4. Complex integration: Cauchy integral theorem and Cauchy integral formulas

Definite integral of a complex-valued function of a real variable

Consider a complex valued function $f(t)$ of a real variable t:

$$f(t) = u(t) + iv(t),$$

which is assumed to be a piecewise continuous function defined in the closed interval $a \leq t \leq b$. The integral of $f(t)$ from $t = a$ to $t = b$, is defined as

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

Properties of a complex integral with real variable of integration

1.
$$\text{Re } \int_a^b f(t)\, dt = \int_a^b \text{Re } f(t)\, dt = \int_a^b u(t)\, dt.$$

2.
$$\text{Im } \int_a^b f(t)\, dt = \int_a^b \text{Im } f(t)\, dt = \int_a^b v(t)\, dt.$$

3.
$$\int_a^b [\gamma_1 f_1(t) + \gamma_2 f_2(t)]\, dt = \gamma_1 \int_a^b f_1(t)\, dt + \gamma_2 \int_a^b f_2(t)\, dt,$$
where γ_1 and γ_2 are any complex constants.

4.
$$\left| \int_a^b f(t)\, dt \right| \leq \int_a^b |f(t)|\, dt.$$

To prove (4), we consider

$$\left|\int_a^b f(t)\ dt\right| = e^{-i\phi}\int_a^b f(t)\ dt = \int_a^b e^{-i\phi} f(t)\ dt,$$

where $\phi = \text{Arg}\left(\int_a^b f(t)\ dt\right)$. Since $\left|\int_a^b f(t)\ dt\right|$ is real, we deduce that

$$\begin{aligned}\left|\int_a^b f(t)\ dt\right| &= \text{Re}\int_a^b e^{-i\phi} f(t)\ dt = \int_a^b \text{Re}\,[e^{-i\phi} f(t)]\ dt \\ &\leq \int_a^b |e^{-i\phi} f(t)|\ dt = \int_a^b |f(t)|\ dt.\end{aligned}$$

Example

Suppose α is real, show that

$$|e^{2\alpha\pi i} - 1| \leq 2\pi|\alpha|.$$

Solution

Let $f(t) = e^{i\alpha t}$, α and t are real. We obtain

$$\left|\int_0^{2\pi} e^{i\alpha t}\, dt\right| \leq \int_0^{2\pi} |e^{i\alpha t}|\, dt = 2\pi.$$

The left-hand side of the above inequality is equal to

$$\left|\int_0^{2\pi} e^{i\alpha t}\, dt\right| = \left|\left.\frac{e^{i\alpha t}}{i\alpha}\right|_0^{2\pi}\right| = \frac{|e^{2\alpha\pi i} - 1|}{|\alpha|}.$$

Combining the results, we obtain

$$|e^{2\alpha\pi i} - 1| \leq 2\pi|\alpha|, \quad \alpha \text{ is real.}$$

Definition of a contour integral

Consider a curve C which is a set of points $z = (x, y)$ in the complex plane defined by

$$x = x(t), \quad y = y(t), \quad a \leq t \leq b,$$

where $x(t)$ and $y(t)$ are continuous functions of the real parameter t. One may write

$$z(t) = x(t) + iy(t), \;\; a \leq t \leq b.$$

- The curve is said to be *smooth* if $z(t)$ has continuous derivative $z'(t) \neq 0$ for all points along the curve.

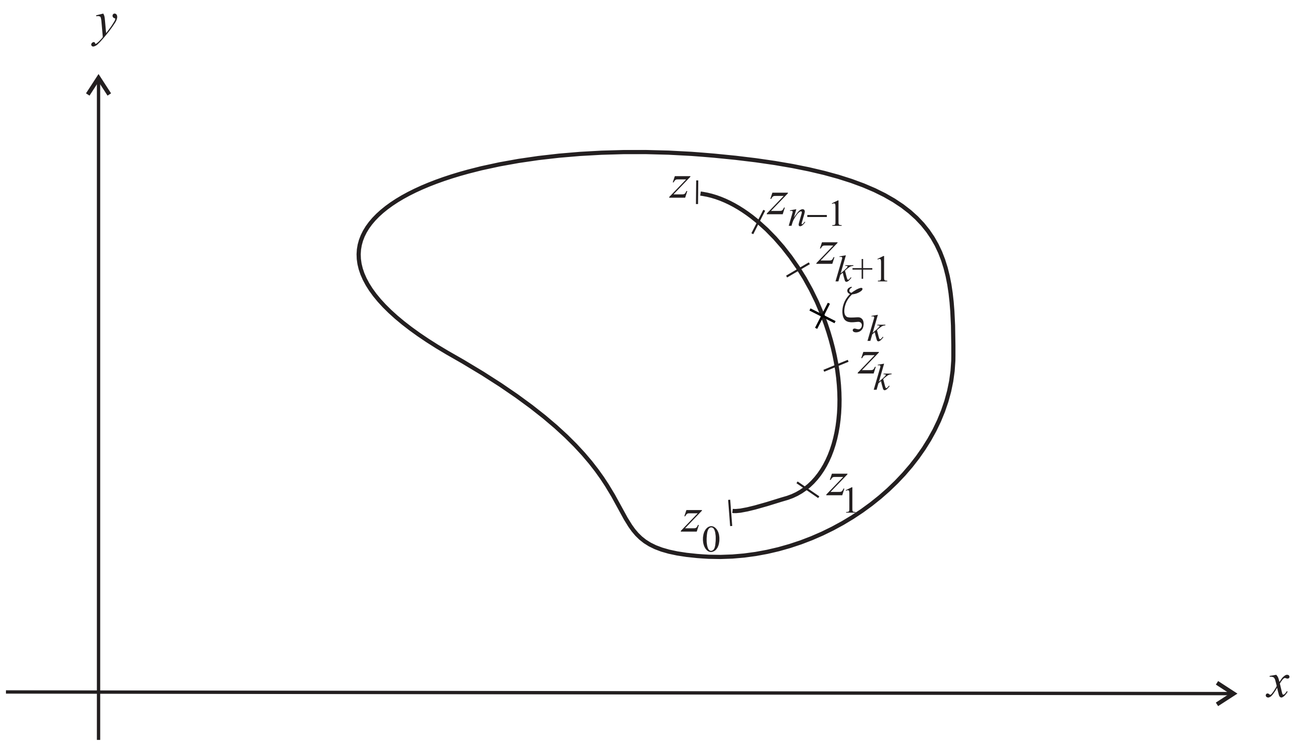
- A *contour* is defined as a curve consisting of a finite number of smooth curves joined end to end. A contour is said to be a *simple closed contour* if the initial and final values of $z(t)$ are the same and the contour does not cross itself.

- Let $f(z)$ be any complex function defined in a domain $\mathcal{D}$ in the complex plane and let C be any contour contained in $\mathcal{D}$ with initial point z_0 and terminal point z.

- We divide the contour C into n subarcs by discrete points $z_0,\ z_1,\ z_2$, $\ldots, z_{n-1}, z_n = z$ arranged consecutively along the direction of increasing t.

- Let ζ_k be an arbitrary point in the subarc $z_k z_{k+1}$ and form the sum

$$\sum_{k=0}^{n-1} f(\zeta_k)(z_{k+1} - z_k).$$



Subdivision of the contour into n subarcs by discrete points $z_0, z_1, \cdots,$ $z_{n-1}, z_n = z$.

We write $\triangle z_k = z_{k+1} - z_k$. Let $\lambda = \max\limits_k |\triangle z_k|$ and take the limit

$$\lim_{\substack{\lambda \to 0 \\ n \to \infty}} \sum_{k=0}^{n-1} f(\zeta_k)\, \triangle z_k.$$

The above limit is defined to be the *contour integral* of $f(z)$ along the contour C.

If the above limit exists, then the function $f(z)$ is said to be *integrable along the contour* C.

If we write

$$\frac{dz(t)}{dt} = \frac{dx(t)}{dt} + i\frac{dy(t)}{dt}, \quad a \le t \le b,$$

then

$$\int_C f(z)\, dz = \int_a^b f(z(t))\, \frac{dz(t)}{dt}\, dt.$$

Writing $f(z) = u(x,y) + iv(x,y)$ and $dz = dx + idy$, we have

$$\begin{aligned}
\int_C f(z)\,dz &= \int_C u\ dx - v\ dy + i\int_C u\ dy + v\ dx \\
&= \int_a^b \left[u(x(t),y(t))\frac{dx(t)}{dt} - v(x(t),y(t))\frac{dy(t)}{dt} \right] dt \\
&\quad + i\int_a^b \left[u(x(t),y(t))\frac{dy(t)}{dt} + v(x(t),y(t))\frac{dx(t)}{dt} \right] dt.
\end{aligned}$$

The usual properties of real line integrals are carried over to their complex counterparts. Some of these properties are:

(i) $\int_C f(z)\,dz$ is independent of the parameterization of C;

(ii) $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$, where $-C$ is the opposite curve of C;

(iii) The integrals of $f(z)$ along a string of contours is equal to the sum of the integrals of $f(z)$ along each of these contours.

