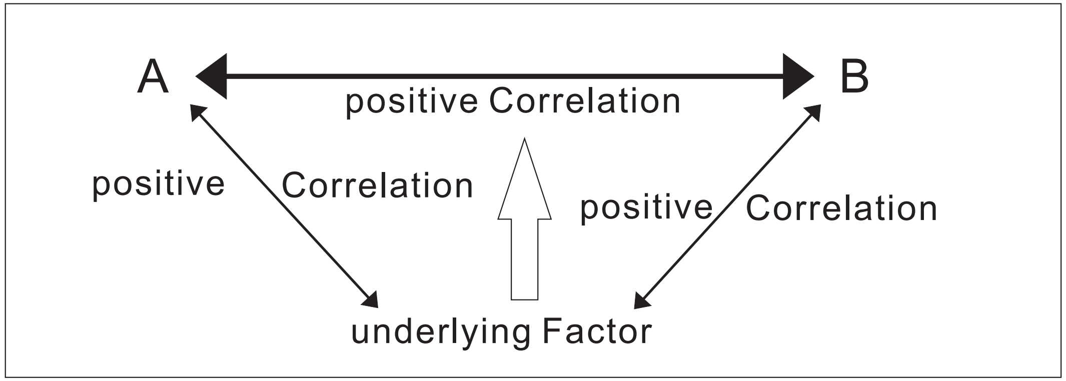


The loss distribution of a credit portfolio

3.2 Global Correlation Model using factor models approach

- Use the factor models approach from multivariate statistics for identifying the underlying drivers of correlated defaults and for reducing the computational effort regarding the calculation of correlated losses.

- Firms A and B are positively correlated via their positive correlation with a common underlying factor.



Correlation induced by an underlying factor.

Example – Automotive Industry

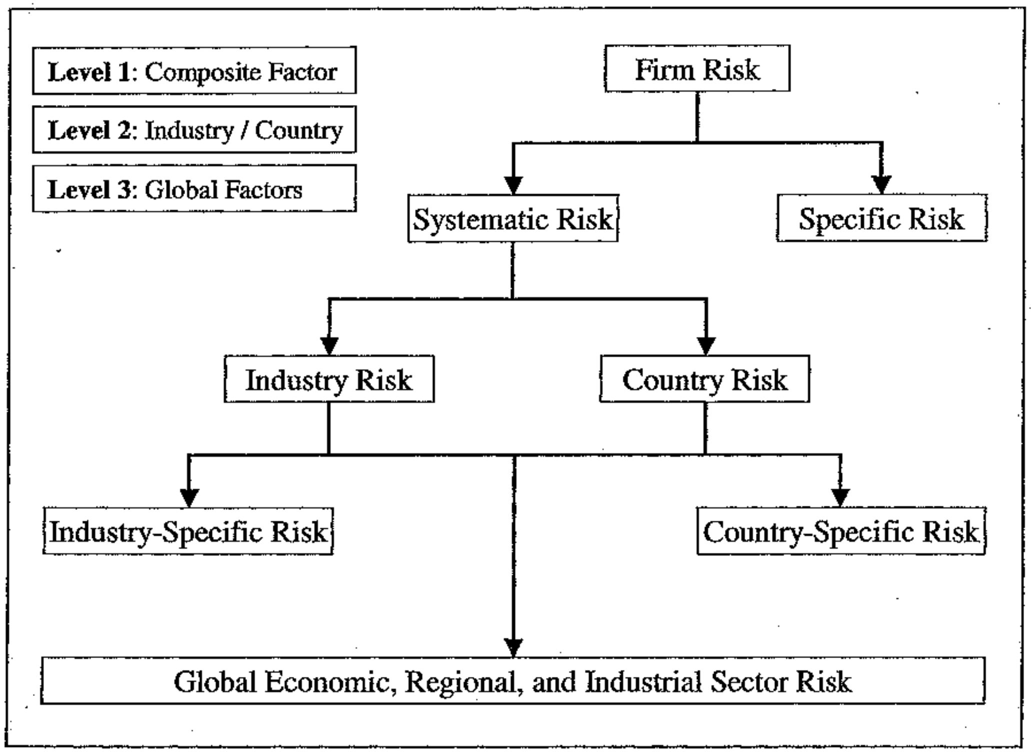
- DaimlerChryler is influenced by a factor for Germany, US and some factors incorporating Aerospace and Financial Companies.

- BMW is correlated with a country factor for Germany and some other factors.

  The underlying factors should be interpretable. For example, suppose the automotive industry faces pressure, DaimlerChryler and BMW are also under pressure at the same time.

- The part of the volatility of a company's asset value process related to systematic factors like industries or countries is called the *systematic risk* of the firm.

- The other part of the firm asset's volatility that cannot be explained by systematic influences is called the specific or idiosyncratic risk of the firm.



Three-level factor structure in the *Global Correlation Model*.

1. First level decomposition

- Decomposition of a firm's variance in a systematic and a specific part.

Consider the asset value log-return r_i of obligor i, as represented by

$$r_i = \beta_i \phi_i + \epsilon_i, \quad i = 1, 2, \ldots, m. \tag{1}$$

Here, ϕ_i is called the composite factor of firm i and β_i is the factor sensitivity. In multi-factor models, ϕ_i typically is a weighted sum of several factors. Also, we assume ϕ_i to be uncorrelated with ϵ_i, the residual part of r_i.

The equation can be interpreted as a standard linear regression equation, where β_i captures the linear correlation of r_i and ϕ_i. This can be easily verified as follows:

$$\text{cov}(r_i, \phi_i) = \beta_i \text{var}(\phi_i) + \text{cov}(\epsilon_i, \phi_i)$$

so that $\beta_i = \text{cov}(r_i, \phi_i)/\text{var}(\phi_i)$.

In CAPM, β_i is called the beta of obligor i.

Taking variances on both sides of (1), we have

$$\text{var}(r_i) = \beta_i^2 \text{var}(\phi_i) + \text{var}(\epsilon_i), \quad i = 1, 2, \ldots, m.$$

The first term captures the variability of r_i coming from the variability of the composite factor (systematic risk). The second term arises from the variability of the residual variable, var(ϵ_i), which is usually called the specific risk.

Define

$$R_i^2 = \frac{\beta_i^2 \text{var}(\phi_i)}{\text{var}(r_i)}$$

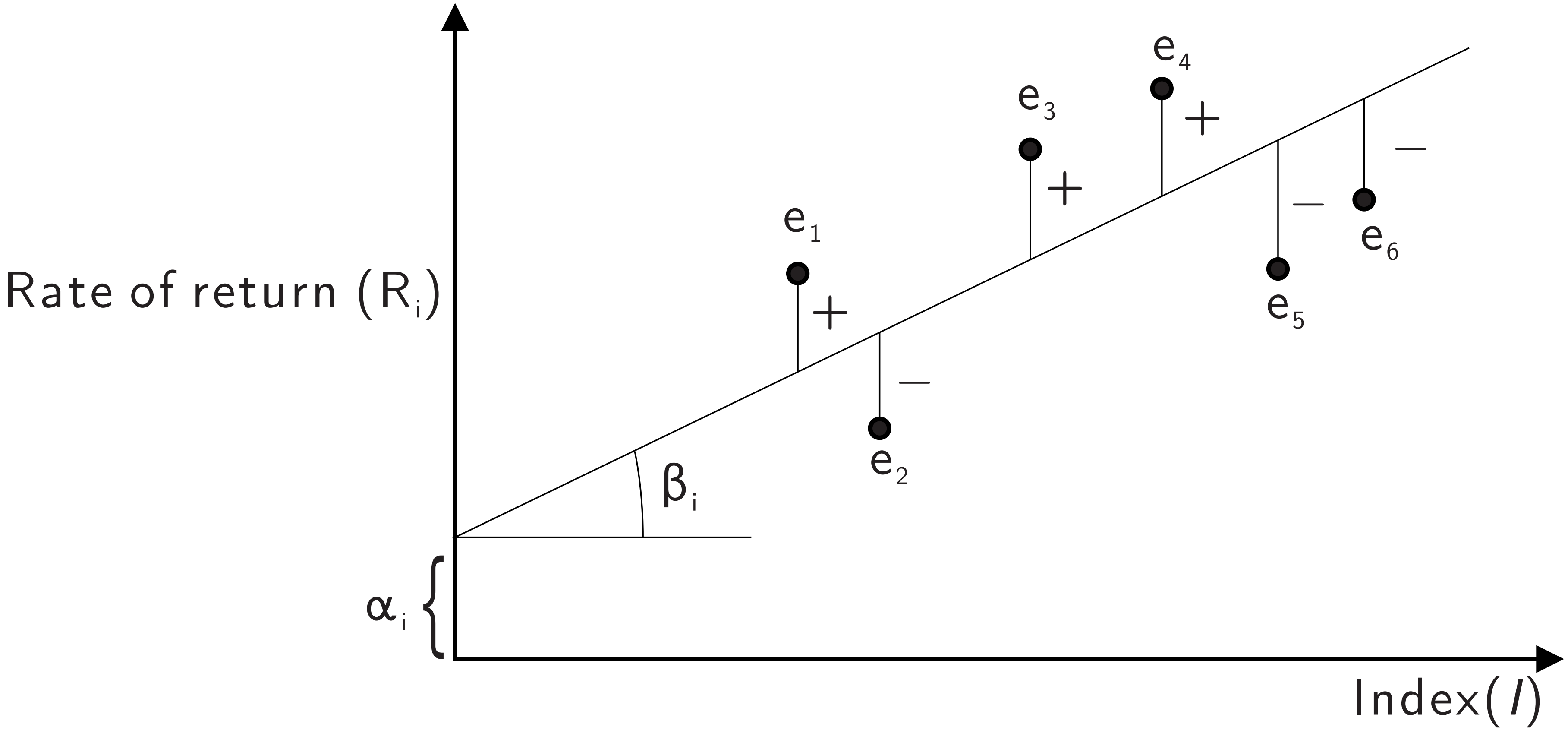
so that the residual part of the total variance of r_i is $1 - R_i^2$.

