Worked examples — Conformal mappings and bilinear transformations

Example 1
Suppose we wish to find a bilinear transformation which maps the circle $|z - i| = 1$ to the circle $|w| = 2$. Since $|w/2| = 1$, the linear transformation $w = f(z) = 2z - 2i$, which magnifies the first circle, and translates its centre, is a suitable choice. (Note that there is no unique choice of bilinear transformation satisfying the given criteria.) Since $f(i) = 0$, $f$ maps the inside of the first circle to the inside of the second.

Suppose now we wish to find a bilinear transformation $g$ which maps the inside of the first circle to the outside of the second circle. Let $g(z) = (\alpha z + \beta)/(\gamma z + \delta)$. We choose $g(i) = \infty$, so that $g(z) = (\alpha z + \beta)/(z - i)$ without the loss of generality. Three points on the first circle are $0, 1 + i$ and $2i$, and $g(0) = i\beta, g(1 + i) = \alpha(1 + i) + \beta$ and $g(2i) = 2\alpha - i\beta$. All three of these points must lie on $|w| = 2$, so the simplest choice is $\alpha = 0$ and $\beta = 2$. Then $g(z) = 2/(z - i)$.

Suppose now we wish to find a bilinear transformation $h$ which maps the circle $|z - i| = 1$ to the real line. Since $0, 1 + i$ and $2i$ lie on the given circle and the given line passes through $0, 1$ and $\infty$, we simply choose $h$ so that $h(0) = 0, h(1 + i) = 1$ and $h(2i) = \infty$ say. Then $h(z) = z/(iz + 2)$. Note that $h(i) = i$, so $h$ maps the region given by $|z - 1| < 1$ to the upper half-plane. This can be shown formally by letting $z = x + iy$. Then

$$h(z) = \frac{2x - i[x^2 + (y - 1)^2 - 1]}{x^2 + (y - 2)^2}$$

and $|z - i| < 1 \Rightarrow x^2 + (y - 1)^2 < 1$.

Example 2  Find a conformal map of the unit disk $|z| < 1$ onto the right half-plane $\text{Re } w > 0$.

Solution
We are naturally led to look for a bilinear transformation that maps the circle $|z| = 1$ onto the imaginary axis. The transformation must therefore have a pole on the circle, according to our earlier remarks. Moreover, the origin $w = 0$ must also lie on the image of the circle. As a first step, let’s look at

$$w = f_1(z) = \frac{z + 1}{z - 1}, \quad (i)$$

which maps $1$ to $\infty$ and $-1$ to $0$.

From the geometric properties of bilinear transformations, we can conclude that (i) maps $|z| = 1$ onto some straight line through the origin. To see which straight line, we plug in $z = i$ and find that the point

$$w = \frac{i + 1}{i - 1} = -i$$

also lies on the line. Hence the image of the circle under $f_1$ must be the imaginary axis.

To see which half-plane is the image of the interior of the circle, we check the point $z = 0$. It is mapped by (i) to the point $w = -1$ in the left half-plane. This not what we want, but it can be corrected by a final rotation of $\pi$, yielding

$$w = f(z) = -\frac{z + 1}{z - 1} = \frac{1 + z}{1 - z}, \quad (ii)$$

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as an answer to the problem. (Of course, any subsequent vertical translation or magnification can be permitted.)

Example 3  Find the image of the interior of the circle \( C : |z - 2| = 2 \) under the bilinear transformation

\[
w = f(z) = \frac{z}{2z - 8}.
\]

**Solution**

First we find the image of the circle \( C \). Since \( f \) has a pole at \( z = 4 \) and this point lies on \( C \), the image has to be a straight line. To specify this line all we need is to determine two of its finite points. The points \( z = 0 \) and \( z = 2 + 2i \) which lie on \( C \) have, as their images,

\[
w = f(0) = 0 \quad \text{and} \quad w = f(2 + 2i) = \frac{2 + 2i}{2(2 + 2i) - i} = -\frac{i}{2}.
\]

Thus the image of \( C \) is the imaginary axis in the \( w \)-plane. From connectivity, we know that the interior of \( C \) is therefore mapped either onto the right half-plane \( \Re w > 0 \) or onto the left half-plane \( \Re w < 0 \). Since \( z = 2 \) lies inside \( C \) and

\[
w = f(2) = \frac{2}{4 - 8} = -\frac{1}{2}
\]

lies in the left half-plane, we conclude that the image of the interior of \( C \) is the left half-plane.

Example 4 – Find a bilinear transformation that maps the region \( D_1 : |z| > 1 \) onto the region \( D_2 : \Re w < 0 \).