Example

Evaluate the integral

$$\oint_C \frac{1}{z - z_0}\, dz,$$

where C is a circle centered at z_0 and of any radius. The path is traced out once in the anticlockwise direction.

Solution

The circle can be parameterized by

$$z(t) = z_0 + re^{it}, \;\; 0 \le t \le 2\pi,$$

where r is any positive real number. The contour integral becomes

$$\oint_C \frac{1}{z - z_0}\, dz = \int_0^{2\pi} \frac{1}{z(t) - z_0}\, \frac{dz(t)}{dt}\; dt = \int_0^{2\pi} \frac{ire^{it}}{re^{it}}\, dt = 2\pi i.$$

The value of the integral is independent of the radius r.

Example

Evaluate the integral

(i) $\int_C |z|^2 \, dz$ and (ii) $\int_C \frac{1}{z^2} \, dz$,

where the contour C is

(a) the line segment with initial point -1 and final point i;

(b) the arc of the unit circle $\text{Im } z \geq 0$ with initial point -1 and final point i.

Do the two results agree?

Solution

(i) Consider $\int_C |z|^2\,dz$,

(a) Parameterize the line segment by

$$z = -1 + (1+i)t, \qquad 0 \le t \le 1,$$

so that

$$|z|^2 = (-1+t)^2 + t^2 \qquad \text{and} \qquad dz = (1+i)\,dt.$$

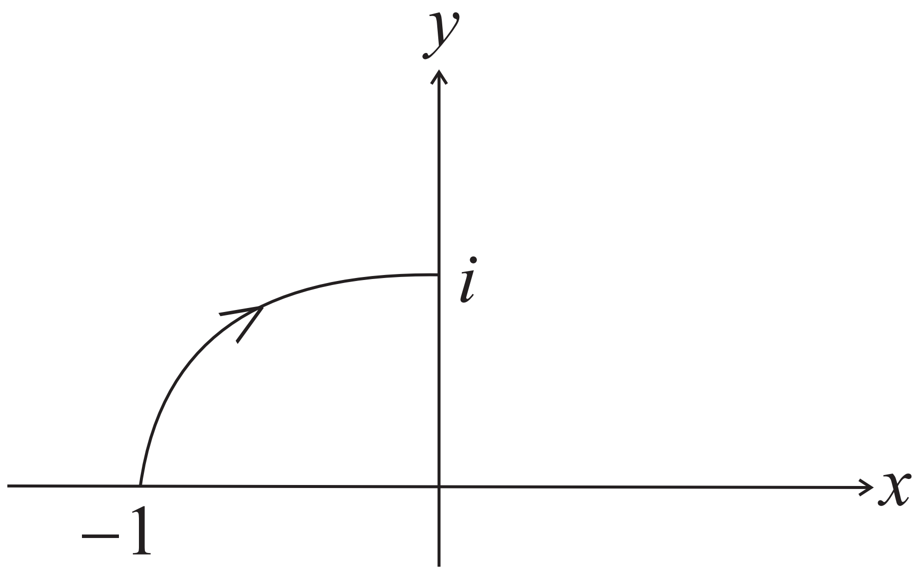
The value of the integral becomes

$$\int_C |z|^2\,dz = \int_0^1 (2t^2 - 2t + 1)(1+i)\,dt = \frac{2}{3}(1+i).$$

(b) Along the unit circle, $|z| = 1$ and $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$. The initial point and the final point of the path correspond to $\theta = \pi$ and $\theta = \frac{\pi}{2}$, respectively. The contour integral can be evaluated as

$$\int_C |z|^2 \, dz = \int_\pi^{\frac{\pi}{2}} ie^{i\theta} \, d\theta = e^{i\theta}\Big|_\pi^{\frac{\pi}{2}} = 1 + i.$$

The results in (a) and (b) do not agree. Hence, the value of this contour integral does depend on the path of integration.



(ii) Consider $\int_C \frac{1}{z^2}\,dz$.

(a) line segment from -1 to i

$$\int_C \frac{1}{z^2}\,dz = \int_0^1 \frac{1+i}{[-1+(1+i)t]^2}\,dt = -\frac{1}{-1+(1+i)t}\bigg|_0^1 = -1-\frac{1}{i} = -1+i.$$

(b) subarc from -1 to i

$$\int_C \frac{1}{z^2}\,dz = \int_\pi^{\frac{\pi}{2}} \frac{1}{e^{2i\theta}} ie^{i\theta}\,d\theta = -e^{-i\theta}\Big]_\pi^{\frac{\pi}{2}} = -1+i.$$

Estimation of the absolute value of a complex integral

The upper bound for the absolute value of a complex integral can be related to the length of the contour C and the absolute value of $f(z)$ along C. In fact,

$$\left|\int_C f(z)\, dz\right| \leq ML,$$

where M is the upper bound of $|f(z)|$ along C and L is the arc length of the contour C.

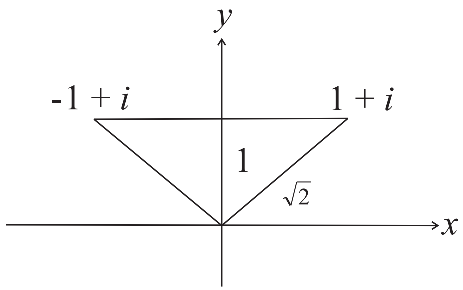
We consider

$$
\begin{aligned}
\left|\int_C f(z)\,dz\right| &= \left|\int_a^b f(z(t))\,\frac{dz(t)}{dt}\,dt\right| \\
&\le \int_a^b |f(z(t))|\,\left|\frac{dz(t)}{dt}\right|\,dt \\
&\le \int_a^b M\,\left|\frac{dz(t)}{dt}\right|\,dt \\
&= M\int_a^b \sqrt{\left(\frac{dx(t)}{dt}\right)^2+\left(\frac{dy(t)}{dt}\right)^2}dt = ML.
\end{aligned}
$$

Example

Show that

$\left|\int_C \frac{1}{z^2}\, dz\right| \leq 2$, where C is the line segment joining $-1+i$ and $1+i$.



Solution

Along the contour C, we have $z = x + i$, $-1 \leq x \leq 1$, so that $1 \leq |z| \leq \sqrt{2}$. Correspondingly, $\frac{1}{2} \leq \frac{1}{|z|^2} \leq 1$. Here, $M = \max_{z \in C} \frac{1}{|z|^2} = 1$ and the arc length $L = 2$. We have

$$\left|\int_C \frac{1}{z^2}\, dz\right| \leq ML = 2.$$

Example

Estimate an upper bound of the modulus of the integral

$$I = \int_C \frac{\text{Log } z}{z - 4i}\, dz$$

where C is the circle $|z| = 3$.

Now, $\left|\dfrac{\text{Log } z}{z-4i}\right| \le \dfrac{\Big|\ln|z|\Big| + |\text{Arg } z|}{\big||z| - |4i|\big|}$ so that

$$\max_{z \in C} \left|\frac{\text{Log } z}{z - 4i}\right| \le \frac{\ln 3 + \pi}{|3 - 4|} = \ln 3 + \pi; \qquad L = (2\pi)(3) = 6\pi.$$

Hence, $\left|\displaystyle\int_C \frac{\text{Log } z}{z - 4i}\, dz\right| \le 6\pi(\pi + \ln 3).$

Example

Find an upper bound for $\left|\int_\Gamma e^z/(z^2+1)\ dz\right|$, where Γ is the circle $|z|=2$ traversed once in the counterclockwise direction.