We may write

$$\mathbf{r} = B\mathbf{\Phi} + \boldsymbol{\epsilon},$$

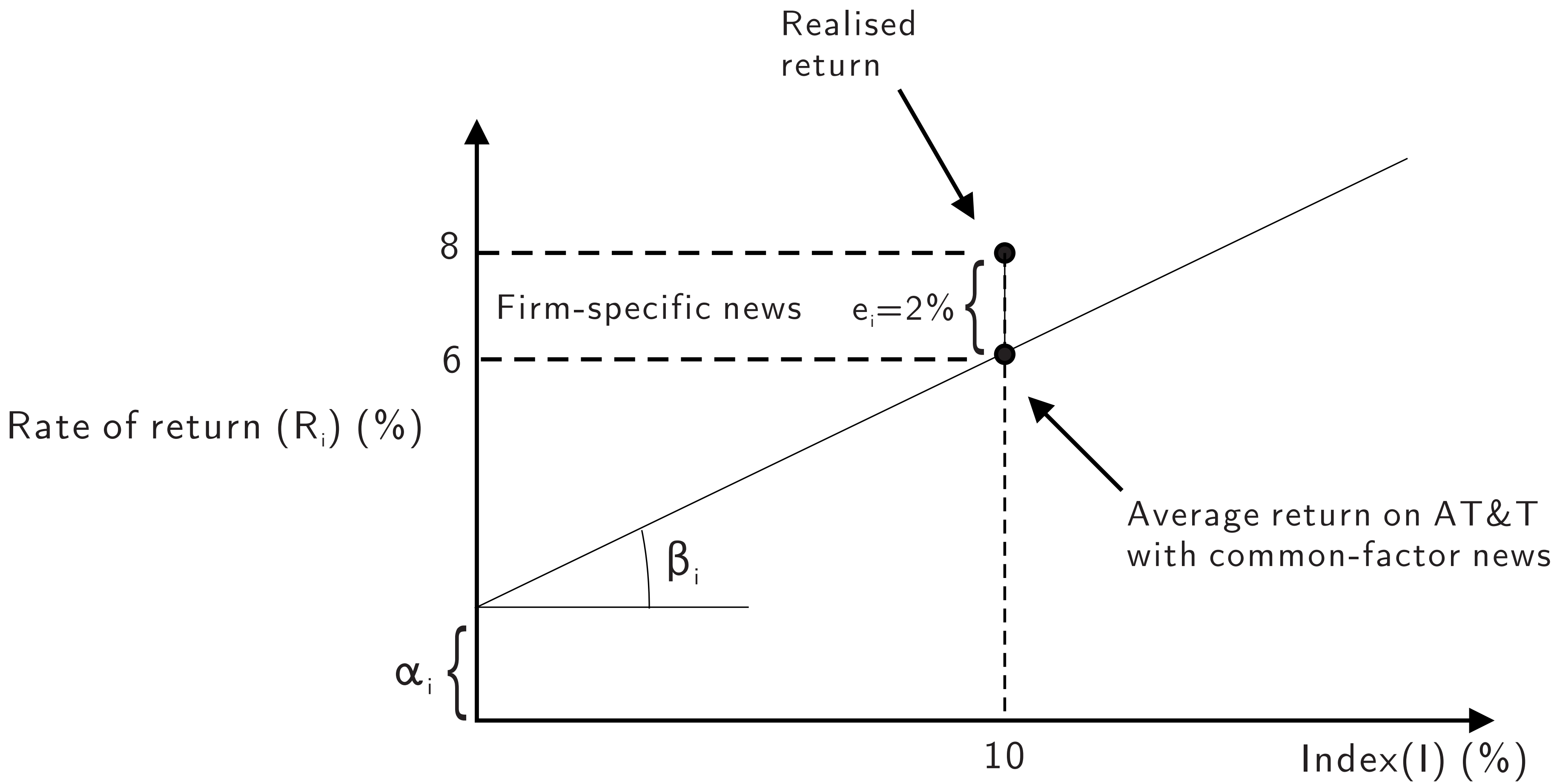
where $\mathbf{\Phi} = (\phi_1 \cdots \phi_m)^T$ and B is a diagonal matrix in $\mathbb{R}^{m \times m}$ whose diagonal elements are $B_{ii} = \beta_i$, $i = 1, 2, \ldots, m$.

Single index model



$$R_i = \alpha_i + \beta_i I + e_i$$

Systematic and unsystematic risk as a function of the number of assets in the portfolio



2. Second-level decomposition

We decompose each composite factor ϕ_i with respect to industry and country breakdown, where

$$\phi_i = \sum_{k=1}^{K} w_{ik}\psi_k, \quad i = 1, 2, \ldots, m.$$

Here, $\psi_1, \ldots, \psi_{K_0}$ are industry indexes and $\psi_{K_0+1}, \ldots, \psi_K$ are country indexes. The industry weights and country weights are summed to one, that is,

$$\sum_{k=1}^{K_0} w_{ik} = \sum_{k=K_0+1}^{K} w_{ik} = 1, \quad i = 1, 2, \ldots, m.$$

Define $W = (w_{ik})_{i=1,\ldots,m;k=1,\ldots,K}$ so that

$$\mathbf{r} = BW\boldsymbol{\psi} + \boldsymbol{\epsilon},$$

where $\boldsymbol{\psi} = (\psi_1 \ldots \psi_K)^T$ is the vector of industry and country indexes.

3. *Third-level decomposition*

We decompose each industry or country index into a weighted sum of independent global factors plus residuals, where

$$\psi_k = \sum_{n=1}^{N} C_{kn}\Gamma_n + \delta_k, \quad k = 1, 2, \ldots, K.$$

In vector notation, we have

$$\boldsymbol{\Psi} = C\boldsymbol{\Gamma} + \boldsymbol{\delta},$$

where $C = (C_{kn})_{k=1,\ldots,K;n=1,\ldots,N}$ denotes the matrix of industry and country betas, $\boldsymbol{\Gamma} = (\Gamma_1 \ldots \Gamma_N)^T$ is the global factor vector, and $\boldsymbol{\delta} = (\delta_1 \ldots \delta_K)^T$ is the vector of industry and country residuals. Putting all relations together:

$$\mathbf{r} = BW(C\boldsymbol{\Gamma} + \boldsymbol{\delta}) + \boldsymbol{\epsilon}.$$

Examples of factors

- S & P index
- US inflation rates
- aggregate sales
- oil prices
- US/Euro exchange rate
- return in long position in small capital stocks less return on large capital stocks

Normalization of asset value log-returns

Define

$$\tilde{r}_i = \frac{r_i - E[r_i]}{\sigma_i}, \quad \text{where } \sigma_i^2 = \text{var}(r_i), \quad i = 1, 2, \ldots, m.$$

We then write

$$\tilde{r}_i = \frac{\beta_i}{\sigma_i}\tilde{\phi}_i + \frac{\tilde{\epsilon}_i}{\sigma_i}, \quad \text{where } E[\tilde{\phi}_i] = E[\tilde{\epsilon}_i] = 0.$$

The asset correlation is given by

$$\text{cov}(\tilde{r}_i, \tilde{r}_j) = E[\tilde{r}_i \tilde{r}_j] = \frac{\beta_i}{\sigma_i}\frac{\beta_j}{\sigma_j}E[\tilde{\phi}_i\tilde{\phi}_j],$$

since the residual $\tilde{\epsilon}_i$ is uncorrelated to the composite factors. Note that

$$R_i^2 = \frac{\beta_i^2}{\sigma_i^2}\text{var}(\phi_i)$$

so that

$$\text{cov}(\tilde{r}_i, \tilde{r}_j) = \frac{R_i}{\sqrt{\text{var}(\phi_i)}}\frac{R_j}{\sqrt{\text{var}(\phi_j)}}E[\tilde{\phi}_i\tilde{\phi}_j].$$

The normalized quantities are related by

$$\tilde{\mathbf{r}} = \tilde{B}W(C\tilde{\mathbf{\Gamma}} + \tilde{\boldsymbol{\delta}}) + \tilde{\boldsymbol{\epsilon}},$$

where $\tilde{B}$ is obtained by scaling every diagonal element in $\mathbf{B}$ by $1/\sigma_i$, and

$$E[\tilde{\mathbf{\Gamma}}] = E[\tilde{\boldsymbol{\epsilon}}] = E[\tilde{\boldsymbol{\delta}}] = \mathbf{0}.$$

The asset correlation matrix is obtained by computing

$$E[\tilde{\mathbf{\Phi}}\tilde{\mathbf{\Phi}}^T] = W(CE[\tilde{\mathbf{\Gamma}}\tilde{\mathbf{\Gamma}}^T]C^T + E[\tilde{\boldsymbol{\delta}}\tilde{\boldsymbol{\delta}}^T])W^T.$$

Note that $E[\tilde{\mathbf{\Gamma}}\tilde{\mathbf{\Gamma}}^T]$ is a diagonal matrix with diagonal elements var(Γ_n), $n = 1, 2, \ldots, N$, and $E[\tilde{\boldsymbol{\delta}}\tilde{\boldsymbol{\delta}}^T]$ is a diagonal matrix with diagonal element var(δ_k), $k = 1, 2, \ldots, K$. This is because we are dealing with uncorrelated global factors and uncorrelated residuals.