**Solution**
We shall take both $D_1$ and $D_2$ to be left regions. This is accomplished for $D_1$ by choosing any three points on the circle $|z| = 1$ that give it a negative (clockwise) orientation, say

$$z_1 = 1, \quad z_2 = -i, \quad z_3 = -1.$$  

Similarly the three points

$$w_1 = 0, \quad w_2 = i, \quad w_3 = \infty$$

on the imaginary axis make $D_2$ it left region. Hence a solution to the problem is given by the transformation that takes

$$1 \text{ to } 0, \quad -i \text{ to } i, \quad -1 \text{ to } \infty.$$  

This we obtain by setting

$$(w, 0, i, \infty) = (z, 1, -i, -1),$$  

that is,

$$\frac{w - 0}{i - 0} = \frac{(z - 1)(-i + 1)}{(z + 1)(-i - 1)},$$

which yields

$$w = \frac{(z - 1)(1 + i)}{(z + 1)(-i - 1)} = \frac{1 - z}{1 + z}.$$  

**Example 5 – Mapping the unit disk onto an infinite horizontal strip**

We will describe a sequence of analytic and one-to-one mappings that takes the unit circle onto an infinite horizontal strip. The first linear fractional transformation, $w_1 = -i\phi(z)$, is obtained by multiplying by $-i$ the linear fractional transformation $\phi(z)$, where $\phi(z) = i\frac{1 - z}{1 + z}$ maps the unit disk onto the upper half-plane, and multiplication by $-i$ rotates by the angle $-\frac{\pi}{2}$, the effect of $-i\phi(z)$ is to map the unit disk onto the right half-pane.

![Figure](image-url)

The principal branch of the logarithm, Log $z$, maps the right half-plane onto an infinite horizontal strip.

In the figure, Log $w_1 = \ln |w_1| + i\text{Arg } w_1$ is the principal branch of the logarithm. As $w_1$ varies in the right half-plane, Arg $w_1$ varies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ which explains the location of the horizontal boundary of the infinite strip. The desired mapping is

$$w = f(z) = \text{Log}(-i\phi(z)) = \text{Log} \frac{1 - z}{1 + z}.$$
Example 6  Find a linear fractional transformation that maps the interior of the circle $|z - i| = 2$ onto the exterior of the circle $|w - 1| = 3$.

We need only three points on the first circle in clockwise order and three on the second circle in counterclockwise order. Let $z_1 = -i, z_2 = -2 + i, z_3 = 3i$ and $w_1 = 4, w_2 = 1 + 3i, w_3 = -2$. We have

$$
\frac{(w - 4)(3 + 3i)}{(w + 2)(-3 + 3i)} = \frac{(z + i)(-2 - 2i)}{(z - 3i)(-2 + 2i)},
$$

which, when solved for $w$, defines the mapping

$$
w = \frac{z - 7i}{z - i}.
$$

The center of the circle, $|z - i| = 2$, is $z = i$. Our transformation maps this point to $w = \infty$, which is clearly in the exterior of the circle. $|w - 1| = 3$.

Example 7  Find a linear fractional transformation that maps the half-plane defined by $\text{Im} \ (z) > \text{Re} \ (z)$ onto the interior of the circle $|w - 1| = 3$.

We shall regard the specified half-plane as the “interior” of the “circle” through $\infty$ defined by the line $\text{Im} \ (z) = \text{Re} \ (z)$. As noted earlier, it is usually convenient to use $\infty$ when possible as a point on a line. Then three points in “clockwise” order are $z_1 = \infty, z_2 = 0, \text{ and } z_3 = -1 - i$. Three points on the circle, $|w - 1| = 3$, in clockwise order are $w_1 = 1 + 3i, w_2 = 4, w_3 = -2$. The unique linear fractional transformation mapping these points in order is defined by

$$
w = \frac{\alpha z + \beta}{\bar{z} + \gamma},
$$

where from the images of $\infty$ and 0, we must have

$$
\alpha = 1 + 3i, \quad \beta = 4, \quad \gamma = i,
$$

and from the image of $z_3 = -1 - i$, we must have

$$
\frac{\alpha(-1 - i) + \beta}{(-1 - i) + \gamma} = -2.
$$

By substituting $\beta = 4\gamma$ into the preceding equation, we have

$$
\beta = 4i \quad \text{and} \quad \gamma = i.
$$
so that

\[ w = \frac{(1 + 3i)z + 4i}{z + i} \]

is the required linear fractional transformation.

Now \( z_0 = i \) is in the half-plane defined by \( \text{Im}(z) > \text{Re}(z) \). Its image under the transformation is \( w_0 = \frac{5}{2} + \frac{3}{2}i \) and as \( w_0 - 1 = \frac{3}{2} + \frac{3}{2}i \) has a modulus of \( \frac{3}{2}\sqrt{2} \approx 2.12132 < 3 \), \( w_0 \) is in the interior of \( |w - 1| = 3 \).