Solution

The path of integration has length $L = 4\pi$. Next we seek an upper bound M for the function $e^z/(z^2+1)$ when $|z| = 2$. Writing $z = x+iy$, we have

$$|e^z| = |e^{x+iy}| = e^x \le e^2, \quad \text{for} \quad |z| = \sqrt{x^2+y^2} = 2,$$

and by the triangle inequality

$$|z^2+1| \ge |z|^2 - 1 = 4 - 1 = 3 \quad \text{for} \quad |z| = 2.$$

Hence, $|e^z/(z^2+1)| \le e^2/3$ for $|z| = 2$, and so

$$\left|\int_\Gamma \frac{e^z}{z^2+1}\ dz\right| \le \frac{e^2}{3} \cdot 4\pi.$$

Path independence

Under what conditions that

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz,$$

where C_1 and C_2 are two contours in a domain $\mathcal{D}$ with the same initial and final points and $f(z)$ is piecewise continuous inside $\mathcal{D}$. The property of path independence is valid for $f(z) = \frac{1}{z^2}$ but it fails when $f(z) = |z|^2$. The above query is equivalent to the question: When does

$$\oint_C f(z)\,dz = 0$$

hold, where C is any closed contour lying completely inside $\mathcal{D}$? The equivalence is revealed if we treat C as $C_1 \cup -C_2$.

We observe that $f(z) = \frac{1}{z^2}$ is analytic everywhere except at $z = 0$ but $f(z) = |z|^2$ is nowhere analytic.

Cauchy integral theorem

Let $f(z) = u(x, y) + iv(x, y)$ *be analytic on and inside a simple closed contour* C *and let* $f'(z)$ *be also continuous on and inside C, then*

$$\oint_C f(z)\, dz = 0.$$

Proof

The proof of the Cauchy integral theorem requires the Green theorem for a positively oriented closed contour C: If the two real functions $P(x, y)$ and $Q(x, y)$ have continuous first order partial derivatives on and inside C, then

$$\oint_C P\, dx + Q\, dy = \iint_{\mathcal{D}} (Q_x - P_y)\, dxdy,$$

where $\mathcal{D}$ is the simply connected domain bounded by C.

Suppose we write $f(z) = u(x, y) + iv(x, y), z = x + iy$; we have

$$\oint_C f(z)\, dz = \oint_C u\, dx - v\, dy + i \oint_C v\, dx + u\, dy.$$

One can infer from the continuity of $f'(z)$ that $u(x, y)$ and $v(x, y)$ have continuous derivatives on and inside C. Using the Green theorem, the two real line integrals can be transformed into double integrals.

$$\oint_C f(z)\, dz = \iint_{\mathcal{D}} (-v_x - u_y)\, dxdy + i \iint_{\mathcal{D}} (u_x - v_y)\, dxdy.$$

Both integrands in the double integrals are equal to zero due to the Cauchy-Riemann relations, hence the theorem.

In 1903, Goursat was able to obtain the same result *without assuming the continuity of* $f'(z)$.

Goursat Theorem

If a function $f(z)$ is analytic throughout a simply connected domain $\mathcal{D}$, then for any simple closed contour C lying completely inside $\mathcal{D}$, we have

$$\oint_C f(z)\ dz = 0.$$

Corollary 1

The integral of a function $f(z)$ which is analytic throughout a simply connected domain $\mathcal{D}$ depends on the end points and not on the particular contour taken. Suppose α and β are inside $\mathcal{D}$, C_1 and C_2 are any contours inside $\mathcal{D}$ joining α to β, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Example

If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining points $(1,1)$ and $(2,3)$, find the value of

$$\int_C (12z^2 - 4iz)\ dz.$$

Method 1. The integral is independent of the path joining $(1,1)$ and $(2,3)$. Hence any path can be chosen. In particular, let us choose the straight line paths from $(1,1)$ to $(2,1)$ and then from $(2,1)$ to $(2,3)$.

Case 1 Along the path from $(1,1)$ to $(2,1), y = 1, dy = 0$ so that $z = x + iy = x + i, dz = dx$. Then the integral equals

$$\int_1^2 \{12(x+i)^2 - 4i(x+i)\}\ dx = \{4(x+i)^3 - 2i(x+i)^2\}\Big|_1^2 = 20 + 30i.$$

Case 2 Along the path from $(2,1)$ to $(2,3), x = 2, dx = 0$ so that $z = x + iy = 2 + iy, dz = idy$. Then the integral equals

$$\int_1^3 \{12(2+iy)^2 - 4i(2+iy)\} i\, dy = \{4(2+iy)^3 - 2i(2+iy)^2\}\Big|_1^3 = -176 + 8i.$$

Then adding, the required value $= (20 + 30i) + (-176 + 8i) = -156 + 38i$.

Method 2. The given integral equals

$$\int_{1+i}^{2+3i} (12z^2 - 4iz)\, dz = (4z^3 - 2iz^2)\Big|_{1+i}^{2+3i} = -156 + 38i.$$

It is clear that Method 2 is easier.

Corollary 2

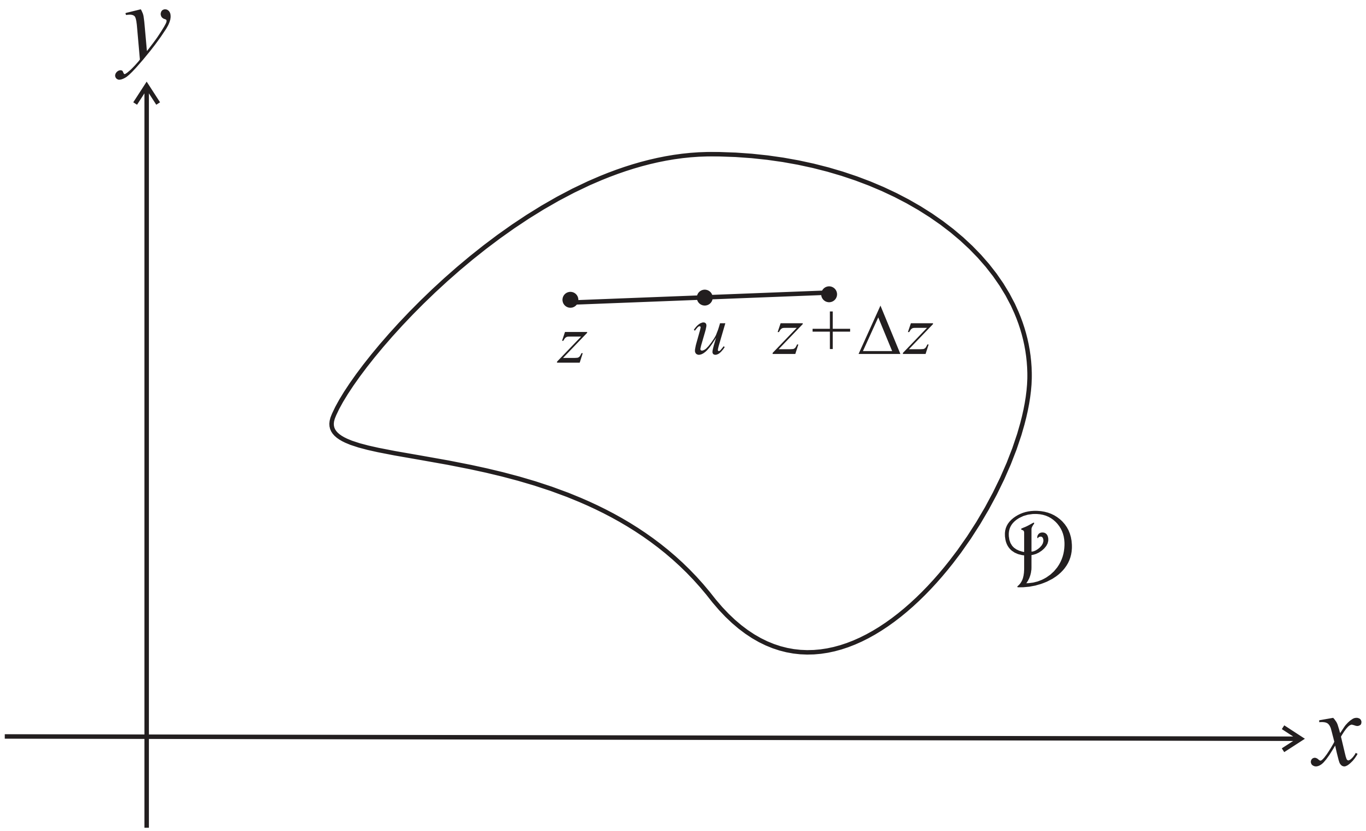
Let $f(z)$ be analytic throughout a simply connected domain $\mathcal{D}$. Consider a fixed point $z_0 \in \mathcal{D}$; by virtue of Corollary 1,

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta, \quad \text{for any } z \in \mathcal{D},$$

is a well-defined function in $\mathcal{D}$. Considering

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} [f(\zeta) - f(z)]\, d\zeta.$$

By the Cauchy Theorem, the last integral is independent of the path joining z and $z+\Delta z$ so long as the path is completely inside $\mathcal{D}$. We choose the path as the straight line segment joining z and $z+\Delta z$ and choose $|\Delta z|$ small enough so that it is completely inside $\mathcal{D}$.