3.3 VaR (value-at-risk), expected shortfall and coherent risk measure

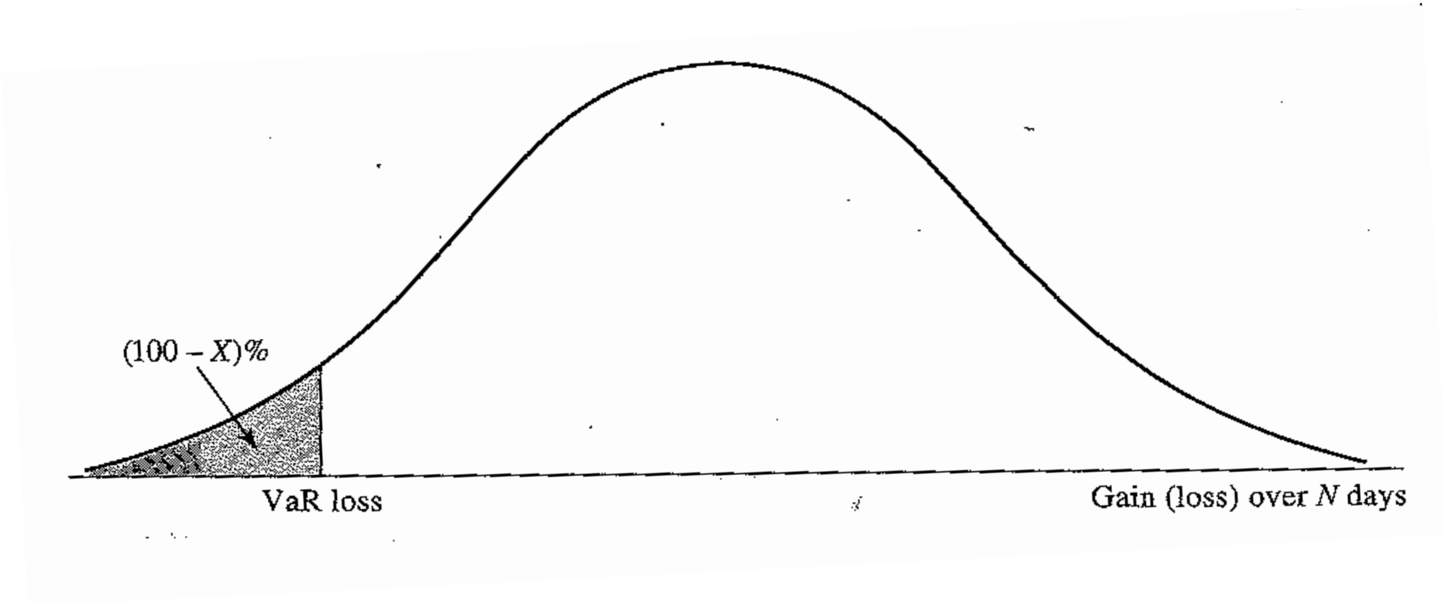
In simple terminology, the value-at-risk measure can be translated as "I am X percent certain there will not be a loss of more than V dollars in the next N days." In essence, it asks the simple question: "How bad can things get?".

- The variable V is the VaR of the portfolio. It is a function of (i) time horizon (N days); (ii) confidence level (X%).

- It is the loss level over N days that has a probability of only $(100-X)$% of being exceeded.

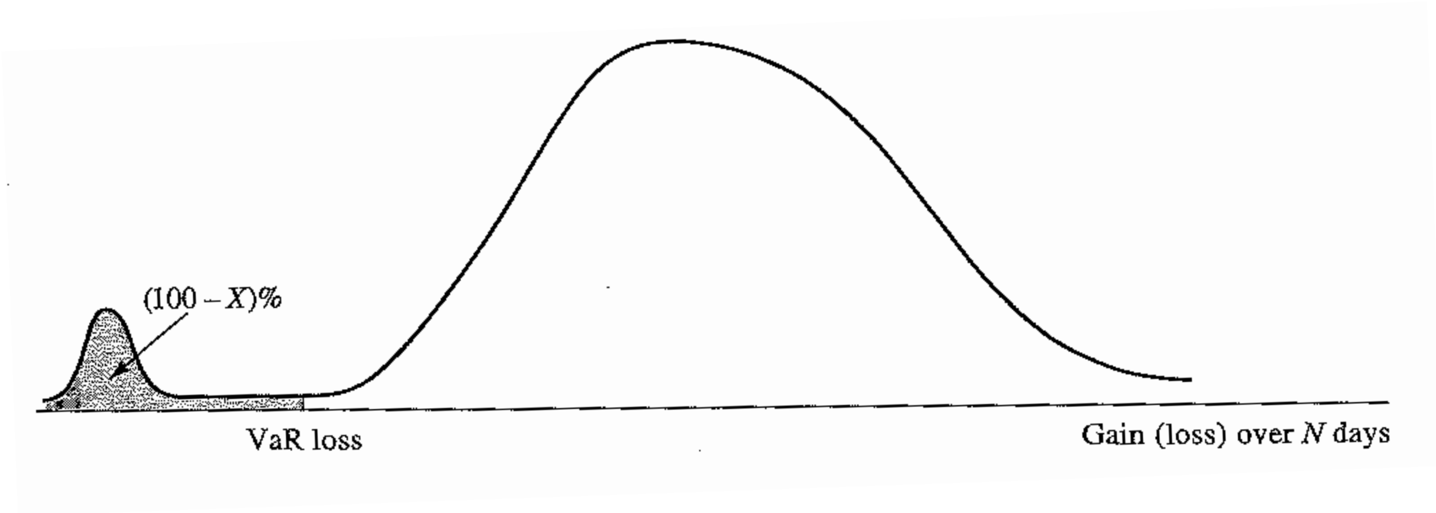
- Bank regulators require banks to calculate VaR for market risk with $N = 10$ and $X = 99$.

- Weakness of volatility as the measure of risk: it does not care about the direction of portfolio value movement.



Calculation of VaR from the probability distribution of the change in the portfolio value; confidence level is $X\%$. Gains in portfolio value are positive; losses are negative.

- VaR disregards any loss beyond the VaR Level (tail risk)



Alternative situation where VaR is the same, but the potential loss is larger.

- VaR is unreliable under market stress. Under extreme asset price fluctuations or an extreme dependence structure of assets (clustering effect), VaR may underestimate risk.

Formal definition

VaR is defined for a probability measure P and some confidence level α as the α-quantile of a loss random variable X

$$\text{var}_\alpha(X) = \text{inf}\{x \geq 0 | P[X \leq x] \geq \alpha\}.$$

For example, take $\alpha = 99\%$ and one-month horizon; the above definition states that with 99% chance that the loss amount (value of X) is less than $\text{VaR}_\alpha(X)$ within the one-month period.

Banks should hold some capital cushion against unexpected losses. Using UL is not sufficient since there might be a significant likelihood that losses will exceed portfolio's EL by more than one standard deviation of the portfolio loss. Unlike VaR, there is no consideration of confidence level in this type of risk measure.

- Unfortunately, all risk measures that rely on one absolute value and a single probability (confidence level) are subject to game playing by fund managers.

Example 1 – Portfolio gain treated as a normal random variable

Suppose that the gain from a portfolio during six months is normally distributed with a mean of \$2 million and a standard deviation of \$10 million.

Recall the cumulative normal distribution:

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt,$$

and $N(-2.33) = 0.01 = 1\%$.

- From the properties of the normal distribution, the one-percentile point of this distribution is $2 - 2.33 \times 10$, or -\$21.3 million.
- The VaR for the portfolio with a time horizon of six months and confidence level of 99% is therefore \$21.3 million.

Example 2

Suppose that for a one-year project all outcomes between a loss \$50 million and a gain of \$50 million are considered equally likely.

- The loss from the project has a uniform distribution extending from $-\$50$ million to $+\$50$ million. There is a 1% chance that there will be a loss greater than \$49 million.

- The VaR with a one-year time horizon and a 99% confidence level is therefore \$49 million.

Calculation of VaR using historical simulation

Suppose that VaR is to be calculated for a portfolio using a 1-day time horizon, a 99% confidence level, and 501 days of data.

- The first step is to identify the market variables affecting the portfolio. These are typically exchange rates, equity prices, interest rates, etc.

- Data is then collected on the movements in these market variables over the most recent 501 days. This provides 500 alternative scenarios for what can happen between today and tomorrow.

- Scenario 1 is where the percentage changes in the values of all variables are the same as they were between Day 0 and Day 1, scenario 2 is where they are the same as they were between Day 1 and Day 2, and so on.

Data for VaR historical simulation calculation.

Day	*Market variable 1*	*Market variable 2*	...	*Market variable n*
0	20.33	0.1132	...	65.37
1	20.78	0.1159	...	64.91
2	21.44	0.1162	...	65.02
3	20.97	0.1184	...	64.90
⋮	⋮	⋮	⋮	⋮
498	25.72	0.1312	...	62.22
499	25.75	0.1323	...	61.99
500	25.85	0.1343	...	62.10

Define v_i as the value of a market variable on Day i and suppose that today is Day m. The ith scenario assumes that the value of the market variable tomorrow will be $v_m \frac{v_i}{v_{i-1}}$.

For the first variable, the value today, v_{500}, is 25.85. Also $v_0 = 20.33$ and $v_1 = 20.78$. It follows that the value of the first market variable in the first scenario is $25.85 \times \frac{20.78}{20.33} = 26.42$. For the second scenario, we have $25.85 \times \frac{21.44}{20.78} = 26.67$.

Scenarios generated for tomorrow (Day 501) using data in the last table.

Scenario number	*Market variable 1*	*Market variable 2*	...	*Market variable n*	*Portfolio value (\$ millions)*	*Change in value (\$ millions)*
1	26.42	0.1375	...	61.66	23.71	0.21
2	26.67	0.1346	...	62.21	23.12	-0.38
3	25.28	0.1368	...	61.99	22.94	-0.56
⋮	⋮	⋮	⋮	⋮	⋮	⋮
499	25.88	0.1354	...	61.87	23.63	0.13
500	25.95	0.1363	...	62.21	22.87	-0.63

Based on today's portfolio value of $23,50$, the change in portfolio in Day 1 and Day 2 are, respectively, $23.71 - 23.50 = 0.21$ and $23.12 - 23.50 = -0.38$.

How to estimate the 1-percentile point of the distribution of changes in the portfolio value?

Since there are a total of 500 scenarios, we can estimate this as the fifth worst number in the final column of the table. With confidence level of 95%, the maximum loss does not exceed the fifth worst number. The N-day VaR for a 99% confidence level is calculated as $\sqrt{N}$ times the 1-day VaR.

Query

Why the choice of $\sqrt{N}$? This is associated with the property of dispersion that grows at square root of time.

Expected shortfall

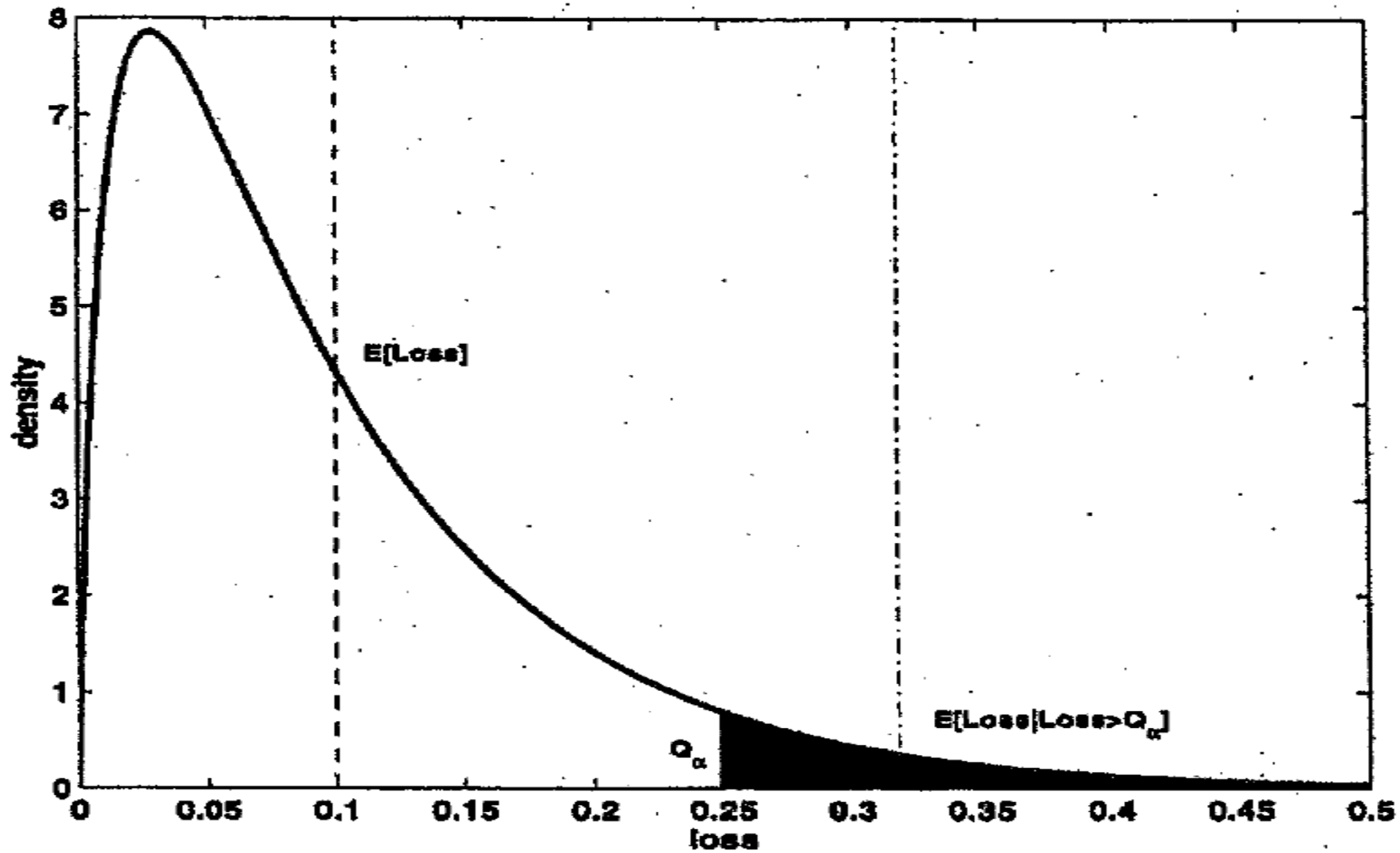
The expected shortfall *(tail conditional expectation)* with respect to a confidence level α is defined as

$$ES_\alpha(X) = \mathbb{E}[X|X > VaR_\alpha(X)].$$

Define by $c = VaR_\alpha(X)$, a *critical loss threshold* corresponding to some confidence level α, expected shortfall capital provides a cushion against the mean value of losses exceeding the critical threshold c.

The expected shortfall focusses on the expected loss in the tail, starting at c, of the portfolio's loss distribution.

- When the loss distribution is normal, VaR and expected shortfall give essentially the same information. Both VaR and ES are multiples of the standard deviation. For example, VaR at the 99% confidence level is 2.33σ while ES of the same level is 2.67σ.



Expected shortfall: $\mathbb{E}[X|X > VaR_\alpha(X)]$.

- The computation of the expected shortfall requires the information on the tail distribution (extreme value distribution).

Coherent risk measures

Let X and Y be two random variables, like the dollar loss amount of two portfolios. A risk measure is called a coherent measure if the following properties hold:

1. monotonicity

$$\text{For } X \leq Y, \quad \gamma(X) \leq \gamma(Y)$$

2. translation invariance

$$\text{For all } X \in \mathbb{R}, \quad \gamma(X + x) = \gamma(X) + x.$$

   This would imply $\gamma(X - \gamma(X)) = 0$ for every loss $X \in L$.

3. positive homogeneity

$$\text{For all } \lambda > 0, \quad \gamma(\lambda X) = \lambda \gamma(X)$$

4. subadditivity

$$\gamma(X + Y) \leq \gamma(X) + \gamma(Y)$$

Financial interpretation of the four properties

1. *Monotonicity*: If a portfolio produces a worse result than another portfolio for every state of the world, its risk measure should be greater.
2. *Translation invariance*: If an amount of cash K is added to a portfolio, its risk measure should go down by K. This is seem by setting $x = -k$, so that $\gamma(X - k) = \gamma(X) - k$.
3. *Positive homogeneity*: Changing the size of a portfolio by a positive factor λ, while keeping the relative amounts of different items in the portfolio the same, should result in the risk measure being multiplied by λ.
4. *Subadditivity*: The risk measure for two portfolio after they have been merged should be no greater than the sum of their risk measures before they were merged. This property reflects the benefit of diversification: $\sigma(X + Y) \leq \sigma(X) + \sigma(Y)$, where $\sigma(X)$ denotes the standard deviation of X.

VaR satisfies the first three conditions. However, it does not always satisfy the fourth one.

Example 3 – Violation of subaddivity in VaR

Suppose each of two independent projects has a probability of 0.02 of loss of \$10 million and a probability of 0.98 of a loss of \$1 million during a one-year period. Suppose we set the confidence level α to be 97.5%. In this example, the loss random variable X of single project can assume two discrete values: \$1 million and \$10 million.

- Since $P[X \leq 1] = 0.98$ and $P[X \leq 10] = 1$, so the one-year 97.5% VaR for each project is \$1 million.

- When the projects are put in the same portfolio, there is a $0.02 \times 0.02 = 0.0004$ probability of a loss of \$20 million, a $2 \times 0.02 \times 0.98 = 0.0392$ probability of a loss of \$11 million, and a $0.98 \times 0.98 = 0.9604$ probability of a loss of \$2 million.

- Let Y denote the loss random variable of the two projects. Since $P[Y \leq 2] = 0.9604$, $P[Y \leq 11] = 0.9996$ and $P[Y \leq 20] = 1$, so the one-year 97.5% VaR for the portfolio is \$11 million.

- The total of the VaRs of the projects considered separately is \$2 million. The VaR of the portfolio is therefore greater than the sum of the VaRs of the projects by \$9 million. This violates the subadditivity condition.

Example 4 – Violation of subaddivity in VaR

A bank had two \$10 million one-year loans, each of which has a 1.25% chance of defaulting. If a default occurs, all losses between 0% and 100% of the principal are equally likely. If the loan does not default, a profit of \$0.2 million is made. To simplify matters, we suppose that if one loan defaults it is certain that the other loan will not default.

1. Consider first a single loan. This has a 1.25% chance of defaulting. When a default occurs, the loss experienced is evenly distributed between zero and \$10 million. Conditional on a loss being made, there is an 80% (0.8) chance that the loss will be greater than \$2 million. Since the probability of a loss is 1.25% (0.0125), the unconditional probability of a loss greater than \$2 million is $0.8 \times 0.0125 = 0.01$ or 1%. The one-year, 99% VaR is therefore \$2 million.

2. Consider next the portfolio of two loans. Each loan defaults 1.25% of the time and they never default together so that

$$P[\text{one loss}] = 2 \times 1.25\% = 2.5\%$$
$$P[\text{two losses}] = 0.$$

Upon occurrence of a loan loss event, the loss amount is uniformly distributed between zero and \$10 million.

- There is a 2.5% (0.025) chance of one of the loans defaulting and conditional on this event there is an 40% (0.4) chance that the loss on the loan that defaults is greater than \$6 million.
- The unconditional probability of a loss from a default being greater than \$6 million is therefore $0.4 \times 0.025 = 0.01$ or 1%.

- In the event that one loan defaults, a profit of \$0.2 million is made on the other loan, showing that the one-year 99% VaR is \$5.8 million.

The total VaR of the loans considered separately is $2 + 2 = \$4$ million. The total VaR after they have been combined in the portfolio is \$1.8 million more at the value \$5.8 million. This shows that the subadditivity condition is violated.

Managers can game the VaR measure to report good risk management while exposing the firm to substantial risks

- Managers evaluated based upon VaR will be more focused on avoiding the intermediate risks but their portfolios can lose far more under the most adverse circumstances. One is prone to the possibility of large losses in the worst scenarios.

- VaR is measured using past data and it does not capture the trending up on market risks in future times.

Example 3 revised – Expected shortfall

In Example 3, the VaR for one of the projects considered on its own is \$1 million. To calculate the expected shortfall for a 97.5% confidence level we note that, of the 2.5% tail of the loss distribution, 2% corresponds to a loss is \$10 million and a 0.5% to a loss of \$1 million.

- Conditional that we are in the 2.5% tail of the loss distribution, there is therefore an 80% probability of a loss of \$10 million and a 20% probability of a loss of \$1 million. The expected loss is $0.8 \times 10 + 0.2 \times 1$ or \$8.1 million.