By continuity of $f(z)$, we have for all points u on this straight line path

$$|f(u) - f(z)| < \epsilon \quad \text{whenever} \quad |u - z| < \delta.$$

Note that $|\Delta z| < \delta$ is observed implicitly.

We have

$$\left| \int_z^{z+\Delta z} [f(u) - f(z)]\, du \right| < \epsilon |\Delta z|$$

so that

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| = \frac{1}{|\Delta z|} \left| \int_z^{z+\Delta z} [f(u) - f(z)]\, du \right| < \epsilon$$

for $|\Delta z| < \delta$. This amounts to say

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z),$$

that is, $F'(z) = f(z)$ for all z in $\mathcal{D}$. Hence, $F(z)$ is analytic in $\mathcal{D}$ since $F'(z)$ exists at all points in $\mathcal{D}$ (which is an open set).

- This corollary may be considered as a complex counterpart of the fundamental theorem of real calculus.
- If we integrate $f(z)$ along any contour joining α and β inside $\mathcal{D}$, then the value of the integral is given by

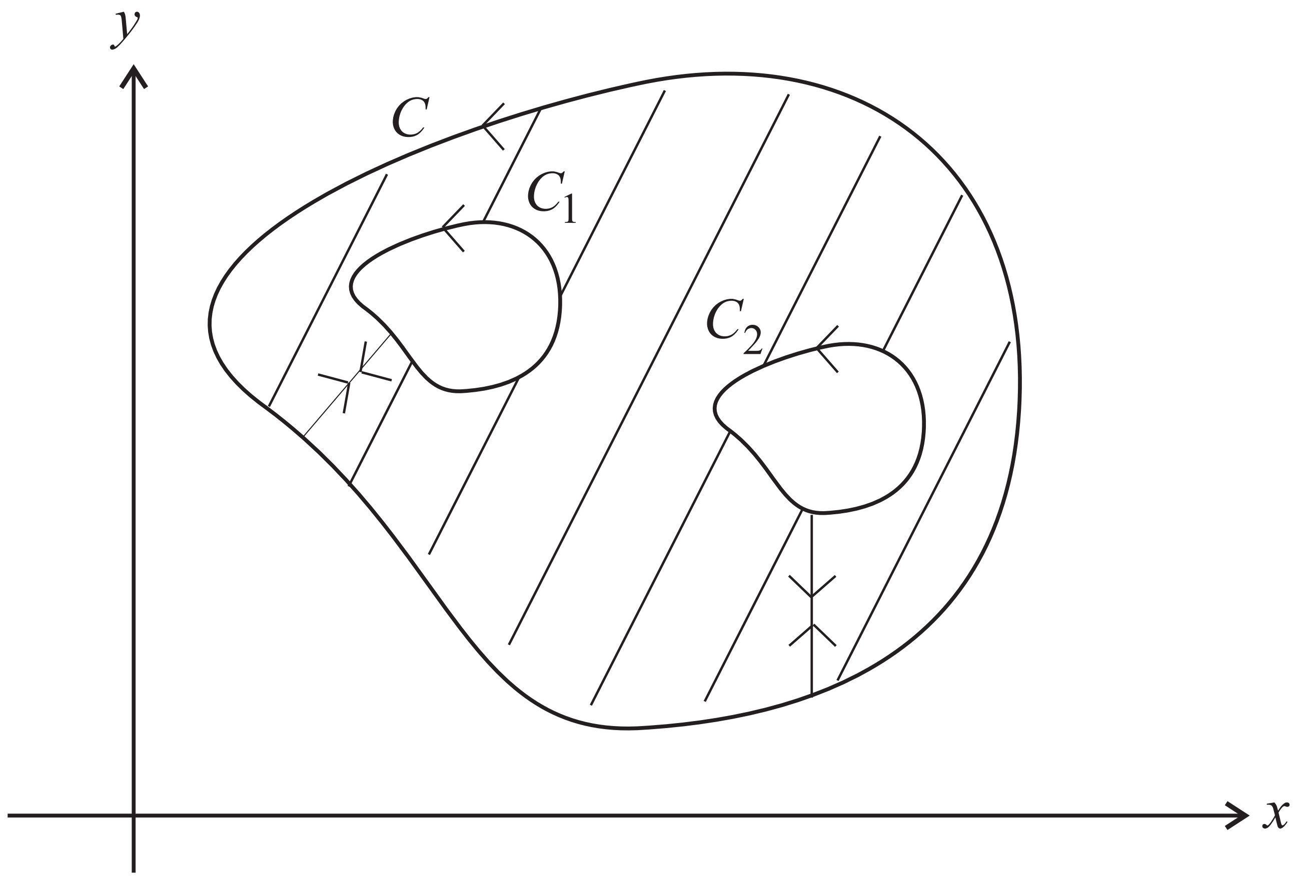
$$\begin{aligned}\int_\alpha^\beta f(z)\,dz &= \int_{z_0}^\beta f(\zeta)\,d\zeta - \int_{z_0}^\alpha f(\zeta)\,d\zeta \\ &= F(\beta) - F(\alpha), \quad \alpha \text{ and } \beta \in \mathcal{D}.\end{aligned}$$

Corollary 3

Let $C,\ C_1,\ C_2,\ldots,C_n$ be positively oriented closed contours, where $C_1,\ C_2,\ldots,C_n$ are all inside C. For $C_1,\ C_2,\ldots,C_n$, each of these contours lies outside of the other contours. Let int C_i denote the collection of all points bounded inside C_i. Let $f(z)$ be analytic on the set $S\colon C\cup$ int $C \setminus$ int $C_1 \setminus$ int $C_2 \setminus\cdots\setminus$ int C_n (see the shaded area in Figure), then

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz.$$

The proof for the case when $n = 2$ is presented below.



Proof

- The constructed boundary curve is composed of $C \cup -C_1 \cup -C_2$ and the cut lines, each cut line travels twice in opposite directions.
- To explain the negative signs in front of C_1 and C_2, we note that the interior contours traverse in the clockwise sense as parts of the positively oriented boundary curve.
- With the introduction of these cuts, the shaded region bounded within this constructed boundary curve becomes a simply connected set.

We have

$$\oint_C f(z)\,dz + \int_{-C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0,$$

so that

$$\oint_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$$

Example

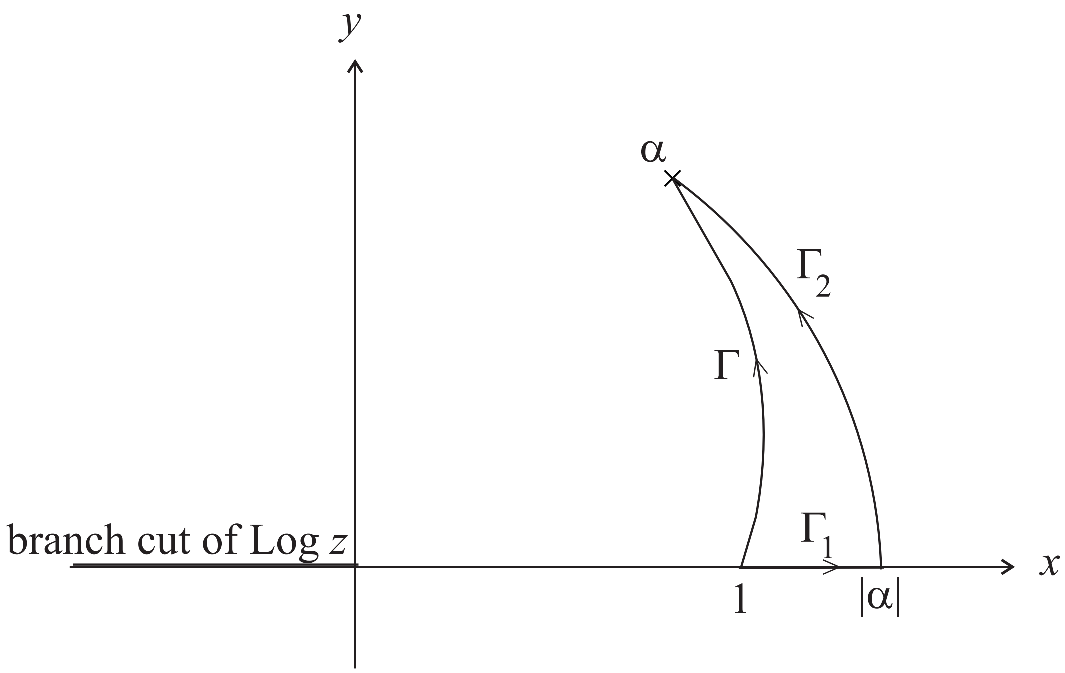
Let $\mathcal{D}$ be the domain that contains the whole complex plane except the origin and the negative real axis. Let Γ be an arbitrary contour lying completely inside $\mathcal{D}$, and Γ starts from 1 and ends at a point α. Show that