- When the two projects are combined, of the 2.5% tail of the loss distribution, 0.04% corresponds to a loss of \$20 million and 2.46% corresponds to a loss of \$11 million. Conditional that we are in the 2.5% tail of the loss distribution, the expected loss is therefore $(0.04/2.5) \times 20 + (2.46/2.5) \times 11$, or \$11.144 million. Since $8.1 + 8.1 >$ 11.144, the expected shortfall measure does satisfy the subadditivity condition for this example.

Example 4 revised – Expected shortfall

We showed that the VaR for a single loan is \$2 million. The expected shortfall from a single loan when the time horizon is one year and the confidence level is 99% is therefore the expected loss on the loan conditional on a loss greater than \$2 million is halfway between \$2 million and \$10 million, or \$6 million.

The VaR for a portfolio consisting of the two loans was calculated in Example 4 as \$5.8 million. The expected shortfall from the portfolio is therefore the expected loss on the portfolio conditional on the loss being greater than \$5.8 million.

- When one loan defaults, the other (by assumption) does not and outcomes are uniformly distributed between a gain of \$0.2 million and a loss of \$9.8 million.

- The expected loss, given that we are in the part of the distribution between \$5.8 million and \$9.8 million, is \$7.8 million. This is therefore the expected shortfall of the portfolio.

- Since \$7.8 million is less than $2 \times$ \$6 million, the expected shortfall measure does satisfy the subadditivity condition.

The subadditivity condition is not of purely theoretical interest. It is not uncommon for a bank to find that, when it combines two portfolios (e.g., its equity portfolio and its fixed income portfolio), the VaR of the combined portfolio goes up.

Impact on risk management with the lack of subadditivity in VaR

- Computing firmwide VaR is often a formidable task to perform. The alternative is to segment the computations by instruments and risk drivers, and to compute separate VaR's on tranches and desks of a company. This is quite necessary in financial institutions since the technological trading platforms are often desk by desk.

- With loss of subadditivity, the firmwide VaR cannot be properly assessed.

Risk control for expected utility-maximizing investors

Utility-maximizing investors with VaR constraint optimally choose to construct vulnerable positions that can result in large losses exceeding the VaR level.

Example

Suppose that an investor invests 100 million yen in the following four mutual funds:

- concentrated portfolio A, consisting of only one defaultable bond with 4% default rate;
- concentrated portfolio B, consisting of only one defaultable bond with 0.5% default rate;
- a diversified portfolio that consists of 100 defaultable bonds with 5% default rate;
- a risk-free asset.

We assume that the profiles of all bonds in these funds are as follows:

- maturity is one year
- occurrences of default events are mutually independent
- recovery rate is 10%, and yield to maturity is equal to the coupon rate.

The yield to maturity, default rate, and recovery rate are fixed until maturity.

Profiles of bonds included in the mutual funds

	Number of bonds included	Coupon (%)	Default rate(%)	Recovery rate(%)
Concentrated portfolio A	1	4.75	4.00	10
Concentrated portfolio B	1	0.75	0.50	10
Diversified portfolio	100	5.50	5.00	10
Risk-free asset	1	0.25	0.00	-

W final wealth,
W_0 initial wealth,
X_1 amount invested in concentrated portfolio A,
X_2 amount invested in concentrated portfolio B,
X_3 amount invested in diversified portfolio.

Assuming logarithmic utility, the expected utility of the investor is given below:

$$
\begin{aligned}
E[u(W)] = &\sum_{n=0}^{100} 0.96 \cdot 0.995 \cdot 0.05^n \cdot 0.95^{100-n} \cdot {}_{100}C_n \cdot \ln \tilde{w}(1,1) \\
&+ \sum_{n=0}^{100} 0.04 \cdot 0.995 \cdot 0.05^n \cdot 0.95^{100-n} \cdot {}_{100}C_n \cdot \ln \tilde{w}(0.1,1) \\
&+ \sum_{n=0}^{100} 0.96 \cdot 0.005 \cdot 0.05^n \cdot 0.95^{100-n} \cdot {}_{100}C_n \cdot \ln \tilde{w}(1,0.1) \\
&+ \sum_{n=0}^{100} 0.04 \cdot 0.005 \cdot 0.05^n \cdot 0.95^{100-n} \cdot {}_{100}C_n \cdot \ln \tilde{w}(0.1,0.1).
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{w}(a,b) = &1.0475 a X_1 + 1.0075 b X_2 + 1.055 X_3 \frac{100 - 0.9n}{100} \\
&+ 1.0025(W_0 - X_1 - X_2 - X_3).
\end{aligned}
$$

We analyze the impact of risk management with VaR and expected shortfall on the rational investor's decisions by solving the following five optimization problems, where the holding period is one year.

1. No constraint

$$\max_{\{X_1 X_2 X_3\}} E[u(W)].$$

2. Constraint with VaR at the 95% confidence level

$$\max_{\{X_1 X_2 X_3\}} E[u(W)]$$

subject to VaR(95% confidence level) $\leq$ 3.

3. Constraint with expected shortfall at the 95% confidence level

$$\max_{\{X_1 X_2 X_3\}} E[u(W)]$$

subject to expected shortfall(95% confidence level) $\leq$ 3.5.

4. Constraint with VaR at the 99% confidence level

$$\max_{\{X_1 X_2 X_3\}} E[u(W)]$$

subject to VaR(99% confidence level) $\leq$ 3.

5. Constraint with expected shortfall at the 99% confidence level

$$\max_{\{X_1 X_2 X_3\}} E[u(W)]$$

subject to expected shortfall(99% confidence level) $\leq$ 3.5.

- We analyze the effect of risk management with VaR and expected shortfall by comparing solutions (2)-(5) with solution (1).

- The solution of the optimization problem with a 95% VaR constraint shows that the amount invested in concentrated portfolio A is greater than that of solution (1): that is, the portfolio concentration is enhanced by risk management with VaR. While VaR is reduced from 3.35 (unconstrained case) to 3, the expected shortfall increases from 5.26 (unconstrained case) to 14.35.

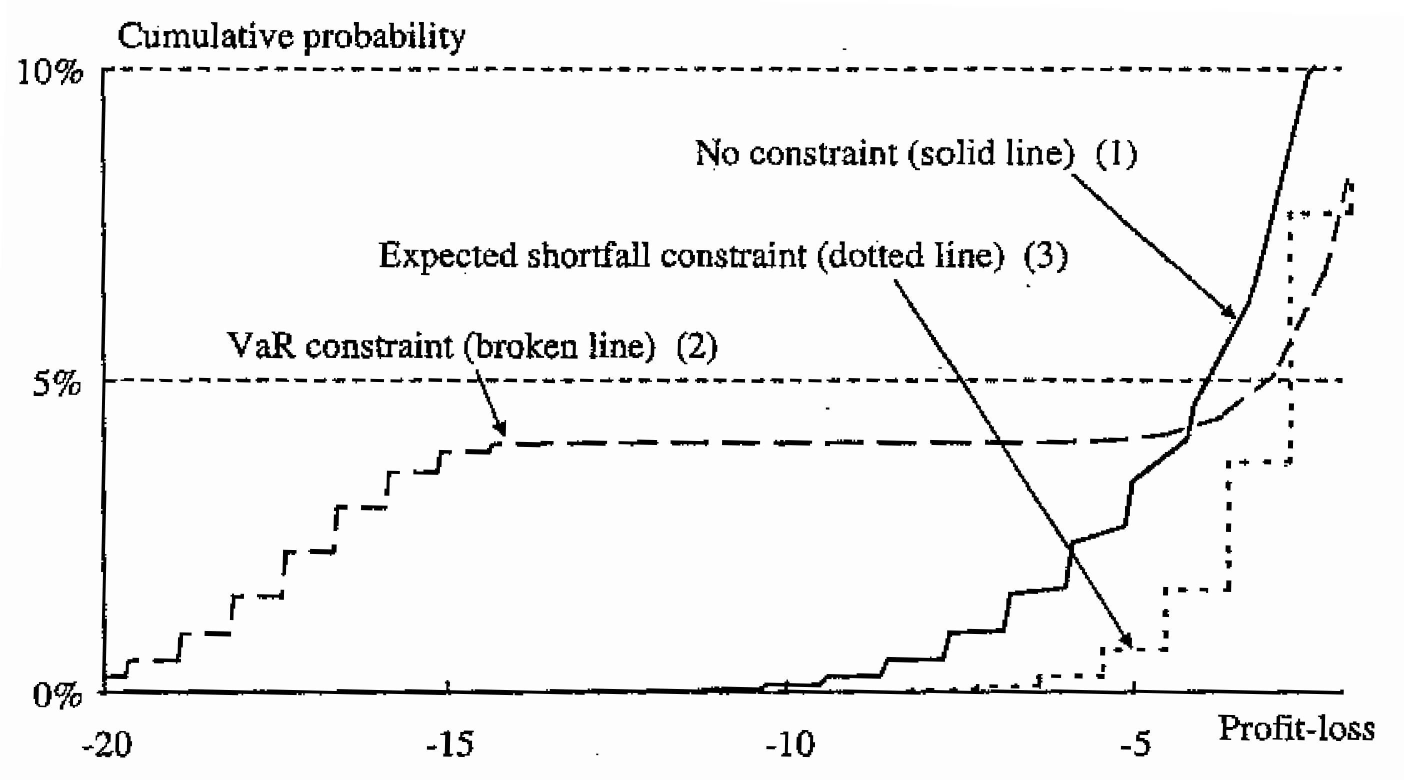
- The figure depicts the tails of the cumulative probability distributions of the profit-loss of the portfolios. The left tail under VaR constraint (95% confidence level) may suffer significant loss when bond A defaults (this risk is not well captured by $\mathrm{VaR}_{95\%}$).