$$\int_\Gamma \frac{dz}{z} = \mathrm{Log}\ \alpha.$$

Solution

Let Γ_1 be the line segment from 1 to $|\alpha|$ along the real axis, and Γ_2 be a circular arc centered at the origin and of radius $|\alpha|$ which extends from $|\alpha|$ to α. The union $\Gamma_1 \cup \Gamma_2 \cup -\Gamma$ forms a closed contour. Since the integrand $\frac{1}{z}$ is analytic everywhere inside $\mathcal{D}$, by the Cauchy integral theorem, we have

$$\int_\Gamma \frac{dz}{z} = \int_{\Gamma_1} \frac{dz}{z} + \int_{\Gamma_2} \frac{dz}{z}.$$



The contour Γ starts from $z = 1$ and ends at $z = \alpha$. The arc Γ_2 is part of the circle $|z| = |\alpha|$.

Since α cannot lie on the negative real axis, so Arg α cannot assume the value π. If we write $\alpha = |\alpha| e^{i\text{Arg } \alpha}$ $(-\pi < \text{Arg } \alpha < \pi)$, then

$$\int_{\Gamma_1} \frac{dz}{z} = \int_1^{|\alpha|} \frac{dt}{t} = \ln|\alpha|$$

$$\int_{\Gamma_2} \frac{dz}{z} = \int_0^{\text{Arg } \alpha} \frac{ire^{i\theta}}{re^{i\theta}}\, d\theta = i \text{ Arg } \alpha.$$

Combining the results,

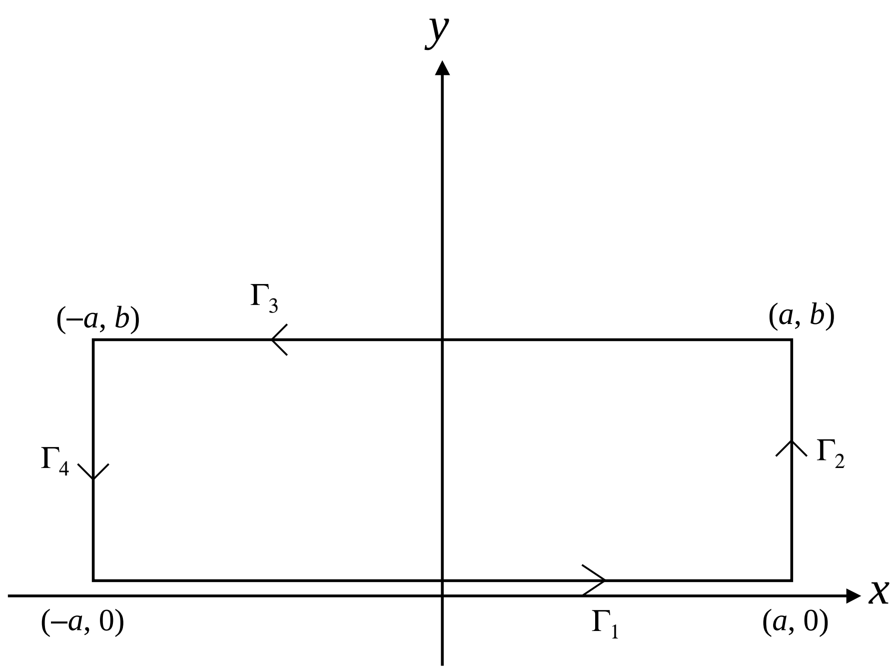
$$\int_{\Gamma} \frac{dz}{z} = \ln|\alpha| + i \text{ Arg } \alpha = \text{ Log } \alpha.$$

Note that the given domain $\mathcal{D}$ is the domain of definition of Log z, the principal branch of the complex logarithm function.

Poisson integral

Consider the integration of the function e^{-z^2} around the rectangular contour Γ with vertices $\pm a$, $\pm a + ib$ and oriented positively. By letting $a \to \infty$ while keeping b fixed, show that

$$\int_{-\infty}^{\infty} e^{-x^2} e^{\pm 2ibx}\, dx = \int_{-\infty}^{\infty} e^{-x^2} \cos 2bx\, dx = e^{-b^2}\sqrt{\pi}.$$



The configuration of the closed rectangular contour Γ.

Solution

Since e^{-z^2} is an entire function, we have

$$\oint_\Gamma e^{-z^2} dz = 0,$$

by virtue of the Cauchy integral theorem. The closed contour Γ consists of four line segments: $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where

$$\begin{aligned}
\Gamma_1 &= \{x : -a \le x \le a\}, \\
\Gamma_2 &= \{a + iy : 0 \le y \le b\}, \\
\Gamma_3 &= \{x + ib : -a \le x \le a\}, \\
\Gamma_4 &= \{-a + iy : 0 \le y \le b\},
\end{aligned}$$

and Γ is oriented in the anticlockwise direction.

The contour integral can be split into four contour integrals, namely,

$$\oint_{\Gamma} e^{-z^2}\,dz = \int_{\Gamma_1} e^{-z^2}\,dz + \int_{\Gamma_2} e^{-z^2}\,dz + \int_{\Gamma_3} e^{-z^2}\,dz + \int_{\Gamma_4} e^{-z^2}\,dz.$$

The four contour integrals can be expressed as real integrals as follows:

$$\begin{aligned}
\int_{\Gamma_1} e^{-z^2}\,dz &= \int_{-a}^{a} e^{-x^2}\,dx, \\
\int_{\Gamma_2} e^{-z^2}\,dz &= \int_{0}^{b} e^{-(a+iy)^2} i\,dy, \\
\int_{\Gamma_3} e^{-z^2}\,dz &= \int_{a}^{-a} e^{-(x+ib)^2}\,dx, \\
&= -e^{b^2}\left[\int_{-a}^{a} e^{-x^2}\cos 2bx\,dx - i\int_{-a}^{a} e^{-x^2}\sin 2bx\,dx\right], \\
\int_{\Gamma_4} e^{-z^2}\,dz &= \int_{b}^{0} e^{-(-a+iy)^2} i\,dy.
\end{aligned}$$

First, we consider the bound on the modulus of the second integral.

$$\begin{aligned}
\left|\int_{\Gamma_2} e^{-z^2}\,dz\right| &\le \int_0^b |e^{-(a^2-y^2+2iay)}i|\,dy \\
&= e^{-a^2}\int_0^b e^{y^2}\,dy \\
&\le e^{-a^2}\int_0^b e^{b^2}\,dy \qquad \text{(since } 0\le y\le b\text{)} \\
&= \frac{be^{b^2}}{e^{a^2}} \to 0 \ \text{ as } \ a\to\infty \ \text{ and } \ b \text{ is fixed.}
\end{aligned}$$

Therefore, the value of $\displaystyle\int_{\Gamma_2} e^{-z^2}\,dz \to 0$ as $a\to\infty$.