Portfolio profiles (95% confidence level)

	No constraint (1)	VaR constraint[a] (2)	Expected shortfall constraint[b] (3)
Portfolio (%)			
Concentrated portfolio A (default rate: 4%)	7.4	20.1	2.9
Concentrated portfolio B (default rate: 0.5%)	0.0	0.0	2.0
Diversified portfolio	92.6	79.9	95.1
Risk-free asset	0.0	0.0	0.0
Risk measure (million yen)			
VaR	3.35	3.00	2.75
Expected shortfall	5.26	14.35	3.50

[a] Optimize with the constraint that VaR at the 95% confidence level is less than or equal to 3.

[b] Optimize with the constraint that expected shortfall at the 95% confidence level is less than or equal to 3.5.



Cumulative probability of profit-loss: the left tail (95% confidence level).

The stepwise increments of the cumulative distribution reflect the discrete loss amount upon default of a name in the portfolio.

Under VaR constraint

When constrained by VaR, the investor must reduce her investment in the diversified portfolio to reduce maximum losses with a 95% confidence level, and she should increase investments either in concentrated portfolio or in a risk-free asset.

- Concentrated portfolio A has little effect on VaR, since the probability of default lies beyond the 95% confidence interval. Concentrated portfolio A also yields a higher return than other assets, except diversified portfolio. Thus, the investor chooses to invest in concentrated portfolio A.

- Although VaR is reduced, the optimal portfolio is vulnerable due to its concentration and larger losses under conditions beyond the VaR level.

Under expected shortfall constraint

When constrained by expected shortfall, there is a higher chance that the investor chooses optimally to reallocate his investment to a risk-free asset, significantly reducing the portfolio risk.

- The investor cannot increase his investment in the concentrated portfolio without affecting expected shortfall, which takes into account the losses beyond the VaR level.

- Unlike risk management with VaR, risk management with expected shortfall does not enhance credit concentration.

Raising the confidence level to 99%

- We examine whether raising the confidence level of VaR solves the problem. The new table gives the results of the optimization problem with a 99% VaR or expected shortfall constraint. It shows that when constrained by VaR at the 99% confidence level, the investor optimally chooses to increase his/her investment in concentrated portfolio B because the default rate of concentrated portfolio B is 0.5%, outside the confidence level of VaR.

- Risk management with expected shortfall reduces the potential loss beyond the VaR level by reducing credit concentration.

- VaR may enhance credit concentration because it disregards losses beyond the VaR level, even at high confidence levels. On the other hand, expected shortfall reduces credit concentration because it takes into account losses beyond the VaR level as a conditional expectation.

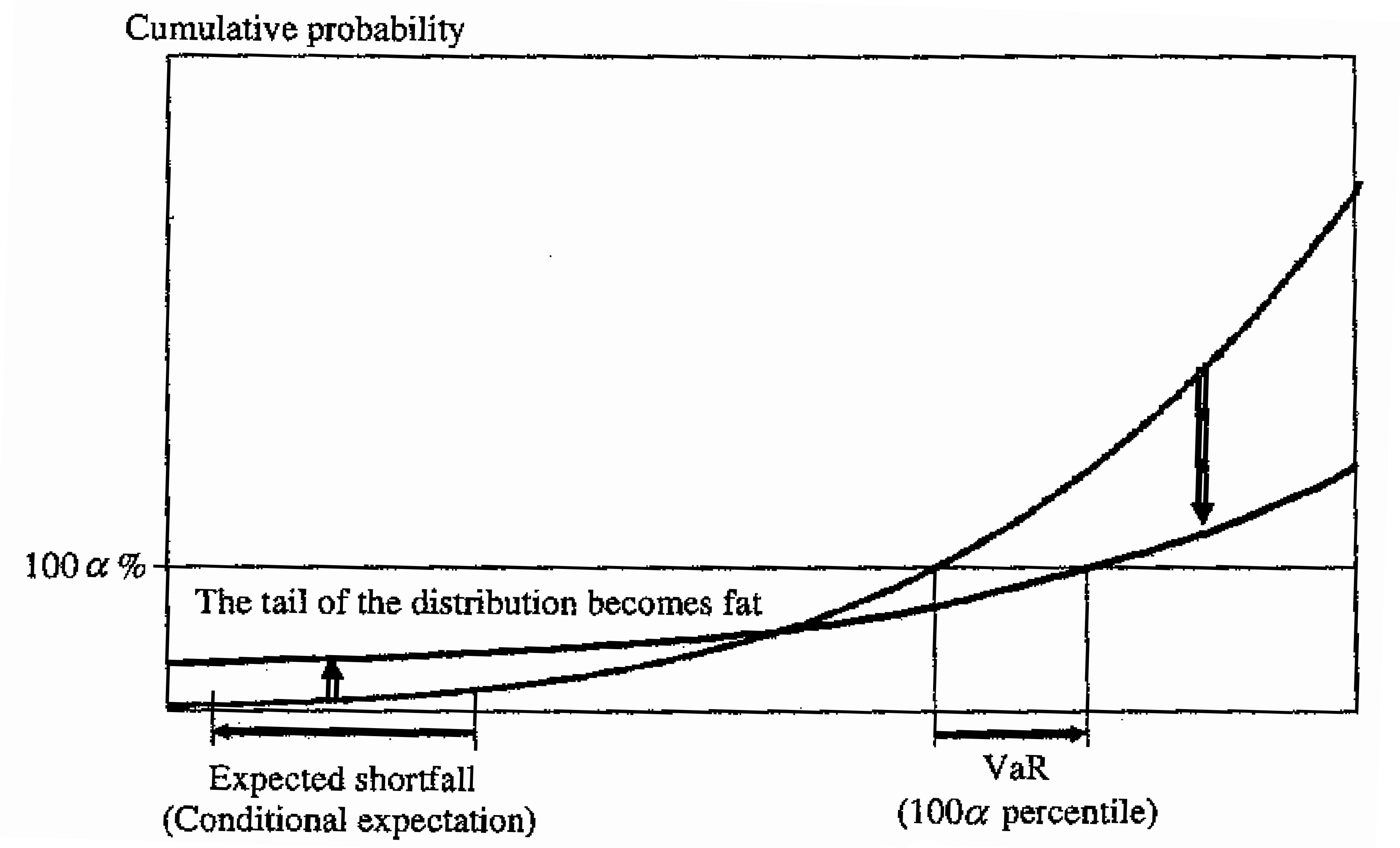
Portfolio profiles (99% confidence level)

	No constraint (1)	VaR constraint[a] (4)	Expected shortfall constraint[b] (5)
Portfolio (%)			
Concentrated portfolio A (default rate: 4%)	7.4	0.7	0.7
Concentrated portfolio B (default rate: 0.5%)	0.0	18.8	0.5
Diversified portfolio	92.6	64.9	65.6
Risk-free asset	0.0	15.6	33.2
Risk measure (million yen)			
VaR	6.77	3.00	3.13
Expected shortfall	7.83	7.33	3.50

[a] Optimize with the constraint that VaR at the 99% confidence level is less than or equal to 3.

[b] Optimize with the constraint that expected shortfall at the 99% confidence level is less than or equal to 3.5.

- With higher confidence level, VaR and ES increase under "no constraint" case.



Cumulative distribution of profit-loss when the tail risk of VaR occurs.

- If investors can invest in assets whose loss is infrequent but large (such as concentrated credit portfolios), the problem of tail risk can be serious. Investors can manipulate the profit-loss distribution using those assets, so that VaR becomes small while the tail becomes fat.

Spectral risk measure

A risk measure can be characterized by the weights it assigns to quantiles of the loss distribution.

- VaR gives a 100% weighting to the Xth quantile and zero to other quantiles.

- Expected shortfall gives equal weight to all quantiles greater than the Xth quantile and zero weight to all quantiles below Xth quantile.

We can define what is known as a *spectral risk measure* by making other assumptions about the weights assigned to quantiles. A general result is that a spectral risk measure is coherent (i.e., it satisfies the subadditivity condition) if the weight assigned to the qth quantile of the loss distribution is a nondecreasing function of q. Expected shortfall satisfies this condition.