By similar argument, the fourth integral can be shown to be zero as $a\to\infty$.

$$
\begin{aligned}
\lim_{a\to\infty}\oint_\Gamma e^{-z^2}\,dz \;=\; & \lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}dx - e^{b^2}\int_{-a}^{a} e^{-x^2}\cos 2bx\,dx\right) \\
& + i\lim_{a\to\infty}\left(e^{b^2}\int_{-a}^{a} e^{-x^2}\sin 2bx\,dx\right) = 0,
\end{aligned}
$$

so that

$$
\int_{-\infty}^{\infty} e^{-x^2}\cos 2bx\,dx - i\int_{-\infty}^{\infty} e^{-x^2}\sin 2bx\,dx = e^{-b^2}\int_{-\infty}^{\infty} e^{-x^2}dx = e^{-b^2}\sqrt{\pi}.
$$

Either by equating the imaginary parts of the above equation or observing that $e^{-x^2}\sin 2bx$ is odd, we deduce

$$
\int_{-\infty}^{\infty} e^{-x^2}\sin 2bx\,dx = 0.
$$

Hence, we obtain

$$
\int_{-\infty}^{\infty} e^{-x^2} e^{\pm 2ibx}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\cos 2bx\,dx = e^{-b^2}\sqrt{\pi}.
$$

Cauchy integral formula

Let the function $f(z)$ be analytic on and inside a positively oriented simple closed contour C and z is any point inside C, then

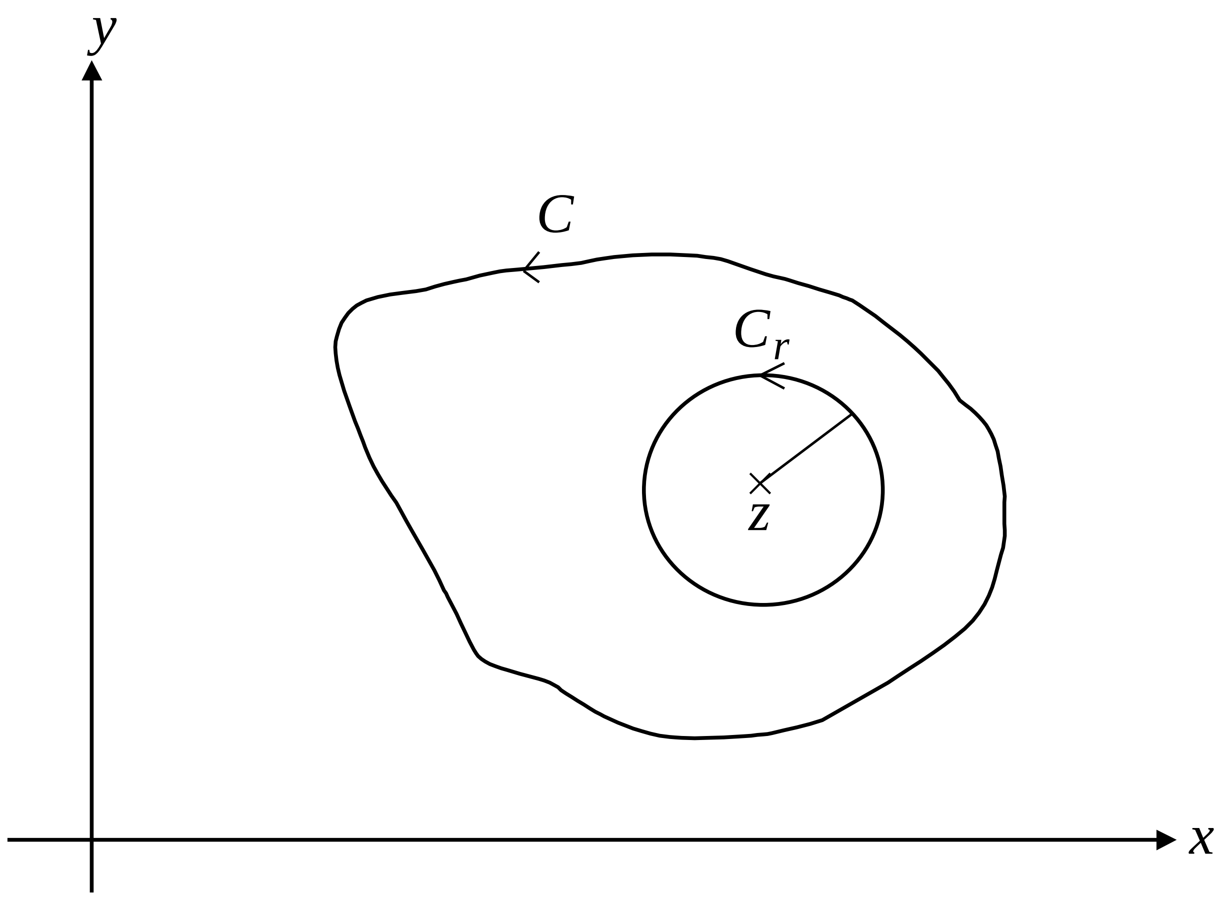
$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Proof

We draw a circle C_r, with radius r around the point z, small enough to be completely inside C. Since $\frac{f(\zeta)}{\zeta - z}$ is analytic in the region lying between C_r and C, we have

$$\begin{aligned}\frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta &= \frac{1}{2\pi i}\oint_{C_r} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i}\oint_{C_r} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta + \frac{f(z)}{2\pi i}\oint_{C_r} \frac{1}{\zeta - z}\, d\zeta.\end{aligned}$$

The last integral is seen to be equal to $f(z)$. To complete the proof, it suffices to show that the first integral equals zero.



The contour C is deformed to the circle C_r, which encircles the point z.

Since f is continuous at z, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(\zeta) - f(z)| < \epsilon \quad \text{whenever} \quad |\zeta - z| < \delta.$$

Now, suppose we choose $r < \delta$ (it is necessary to guarantee that C_r lies completely inside the contour C), the modulus of the first integral is bounded by

$$\begin{aligned}
\left|\frac{1}{2\pi i}\oint_{C_r}\frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta\right| &\le \frac{1}{2\pi}\oint_{C_r}\frac{|f(\zeta) - f(z)|}{|\zeta - z|}\,|d\zeta| \\
&= \frac{1}{2\pi r}\oint_{C_r}|f(\zeta) - f(z)|\,|d\zeta| \\
&< \frac{\epsilon}{2\pi r}\oint_{C_r}|d\zeta| = \frac{\epsilon}{2\pi r}2\pi r = \epsilon.
\end{aligned}$$

Since the modulus of the above integral is less than any positive number ϵ, however small, so the value of that integral is zero.

By the Cauchy integral formula, the value of $f(z)$ at any point inside the closed contour C is determined by the values of the function along the boundary contour C.

Example

Apply the Cauchy integral formula to the integral

$$\oint_{|z|=1} \frac{e^{kz}}{z}\, dz, \quad k \text{ is a real constant,}$$

to show that

$$\int_0^{2\pi} e^{k\cos\theta} \sin(k\sin\theta)\, d\theta = 0$$
$$\int_0^{2\pi} e^{k\cos\theta} \cos(k\sin\theta)\, d\theta = 2\pi.$$

Solution

By Cauchy's integral formula: $\displaystyle\oint_{|z|=1} \frac{e^{kz}}{z}\, dz = (2\pi i)e^{kz}\Big|_{z=0} = 2\pi i.$

On the other hand,

$$\begin{aligned} 2\pi i &= \oint_{|z|=1} \frac{e^{kz}}{z}\, dz = \int_0^{2\pi} \frac{e^{k(\cos\theta + i\sin\theta)}}{e^{i\theta}} ie^{i\theta}\, d\theta \\ &= i\int_0^{2\pi} e^{k\cos\theta}[\cos(k\sin\theta) + i\sin(k\sin\theta)]\, d\theta. \end{aligned}$$

Equating the real and imaginary parts, we obtain

$$\begin{aligned} 0 &= \int_0^{2\pi} e^{k\cos\theta}\sin(k\sin\theta)\, d\theta \\ 2\pi &= \int_0^{2\pi} e^{k\cos\theta}\cos(k\sin\theta)\, d\theta. \end{aligned}$$

Example

Evaluate

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz,$$

where C is the circle: $|z-i| = 3$.

Solution

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\, dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\, dz$$

By Cauchy's integral formula, we have

$$\begin{aligned}
\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\, dz &= 2\pi i\{\sin \pi(2)^2 + \cos \pi(2)^2\} = 2\pi i \\
\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\, dz &= 2\pi i\{\sin \pi(1)^2 + \cos \pi(1)^2\} = -2\pi i
\end{aligned}$$

since $z = 1$ and $z = 2$ are inside C and $\sin \pi z^2 + \cos \pi z^2$ is analytic on and inside C. The integral has the value $2\pi i - (-2\pi i) = 4\pi i$.

Remark

Alternately, by Corollary 3 of the Cauchy Integral Theorem, we have

$$\begin{aligned}\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz \;=\; & \oint_{C_1} \frac{(\sin \pi z^2 + \cos \pi z^2)/(z-2)}{z-1}\,dz \\ & + \oint_{C_2} \frac{(\sin \pi z^2 + \cos \pi z^2)/(z-1)}{z-2}\,dz,\end{aligned}$$

where C_1 and C_2 are closed contours completely inside C, C_1 encircles the point $z = 1$ while C_2 encircles the point $z = 2$.