Proof of subadditivity for the expected shortfall

Let $F(x)$ be a probability distribution of a random variable X, where

$$F(x) = P[X \leq x].$$

For some probability $q \in (0, 1)$, we define the q-quantile as

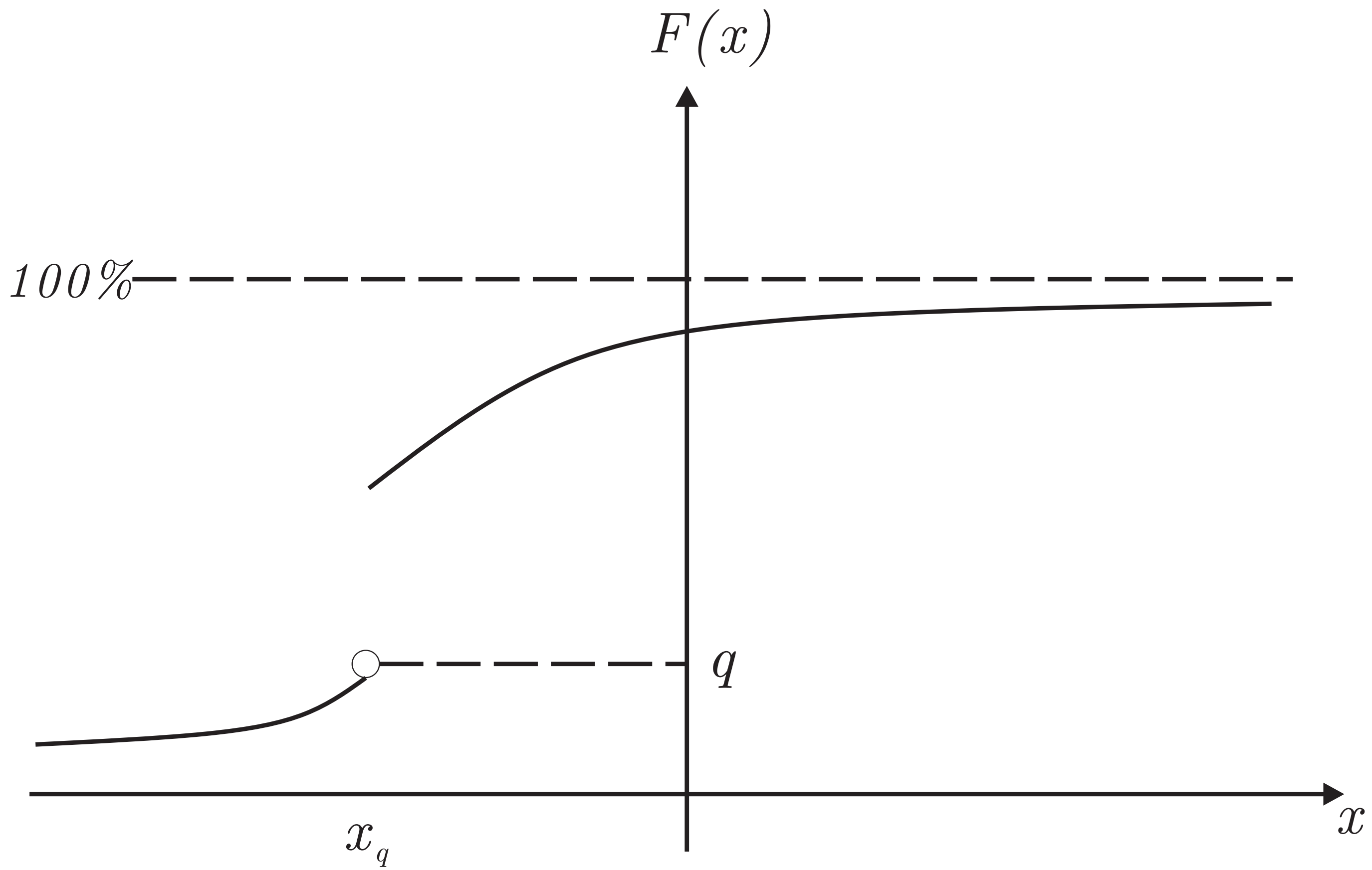
$$x_q = \text{inf}\{x|F(x) \geq q\}.$$

If $F(\cdot)$ is continuous, we have $F(x_q) = q$, while if $F(\cdot)$ is discontinuous in x_q, with $P[X = x_q] > 0$, we may have $F(x_q) = P[X \leq x_q] > q$. The definition of q-Tail Mean $\bar{x}_q$ as the "expected value of the distribution in the q-quantile" must take this fact into account.

In the current context, we take X to be the profit / gain (random) variable of the portfolio. Suppose we let $q = 0.05$, x_q refers to the profit level (negative in value) that

$$x_q = \text{inf}\{x|P[X \leq x] \geq q\}.$$

That is, with confidence level of $1 - q = 95\%$, the loss amount is not more than $-x_q$.



For a random variable X and for a specified level of probability q, we define the q-Tail Mean:

$$\bar{x}_q = \frac{1}{q} E[X \mathbf{1}_{X \le x_q}] + \left[1 - \frac{F(x_q)}{q}\right] x_q = \frac{1}{q} E[X \mathbf{1}^q_{X \le x_q}],$$

where

$$\mathbf{1}^q_{X \le x_q} = \mathbf{1}_{X \le x_q} + \frac{q - F(x_q)}{P[X = x_q]} \mathbf{1}^q_{X = x_q}.$$

The second term in the sum is zero if $P[X = x_q] = 0$. By virtue of the above definition, we have

$$E[\mathbf{1}^q_{X \le x_q}] = q$$
$$0 \le \mathbf{1}^q_{X \le z_q} \le 1.$$

Note that in the case $X = x_q$:

$$\mathbf{1}^q_{X \le x_q}|_{X=x_q} = 1 + \frac{q - F(x_q)}{P[X = x_q]} = \frac{q - F(x_q^-)}{P[X = x_q]} \in [0, 1]$$

since $F(x_q^-) \le q \le F(x_q^+) = F(x_q)$ and $P[X = x_q] = F(x_q) - F(x_q^-)$ by definition of x_q.

Given two random variables X and Y and defining $Z = X + Y$, we would like to establish:

$$\bar{z}_q \ge \bar{x}_q + \bar{y}_q.$$

Consider

$$\begin{aligned} q(\bar{z}_q - \bar{x}_q - \bar{y}_q) &= E[Z\mathbf{1}^q_{Z\le z_q} - X\mathbf{1}^q_{X\le x_q} - Y\mathbf{1}^q_{Y\le y_q}] \\ &= E[X(\mathbf{1}^q_{Z\le z_q} - \mathbf{1}^q_{X\le x_q}) + Y(\mathbf{1}^q_{Z\le z_q} - \mathbf{1}^q_{Y\le y_q})] \\ &\ge x_q E[(\mathbf{1}^q_{Z\le z_q} - \mathbf{1}^q_{X\le x_q})] + y_q E[(\mathbf{1}^q_{Z\le z_q} - \mathbf{1}^q_{Y\le y_q})] \\ &= x_q(q-q) + y_q(q-q) = 0. \end{aligned}$$

In the inequality, we have used

$$\begin{cases} \mathbf{1}^q_{Z\le z_q} - \mathbf{1}^q_{X\le x_q} \ge 0, & \text{if } X > x_q \\ \mathbf{1}^q_{Z\le z_q} - \mathbf{1}^q_{X\le x_q} \le 0, & \text{if } X < x_q \end{cases}.$$

For any risk measure R defined as

$$R(X) = f(X) - \bar{x}_q,$$

where f is a linear function, the subadditive property holds

$$R(X+Y) \le R(X) + R(Y).$$

3.4 Economic capital and risk-adjusted return on capital

Definition of economic capital (also called risk capital)

- This is the amount of capital a financial institution needs in order to absorb losses over a certain time horizon (usually one year) with a certain confidence level.

- The confidence level depends on financial institutions' objectives. Corporations rated AA have a one-year probability of default less than 0.1%. This suggests that the confidence level should be 99.9%, or even higher.

Take a *target level of statistical confidence* into account. For a given level of confidence α, let $\tilde{L}_p$ denote the random portfolio loss amount, we define the credit VaR by the α-quartile of $\tilde{L}_p$:

$$q_\alpha = \text{inf}\{q > 0 | P[\tilde{L}_P \leq q] \geq \alpha\}.$$

Also, we define

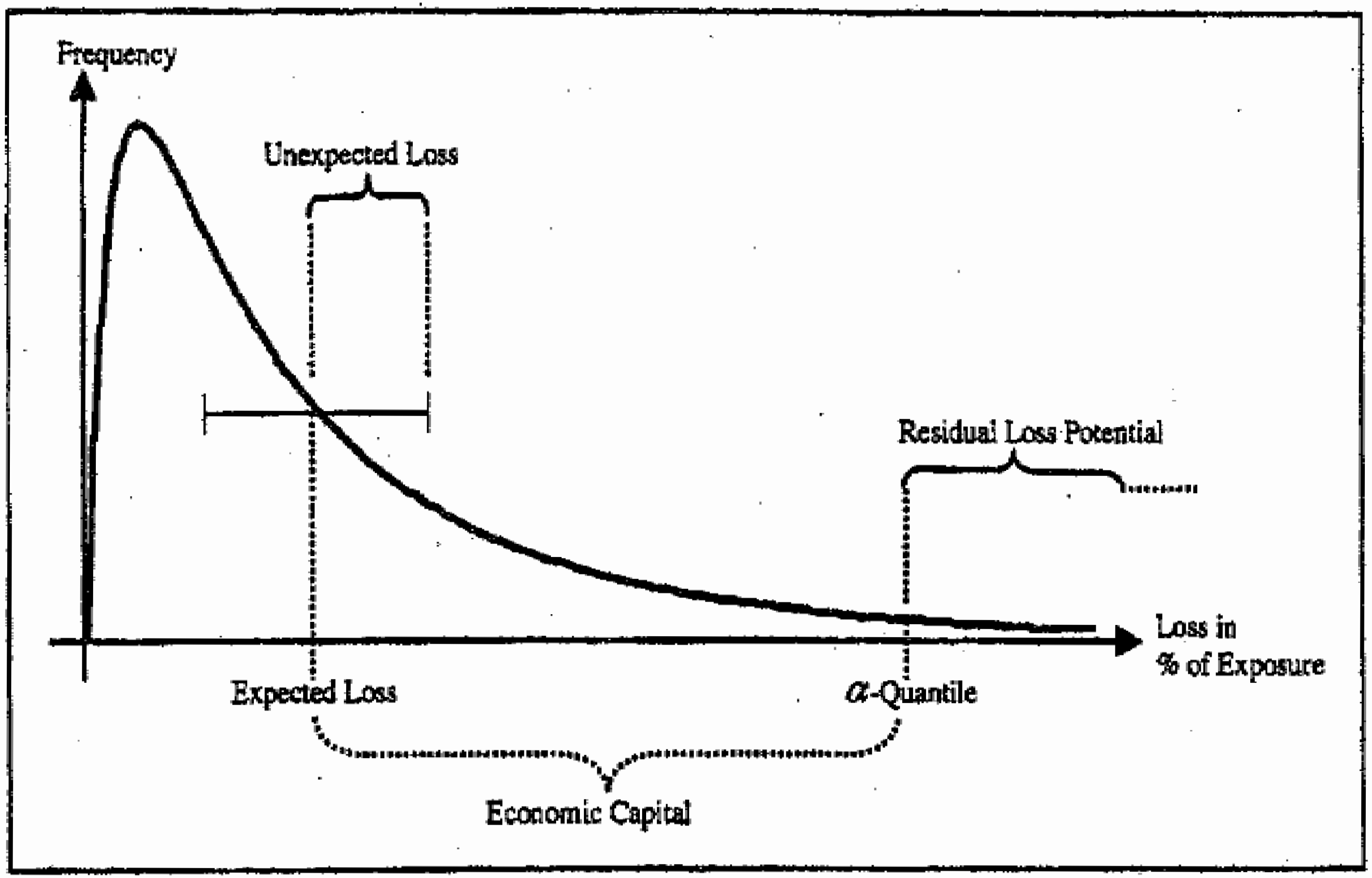
$$EC = \text{economic capital} = q_\alpha - EL_P.$$

Say, $\alpha = 99.98\%$, this would mean EC_α will be sufficient to cover unexpected losses in 9,998 out of 10,000 years (一万年两遇), assuming a planning horizon of one year.