By the Cauchy Integral formula, choosing $f(z) = \dfrac{\sin \pi z^2 + \cos \pi z^2}{z-2}$, we obtain

$$\oint_{C_1} \frac{f(z)}{z-1}\,dz = 2\pi i f(1) = 2\pi i \frac{\sin \pi + \cos \pi}{-1} = 2\pi i.$$

In a similar manner

$$\oint_{C_2} \frac{(\sin \pi z^2 + \cos \pi z^2)/(z-1)}{z-2}\,dz = 2\pi i \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\bigg|_{z=2} = 2\pi i.$$

Hence, the integral is equal to $2\pi i + 2\pi i = 4\pi i$.

The Cauchy integral formula can be extended to the case where the simple closed contour C can be replaced by the oriented boundary of a multiply connected domain.

Suppose C, C_1, $C_2, \ldots, C_n$ and $f(z)$ are given the same conditions as in Corollary 3, then for any point $z \in C \ \cup \ $ int $C \setminus$ int $C_1 \setminus$ int $C_2 \setminus \cdots \setminus$ int C_n, we have

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta - \sum_{k=1}^{n} \frac{1}{2\pi i}\oint_{C_k} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Derivatives of contour integrals

Suppose we differentiate both sides of the Cauchy integral formula formally with respect to z (holding ζ fixed), assuming that differentiation under the integral sign is legitimate, we obtain

$$f'(z) = \frac{1}{2\pi i}\,\frac{d}{dz}\oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\oint_C \frac{d}{dz}\frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta.$$

How to justify the legitimacy of direct differentiation of the Cauchy integral formula? First, consider the expression

$$\begin{aligned}
& \frac{f(z+h) - f(z)}{h} - \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta \\
= \;& \frac{1}{h}\left\{\frac{1}{2\pi i}\oint_C \left[\frac{f(\zeta)}{\zeta - z - h} - \frac{f(\zeta)}{\zeta - z} - h\frac{f(\zeta)}{(\zeta - z)^2}\right]\,d\zeta\right\} \\
= \;& \frac{h}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z - h)\,(\zeta - z)^2}\,d\zeta.
\end{aligned}$$

It suffices to show that the value of the last integral goes to zero as $h \to 0$. To estimate the value of the last integral, we draw the circle C_{2d}: $|\zeta - z| = 2d$ inside the domain bounded by C and choose h such that $0 < |h| < d$.

For every point ζ on the curve C, it is outside the circle C_{2d} so that

$$|\zeta - z| > d \quad \text{and} \quad |\zeta - z - h| > d.$$

Let M be the upper bound of $|f(z)|$ on C and L is the total arc length of C. Using the modulus inequality and together with the above inequalities, we obtain

$$\left| \frac{h}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z - h)\,(\zeta - z)^2}\, d\zeta \right| \leq \frac{|h|}{2\pi}\,\frac{ML}{d^3}.$$

In the limit $h \to 0$, we observe that

$$\lim_{h \to 0} \left| \frac{h}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)^2}\, d\zeta \right| \le \lim_{h \to 0} \frac{|h|}{2\pi} \frac{ML}{d^3} = 0;$$

therefore,

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

By induction, we can show the general result

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta, \qquad k = 1, 2, 3, \cdots,$$

for any z inside C. This result is called the *generalized Cauchy Integral Formula*.

Theorem

If a function $f(z)$ is analytic at a point, then its derivatives of all orders are also analytic at the same point.

Proof

Suppose f is analytic at a point z_0, then there exists a neighborhood of $z_0 : |z - z_0| < \epsilon$ throughout which f is analytic. Take C_0 to be a positively oriented circle centered at z_0 and with radius $\epsilon/2$ such that f is analytic inside and on C_0. We then have

$$f''(z) = \frac{1}{\pi i} \oint_{C_0} \frac{f(\zeta)}{(\zeta - z)^3} \, d\zeta$$

at each point z interior to C_0. The existence of $f''(z)$ throughout the neighborhood: $|z - z_0| < \epsilon/2$ means that f' is analytic at z_0. Repeating the argument to the analytic function f', we can conclude that f'' is analytic at z_0.

Remarks

(i) The above theorem is limited to complex functions only. In fact, no similar statement can be made on real differentiable functions. It is easy to find examples of real valued function $f(x)$ such that $f'(x)$ exists but not so for $f''(x)$ at certain points.

(ii) Suppose we express an analytic function inside a domain $\mathcal{D}$ as $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$. Since its derivatives of all orders are analytic functions, it then follows that the partial derivatives of $u(x, y)$ and $v(x, y)$ of all orders exist and are continuous.

To see this, since $f''(z)$ exists, we consider

$$f''(z) = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} - i\frac{\partial^2 u}{\partial y \partial x} \left[\text{from } f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]$$

or

$$f''(z) = \frac{\partial^2 v}{\partial x \partial y} - i\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2} - i\frac{\partial^2 v}{\partial y^2} \left[\text{from } f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right].$$

The continuity of f'' implies that all second order partials of u and v are continuous at points where f is analytic. Continuing with the process, we obtain the result.

The mere assumption of the analyticity of $f(z)$ on and inside C is sufficient to guarantee the existence of the derivatives of $f(z)$ of all orders. Moreover, the derivatives are all continuous on and inside C.

Example

Suppose $f(z)$ is defined by the integral

$$f(z) = \oint_{|\zeta|=3} \frac{3\zeta^2 + 7\zeta + 1}{\zeta - z}\, d\zeta,$$

find $f'(1+i)$.

Solution

Setting $k = 1$ in the generalized Cauchy integral formula,

$$\begin{aligned} f'(z) &= \oint_{|\zeta|=3} \frac{3\zeta^2 + 7\zeta + 1}{(\zeta - z)^2}\, d\zeta \\ &= \oint_{|\zeta|=3} \frac{3(\zeta - z)^2 + (6z+7)(\zeta - z) + 3z^2 + 7z + 1}{(\zeta - z)^2}\, d\zeta \\ &= \oint_{|\zeta|=3} 3\ d\zeta + (6z+7) \oint_{|\zeta|=3} \frac{1}{\zeta - z}\, d\zeta \\ &\quad + (3z^2 + 7z + 1) \oint_{|\zeta|=3} \frac{1}{(\zeta - z)^2}\, d\zeta. \end{aligned}$$

The first integral equals zero since the integrand is entire (a constant function). For the second integral, we observe that

$$\oint_{|\zeta|=3} \frac{1}{\zeta - z}\, d\zeta = \begin{cases} 0 & \text{if } |z| > 3 \\ 2\pi i & \text{if } |z| < 3 \end{cases}.$$

Furthermore, we deduce that the third integral is zero since

$$\oint_{|\zeta|=3} \frac{1}{(\zeta - z)^2}\, d\zeta = \frac{d}{dz}\left[\oint_{|\zeta|=3} \frac{1}{\zeta - z}\, d\zeta\right] = 0.$$

Combining the results, we have

$$f'(z) = (2\pi i)(6z + 7) \quad \text{if } |z| < 3.$$

We observe that $1 + i$ is inside $|z| < 3$ since $|1 + i| = \sqrt{2} < 3$. Therefore, we obtain

$$f'(1+i) = 2\pi i\,[6(1+i) + 7] = -12\pi + 26\pi i.$$

Example

Evaluate

$\oint_C \frac{e^{2z}}{(z+1)^4}\,dz$, where C is the circle $|z| = 3$.

Solution

Let $f(z) = e^{2z}$ and $a = -1$ in the Cauchy integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz.$$

If $n = 3$, then $f'''(z) = 8e^{2z}$ and $f'''(-1) = 8e^{-2}$. Hence,

$$8e^{-2} = \frac{3!}{2\pi i}\oint \frac{e^{2z}}{(z+1)^4}\,dz$$

from which we see the required integral has the value $8\pi i e^{-2}/3$.

Cauchy inequality

Suppose $f(z)$ is analytic on and inside the disc $|z-z_0| = r,\ 0 < r < \infty$, and let

$$M(r) = \max_{|z-z_0|=r} |f(z)|,$$

then

$$\frac{|f^{(k)}(z)|}{k!} \leq \frac{M(r)}{r^k}, \quad k = 0, 1, 2, \ldots.$$

This inequality follows from the generalized Cauchy integral formula.