Why reducing the quantile q_α by the EL? This is the usual practice of decomposing the total risk capital into (i) expected loss (ii) cushion against unexpected losses.

Note that EL charges are *portfolio independent* (diversification has no impact) while EC charges are *portfolio dependent*. New loans may add a lot or little risk contributions (risk concentration).

- When lending in a certain region of the world a AA-rated bank estimates its losses as 1% of outstanding loans per year on average. The 99.9% worst-case loss (i.e., the loss exceeded only 0.1% of the time) is estimated as 5% of outstanding loans. The economic capital required per \$100 of loans made is therefore \$4.0. This is the difference between the 99.9% worst-case loss ($\mathrm{VaR}_{99.9\%} = 5$) and the expected loss.


Frequency
Unexpected Loss
Residual Loss Potential
Loss in
% of Exposure
Expected Loss
α-Quantile
Economic Capital

Bottom-up approach

The loss distributions are estimated for different types of risk and different business units and then aggregated.

- The first step in the aggregation is to calculate the probability distributions for losses by risk type or losses by business unit.
- A final aggregation gives a probability distribution of total losses for the whole financial institution.

Operational risk as "the risk of loss resulting from inadequate or failed internal processes, people, and systems or from external events". Operational risk includes model risk and legal risk, but it does not include risk arising from strategic decisions or reputational risk. This type of risk is collectively referred as *business risk*. Regulatory capital is not required for business risk under Basel II, but some banks do assess economic capital for business risk.

Aggregating economic capital

Suppose a financial institution has calculated market, credit, operational, and (possibly) business risk loss distributions for a number of different business units.

Question

How to aggregate the loss distributions to calculate a total economic capital for the whole enterprise?

According to Basel II,

$$\text{total economic capital, } E_{\text{total}} = \sum_{i=1}^{n} E_i.$$

This is based on the assumption of perfect correlation between the different types of risks – too conservative.

Hybrid approach

$$\begin{aligned} E_{\text{total}} &= \text{cov}(\tilde{x}_1 + \cdots + \tilde{x}_n, \tilde{x}_1 + \cdots + \tilde{x}_n) \\ &= \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} E_i E_j \rho_{ij}}, \end{aligned}$$

where we take $E_i = \sigma(\tilde{x}_i)$ as proxy and ρ_{ij} is the correlation between risk i and risk j.

- When the distributions are normal, this approach is exact.

Economic capital estimates

Type of risk	*Business Unit* 1	*Business Unit* 2
Market risk	30	40
Credit risk	70	80
Operational risk	30	90

Correlations between losses:
MR, CR, and OR refer to market risk, credit risk, and operational risk; 1 and 2 refer to the business units.

	MR-1	CR-1	OR-1	MR-2	CR-2	OR-2
MR-1	1.0	0.5	0.2	0.4	0.0	0.0
CR-1	0.5	1.0	0.2	0.0	0.6	0.0
OR-1	0.2	0.2	1.0	0.0	0.0	0.0
MR-2	0.4	0.0	0.0	1.0	0.5	0.2
CR-2	0.0	0.6	0.0	0.5	1.0	0.2
OR-2	0.0	0.0	0.0	0.2	0.2	1.0

- Correlation between 2 different risk types in 2 different business units $= 0$.
- Correlation between market risks across business units $= 0.4$.
- Correlation between credit risks across business units $= 0.6$.
- Correlation between operational risks across business units $= 0$.

- Total market risk economic capital is

$$\sqrt{30^2 + 40^2 + 2 \times 0.4 \times 30 \times 40} = 58.8.$$

- Total credit risk economic capital is

$$\sqrt{70^2 + 80^2 + 2 \times 0.6 \times 70 \times 80} = 134.2.$$

- Total operational risk economic capital is

$$\sqrt{30^2 + 90^2} = 94.9.$$

- Total economic capital for Business Unit 1 is

$$\sqrt{30^2+70^2+30^2+2\times0.5\times30\times70+2\times0.2\times30\times30+2\times0.2\times70\times30} = 100.0.$$

- Total economic capital for Business Unit 2 is

$$\sqrt{40^2+80^2+90^2+2\times0.5\times40\times80+2\times0.2\times40\times90+2\times0.2\times80\times90} = 153.7.$$

The total enterprise-wide economic capital is the square root of

$$30^2+40^2+70^2+80^2+30^2+90^2+2\times0.4\times30\times40+2\times0.5\times30\times70$$
$$+2\times0.2\times30\times30+2\times0.5\times40\times80+2\times0.2\times40\times90$$
$$+2\times0.6\times70\times80+2\times0.2\times70\times30+2\times0.2\times80\times90$$

which is 203.224.

There are significant diversification benefits. The sum of the economic capital estimates for market, credit, and operational risk is

$$58.8+134.2+94.9=287.9,$$

and the sum of the economic capital estimates for two business units is

$$100+153.7=253.7.$$

Both of these are greater than the total economic capital estimate of 203.2.

Risk-adjusted return on capital (RAROC)

Risk-adjusted performance measurement (RAPM) has become an important part of how business units are assessed. There are many different approaches, but all have one thing in common. They compare return with capital employed in a way that incorporates an adjustment for risk.

The most common approach is to compare expected return with economic capital. The formula for RAROC is

$$\text{RAROC} = \frac{\text{Revenues} - \text{Costs} - \text{Expected losses}}{\text{Economic capital}}.$$

The numerator may be calculated on a pre-tax or post-tax basis. Sometimes, a risk-free rate of return on the economic capital is calculated and added to the numerator.

Example

When lending in a certain region of the world, a AA-rated bank estimates its losses as 1% of outstanding loans per year on average. The 99.9% worst-case loss (i.e., the loss exceeded only 0.1% of the time) is 5% of outstanding loans.

- The economic capital required per \$100 of loans is \$4, which is the difference between the 99.9% worst-case loss and the expected loss. This ignores diversification benefits that would in practice be allocated to the lending unit.

- The spread between the cost of funds and the interest charged is 2.5%. Subtracting from this the expected loan loss of 1%, the expected contribution per \$100 of loans is \$1.50.

- Assuming that the lending unit's administrative costs total 0.7% of the amount loaned, the expected profit is reduced to \$0.80 per \$100 in the loan portfolio. RAROC is therefore

$$\frac{0.80}{4} = 20\%.$$

- An alternative calculation would add the interest on the economic capital to the numerator. Suppose that the risk-free interest rate is 2%. Then $0.02 \times 4 = 0.08$ is added to the numerator, so that RAROC becomes

$$\frac{0.88}{4} = 22\%.$$

Deutsche Bank's economic capital and regulatory capital (millions of euros).

Credit risk	8,506
Market risk	3,481
Operational risk	3,974
Diversification benefit across credit, market, and operational risk	(2,651)
Business risk	301
Total economic capital	13,611

Allocation of Deutsche Bank's economic capital (millions of euros).

Corporate banking and securities	10,533
Global transaction banking	430
Asset and wealth management	871
Private business clients	1,566
Corporate investments	207
Consolidation and adjustments	5
Total	13,611

- RAROC can be calculated *ex-ante* (before the start of the year) or *ex-post* (after the end of the year). *Ex-ante* calculations are based on estimates of expected profit. *Ex-post* calculations are based on actual profit results. *Ex-ante* calculations are typically used to decide whether a particular business unit should be expanded or contracted. *Ex-post* calculations are typically used for performance evaluation and bonus calculations.

- It is usually not appropriate to base a decision to expand or contract a particular business unit on an *ex-post* analysis (although there is a natural temptation to do this). It may be that results were bad for the most recent year because credit losses were much larger than average or because there was an unexpectedly large operational risk loss. Key strategic decisions should be based on expected long-term results.

Appendix: Estimation of the annualized realized volatility of stock prices

The stock price process S_t is governed by

$$\frac{dS_t}{S_t} = \mu\ dt + \sigma\ dZ_t \quad \text{or} \quad d\ln S_t = (\mu - \frac{\sigma^2}{2})dt + \sigma\ dZ_t,$$

where μ is the expected growth rate, σ is the volatility, and Z_t is the standard Brownian process; dZ_t can be loosely interpreted as $\phi\sqrt{\delta t}$, where ϕ is the standard normal random variable and δt is the infinitesimal time increment.

Discrete approximations Write $S_i = S(t_i)$ and $S_{i-1} = S(t_{i-1})$

$$\left.\frac{dS_t}{S_t}\right|_{t=t_i} \approx \frac{S_i - S_{i-1}}{S_i},$$

$$d\ln S_t|_{t=t_i} \approx \ln S_i - \ln S_{i-1} = \ln\frac{S_i}{S_{i-1}}.$$

Write $\delta t = t_i - t_{i-1}$ so that

$$\ln \frac{S_i}{S_{i-1}} \approx (\mu - \frac{\sigma^2}{2})\delta t + \sigma \phi \sqrt{\delta t}.$$

Estimate the volatility of log-return by sampling the logarithm of stock return over the tenor: $[t_0, t_1, \ldots, t_N]$.

Recall $\text{var}(X) = E[X^2] - E[X]^2$. The market convention of measuring the annualized variance of log-return is given by

$$\frac{A}{N} \sum_{i=1}^{N} (\ln \frac{S_i}{S_{i-1}})^2.$$

Here, the square of the sampled mean is assumed to be negligibly small.

The factor A is called the annualization factor that converts daily log-return into annualized quantities; its value is typically taken to be 252 (number of trading days per year). The realized volatility of the stock price process is measured by

$$\sqrt{\frac{A}{N}\sum_{i=1}^{N}(\ln\frac{S_i}{S_{i-1}})^2}.$$

The conversion factor in computing annualized volatility is $\sqrt{A}$ instead of A since the diffusion term is $\sigma\sqrt{\delta t}\phi$. Here, $\sqrt{\delta t} \approx \frac{1}{\sqrt{A}}$, where δt is the small time step corresponding to one trading day.