Example

Suppose $f(z)$ is analytic inside the unit circle $|z| = 1$ and

$$|f(z)| \leq \frac{1}{1 - |z|},$$

show that

$$|f^{(n)}(0)| \leq (n+1)! \left(1 + \frac{1}{n}\right)^n.$$

Solution

We integrate $\frac{f(\zeta)}{\zeta^{n+1}}$ around the circle $|\zeta| = \frac{n}{n+1}$, where $f(\zeta)$ is analytic on and inside the circle. Using the generalized Cauchy integral formula, we have

$$f^{(n)}(0) = \frac{n!}{2\pi i} \oint_{|\zeta| = \frac{n}{n+1}} \frac{f(\zeta)}{\zeta^{n+1}} \, d\zeta$$

$$
\begin{aligned}
&= \frac{n!}{2\pi i}\int_0^{2\pi} \frac{f\left(\frac{n}{n+1}e^{i\theta}\right)}{\left(\frac{n}{n+1}\right)^{n+1} e^{i(n+1)\theta}} \left(\frac{n}{n+1}\right) e^{i\theta}\, i\, d\theta \\
&= \left(1+\frac{1}{n}\right)^n \frac{n!}{2\pi}\int_0^{2\pi} f\left(\frac{n}{n+1}e^{i\theta}\right) e^{-in\theta}\, d\theta.
\end{aligned}
$$

The modulus $|f^{(n)}(0)|$ is bounded by

$$
\begin{aligned}
|f^{(n)}(0)| &\leq \left(1+\frac{1}{n}\right)^n \frac{n!}{2\pi}\int_0^{2\pi} \left| f\left(\frac{n}{n+1}e^{i\theta}\right)\right| d\theta \\
&\leq \left(1+\frac{1}{n}\right)^n \frac{n!}{2\pi}\int_0^{2\pi} \frac{1}{1-\frac{n}{n+1}}\, d\theta \\
&= \left(1+\frac{1}{n}\right)^n \frac{n!}{2\pi}\left[(n+1)\, 2\pi\right] \\
&= (n+1)!\left(1+\frac{1}{n}\right)^n.
\end{aligned}
$$

Gauss' mean value theorem

If $f(z)$ is analytic on and inside the disc $C_r\colon |z - z_0| = r$, then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta.$$

Proof

From the Cauchy integral formula, we have

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\oint_{C_r} \frac{f(z)}{z - z_0}\, dz \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta}) ire^{i\theta}}{re^{i\theta}}\, d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta. \end{aligned}$$

Write $u(z) = \text{Re } f(z)$, it is known that u is harmonic. We have

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta.$$

Example

Find the mean value of $x^2 - y^2 + x$ on the circle $|z - i| = 2$.

Solution

First, we observe that $x^2 - y^2 + x = \text{Re}(z^2 + z)$. The mean value is defined by

$$\frac{1}{2\pi}\int_0^{2\pi} u(i + 2e^{i\theta})\, d\theta,$$

where $u(z) = \text{Re}(z^2 + z)$. By Gauss' mean value theorem,

$$\frac{1}{2\pi}\int_0^{2\pi} u(i + 2e^{i\theta})\, d\theta = \text{Re}(z^2 + z)\Big|_{z=i} = \text{Re}(-1 + i) = -1.$$

Maximum modulus theorem

If $f(z)$ is analytic on and inside a domain $\mathcal{D}$ bounded by a simple closed curve C, then the maximum value of $|f(z)|$ occurs on C, unless $f(z)$ is a constant.

Example

Find the maximum value of $|z^2 + 3z - 1|$ in the disk $|z| \leq 1$.

Solution

The triangle inequality gives

$$|z^2 + 3z - 1| \leq |z^2| + 3|z| + 1 \leq 5, \quad \text{for } |z| \leq 1.$$

However, the maximum value is actually smaller than this, as the following analysis shows.

The maximum of $|z^2+3z-1|$ must occur on the boundary of the disk ($|z| = 1$). The latter can be parameterized as $z = e^{it}, 0 \leq t \leq 2\pi$; whence

$$|z^2 + 3z - 1|^2 = (e^{i2t} + 3e^{it} - 1)(e^{-i2t} + 3e^{-it} - 1).$$

Expanding and gathering terms reduces this to $(11-2\cos 2t)$, whose maximum value is 13. The maximum value is obtained by taking $t = \pi/2$ or $t = 3\pi/2$.

Thus the maximum value of $|z^2 + 3z - 1|$ is $\sqrt{13}$, which occurs at $z = \pm i$.

Example

Let R denote the rectangular region:

$$0 \leq x \leq \pi, \quad 0 \leq y \leq 1,$$

the modulus of the entire function

$$f(z) = \sin z$$

has a maximum value in R that occurs on the boundary.

To verify the claim, consider

$$|f(z)| = \sqrt{\sin^2 x + \sinh^2 y},$$

the term $\sin^2 x$ is greatest at $x = \pi/2$ and the increasing function $\sinh^2 y$ is greatest when $y = 1$. The maximum value of $|f(z)|$ in $\mathcal{R}$ occurs at the boundary point $\left(\frac{\pi}{2}, 1\right)$ and at no other point in $\mathcal{R}$.

Proof of the Maximum Modulus Theorem

Proof by contradiction. Suppose $|f(z)|$ attains its maximum at $\alpha \in \mathcal{D}$. Using the Gauss Mean Value Theorem:

$$f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\theta})\, d\theta$$

where the neighborhood $N(\alpha; r)$ lies inside $\mathcal{D}$. By the modulus inequality,

$$|f(\alpha)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(\alpha + re^{i\theta})|\, d\theta.$$

Since $|f(\alpha)|$ is a maximum, then $|f(\alpha+re^{i\theta})| \le |f(\alpha)|$ for all θ, giving

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\alpha + re^{i\theta})|\, d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |f(\alpha)|\, d\theta = |f(\alpha)|.$$

Combining the results, we obtain

$$\int_0^{2\pi} \underbrace{\left[|f(\alpha)| - |f(\alpha + re^{i\theta})|\right]}_{\text{non-negative}} d\theta = 0.$$

One then infer that $|f(\alpha)| = |f(\alpha + re^{i\theta})|$. However, it may be possible to have $|f(\alpha + re^{i\theta})| < |f(\alpha)|$ at isolated points. We argue that this is not possible due to continuity of $f(z)$.

If $|f(\alpha + re^{i\theta})| < |f(\alpha)|$ at a single point, then $|f(\alpha + re^{i\theta})| < |f(\alpha)|$ for a finite arc on the circle, giving

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\alpha + re^{i\theta})|\, d\theta < |f(\alpha)|,$$

a contradiction. We can then deduce that

$$|f(\alpha)| = |f(\alpha + re^{i\theta})|$$

for all points on the circle.

- Since r can be any value, $|f(z)|$ is constant in any neighborhood of α lying inside $\mathcal{D}$.

Finally, we need to show that $|f(z)|$ is constant at any point in $\mathcal{D}$. Take any $z \in \mathcal{D}$, we can join α to z by a curve lying completely inside $\mathcal{D}$. Taking a sequence of points $z_0 = \alpha, z_1, \cdots, z_n = z$ such that each of these points is the center of a disc (plus its boundary) lying completely inside $\mathcal{D}$ and z_k is contained in the disk centered at $z_{k-1}, k = 1, 2, \cdots, n$.

We then have $|f(z_1)| = |f(\alpha)|$. Also z_2 is contained inside the disc centered at z_1, so $|f(z_2)| = |f(z_1)|, \cdots$, and deductively $|f(z)| = |f(\alpha)|$.

Lastly, we use the result that if $|f(z)| =$ constant throughout $\mathcal{D}$, then $f(z) =$ constant throughout $\mathcal{D}$.

Liouville's Theorem

If f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

Proof

It suffices to show that $f'(z) = 0$ for all $z \in \mathbb{C}$. We integrate $\dfrac{f(\zeta)}{(\zeta - z)^2}$ around $C_R : |\zeta - z| = R$. By the generalized Cauchy integral formula

$$f'(z) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(\zeta)}{(\zeta - z)^2}\, d\xi,$$

which remains valid for any sufficiently large R since $f(z)$ is entire. Since $f(z)$ is bounded, so $|f(z)| \leq B$ for all $z \in \mathbb{C}$,

$$|f'(z)| = \frac{1}{2\pi} \left| \oint_{C_R} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \right| \leq \frac{1}{2\pi} \frac{B}{R^2} 2\pi R = \frac{B}{R}.$$

Now, B is independent of R and R can be arbitrarily large. The inequality can hold for arbitrarily large values of R only if $f'(z) = 0$.

Since the choice of z is arbitrary, this means that $f'(z) = 0$ everywhere in the complex plane. Consequently, f is a constant function.

Remark

Non-constant entire functions must be unbounded. For example, $\sin z$ and $\cos z$ are unbounded, unlike their real counterparts.